

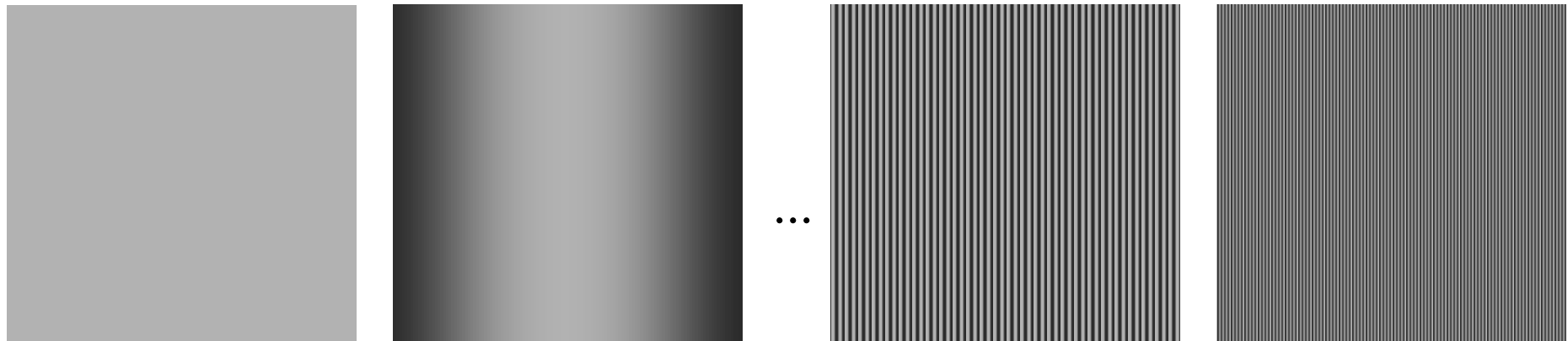
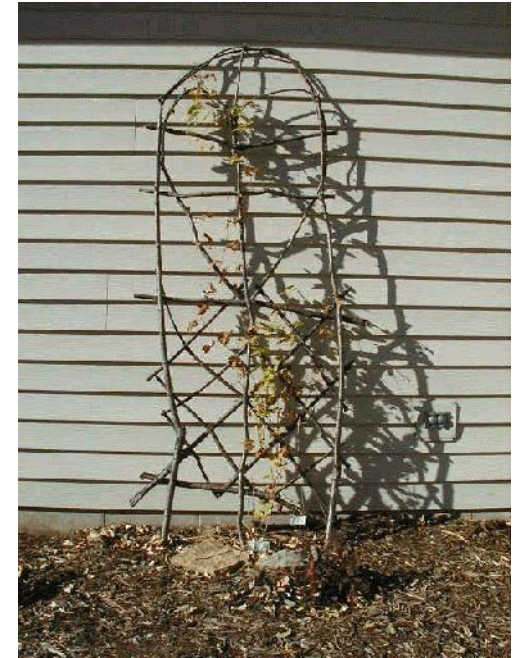
Computer Graphics

Spectral Analysis

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Spatial Frequency

- **Frequency**
 - Inverse of period length of some structure in an image
 - Unit [1/pixel]
- **Lowest frequency**
 - Image average
- **Highest representable frequency**
 - Nyquist frequency (1/2 the sampling frequency)
 - Defined by half the image resolution
- **Phase allows shifting of the pattern**



Fourier Transformation

- Any continuous function $f(x)$ can be expressed as an integral over sine and cosine waves:

$$\text{Analysis: } F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi kx} dx$$

$$\text{Synthesis: } f(x) = F_x^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k)e^{i2\pi kx} dk$$

- **Representation via complex exponential**

- $e^{ix} = \cos(x) + i \sin(x)$ (see Taylor expansion)

- **Division into even and odd parts**

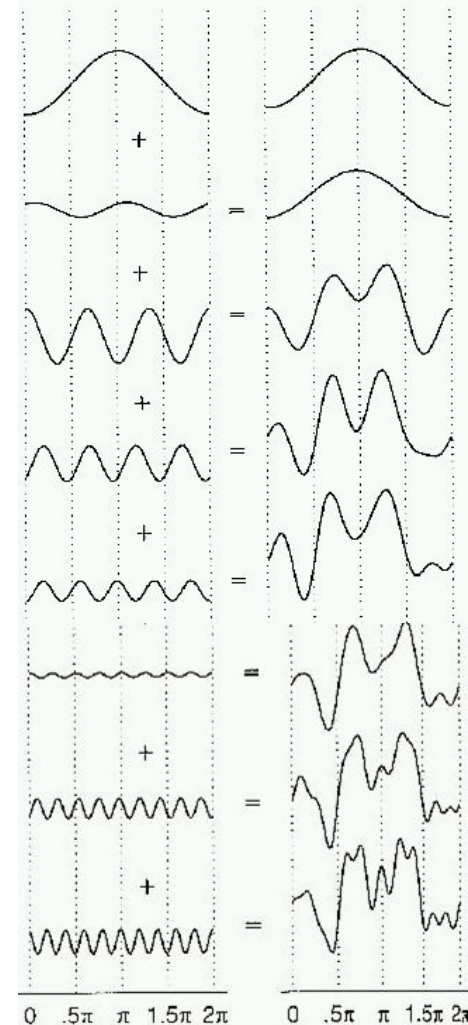
- Even: $f(x) = f(-x)$ (symmetry about y axis)

- Odd: $f(x) = -f(-x)$ (symmetry about origin)

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x)$$

- **Transform of each part**

- Even: *cosine* only; odd: *sine* only



Analysis & Synthesis

Symmetric integral $([-a, a])$
over an odd function is zero

- **Analysis**

$$F(k) = \int_{-\infty}^{\infty} f(x) (\cos(-2\pi kx) + i \sin(-2\pi kx)) dx = b(k) - i a(k)$$

– Even term

$$b(k) = \int_{-\infty}^{\infty} f(x) \cos(2\pi kx) dx = \int_{-\infty}^{\infty} (E(x) + O(x)) \cos(2\pi kx) dx = \int_{-\infty}^{\infty} E(x) \cos(2\pi kx) dx$$

– Odd term

$$a(k) = \int_{-\infty}^{\infty} f(x) \sin(2\pi kx) dx = \int_{-\infty}^{\infty} (E(x) + O(x)) \sin(2\pi kx) dx = \int_{-\infty}^{\infty} O(x) \sin(2\pi kx) dx$$

- **Synthesis**

$$f(x) = \int_{-\infty}^{\infty} F(k) (\cos(2\pi kx) + i \sin(2\pi kx)) dk = E(x) + O(x)$$

– Even term

$$E(x) = \int_{-\infty}^{\infty} F(k) \cos(2\pi kx) dk = \int_{-\infty}^{\infty} (b(k) - i a(k)) \cos(2\pi kx) dk = \int_{-\infty}^{\infty} b(k) \cos(2\pi kx) dk$$

– Odd term

$$O(x) = \int_{-\infty}^{\infty} F(k) i \sin(2\pi kx) dk = \int_{-\infty}^{\infty} (b(k) - i a(k)) i \sin(2\pi kx) dk = \int_{-\infty}^{\infty} a(k) \sin(2\pi kx) dk$$

Spatial vs. Frequency Domain

- **Important basis functions:**

- Box \leftrightarrow (normalized) *sinc*

$$\text{sinc}(x) = \frac{\sin(x\pi)}{x\pi}$$

$$\text{sinc}(0) = 1$$

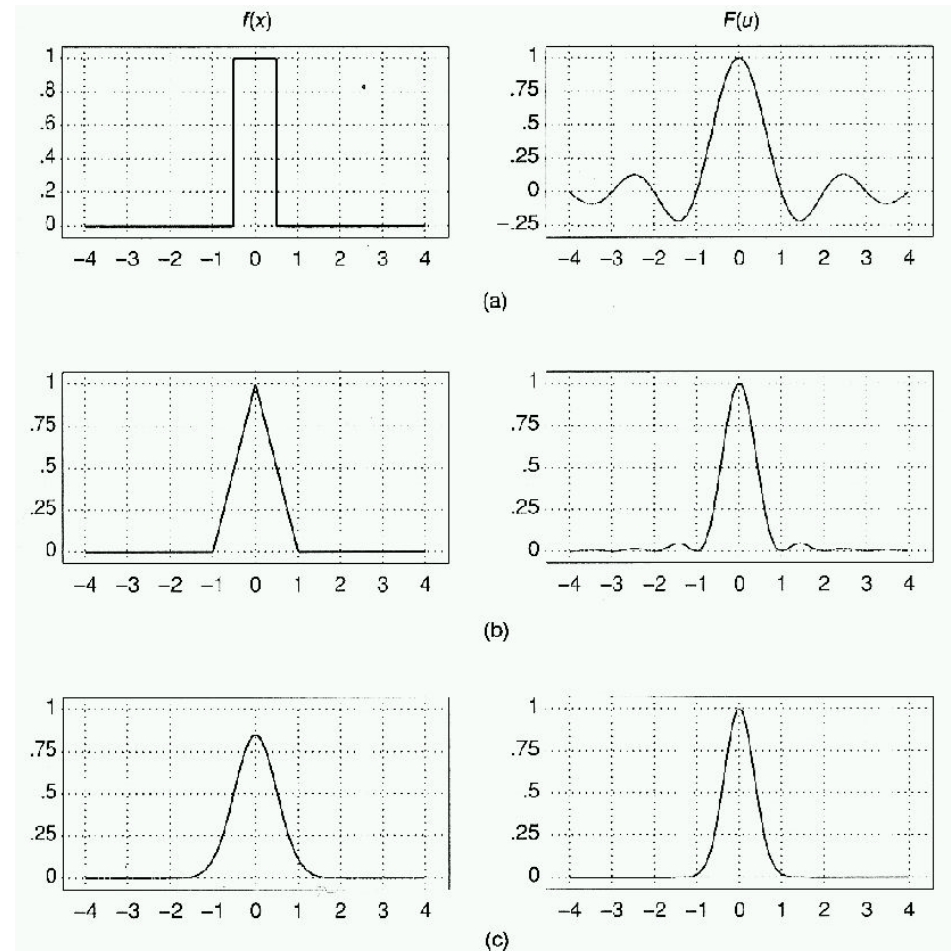
$$\int \text{sinc}(x) dx = 1$$

- Negative values
- Infinite support

- Tent $\leftrightarrow \text{sinc}^2$

- Tent == Convolution of box function with itself

- Gaussian \leftrightarrow Gaussian



Spatial vs. Frequency Domain

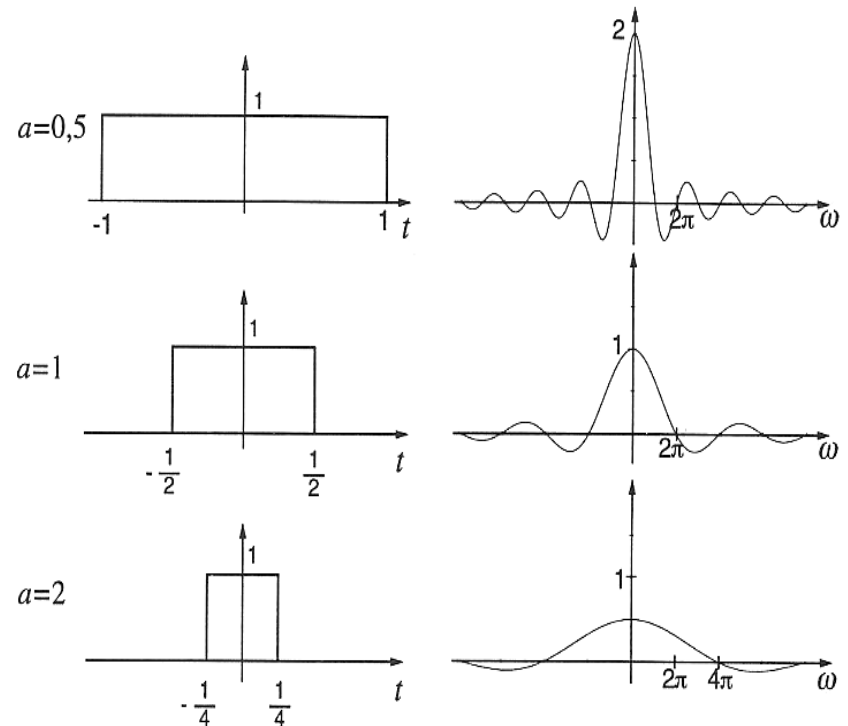
- Transform behavior
- Example: Fourier transform of a box function

$$\text{rect}(at) \quad \longleftrightarrow \quad \frac{1}{|a|} \text{si}\left(\frac{\omega}{2a}\right)$$

– Wide box \rightarrow narrow *sinc*

– Box \rightarrow *sinc*

– Narrow box \rightarrow wide *sinc*



Fourier Transformation

- **Periodic in space \Leftrightarrow discrete in frequency (vice ver.)**

- Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:

$$f(x) = \sum_k a_k \sin(2\pi k x) + b_k \cos(2\pi k x)$$

- Any finite interval can be made periodic by concatenating with itself

- **Decomposition of signal into different frequency bands: Spectral Analysis**

- Frequency band: k
 - $k = 0$: mean value
 - $k = 1$: function period, lowest possible frequency
 - $k = 1.5$? : not possible, periodic function, e.g. $f(x) = f(x+1)$
 - k_{max} ? : band limit, no higher frequency present in signal
- Fourier coefficients: a_k, b_k (real-valued, as before)
 - Even function $f(x) = f(-x)$: $a_k = 0$
 - Odd function $f(x) = -f(-x)$: $b_k = 0$

Fourier Synthesis Example

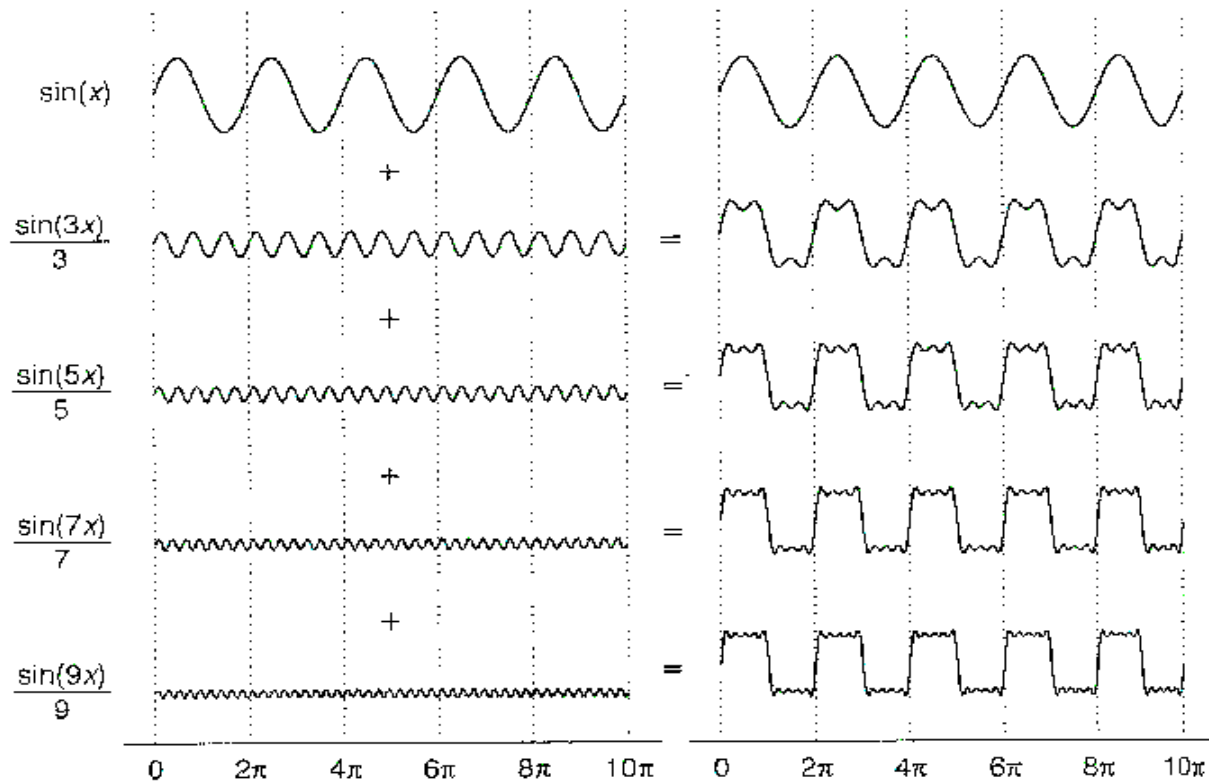
- **Square wave: periodic, uneven function**

$$f(x) = 0.5 \quad \forall 0 < (x \bmod 2\pi) < \pi$$

$$= -0.5 \quad \forall \pi < (x \bmod 2\pi) < 2\pi$$

$$a_k = \int \sin(2\pi kx) f(x) dx \quad f(x) = \sum_k a_k \sin(2\pi kx)$$

- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = 1/3$
- $a_4 = 0$
- $a_5 = 1/5$
- $a_6 = 0$
- $a_7 = 1/7$
- $a_8 = 0$
- $a_9 = 1/9$
- ...



Discrete Fourier Transform

- **Equally-spaced function samples (N samples)**
 - Function values known only at discrete points, e.g.
 - Physical measurements
 - Pixel positions in an image!
 - Represented via sum of Delta distribution (Fourier integrals \rightarrow sums)

- **Fourier analysis**

$$a_k = \sum_i \sin\left(\frac{2\pi ki}{N}\right) f_i$$

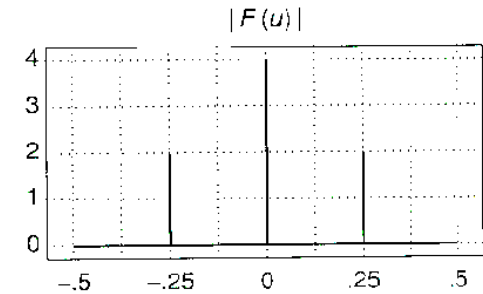
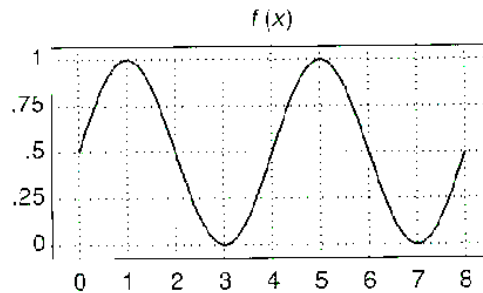
$$b_k = \sum_i \cos\left(\frac{2\pi ki}{N}\right) f_i$$

- Sum over all N measurement points
- $k = 0, 1, 2, \dots$? Highest possible frequency?
 - Nyquist frequency: highest frequency that can be represented
 - Defined as 1/2 the sampling frequency
 - Sampling rate N : determined by image resolution (pixel size)
 - 2 samples / period \leftrightarrow 0.5 cycles per pixel $\Rightarrow k_{max} \leq N / 2$

Spatial vs. Frequency Domain

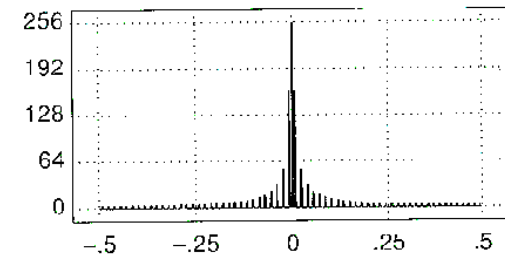
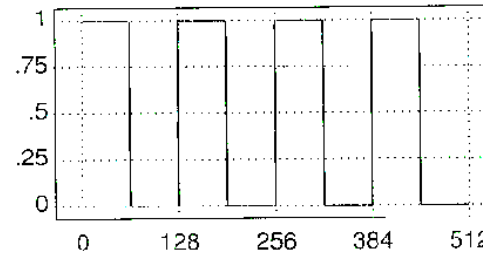
- **Examples (pixels vs. cycles per pixel)**

- Sine wave with positive offset



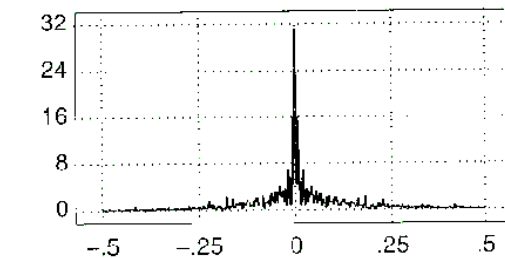
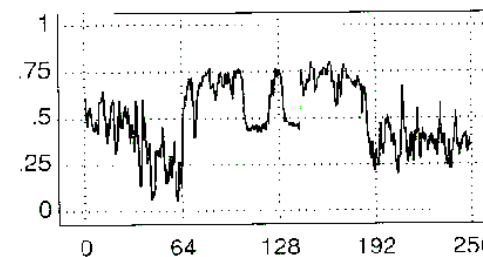
(a)

- Square wave with offset



(b)

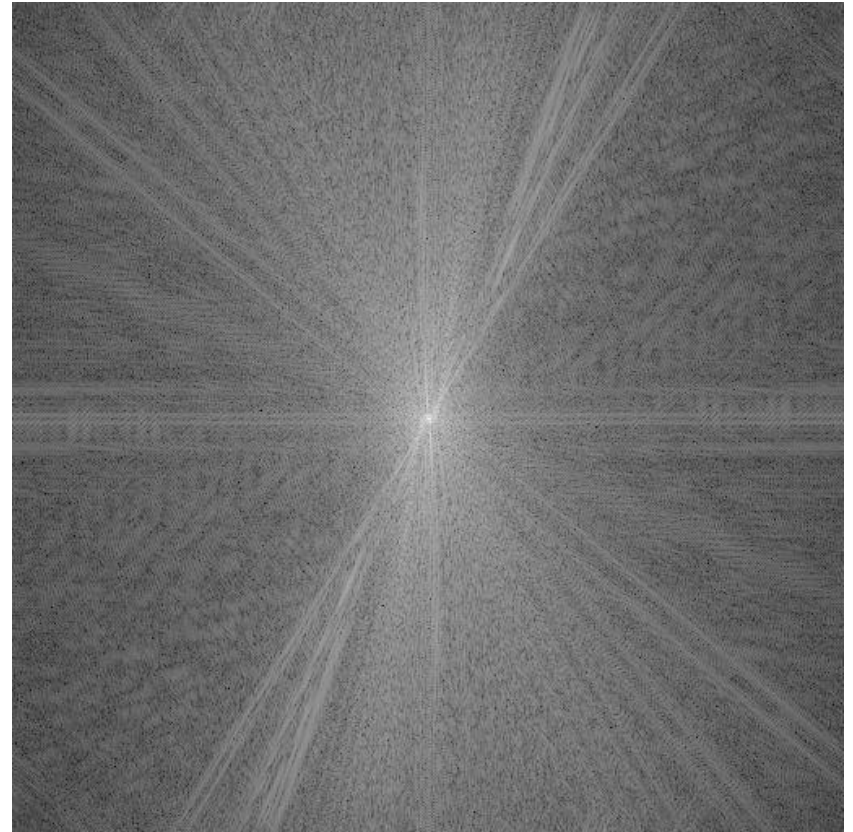
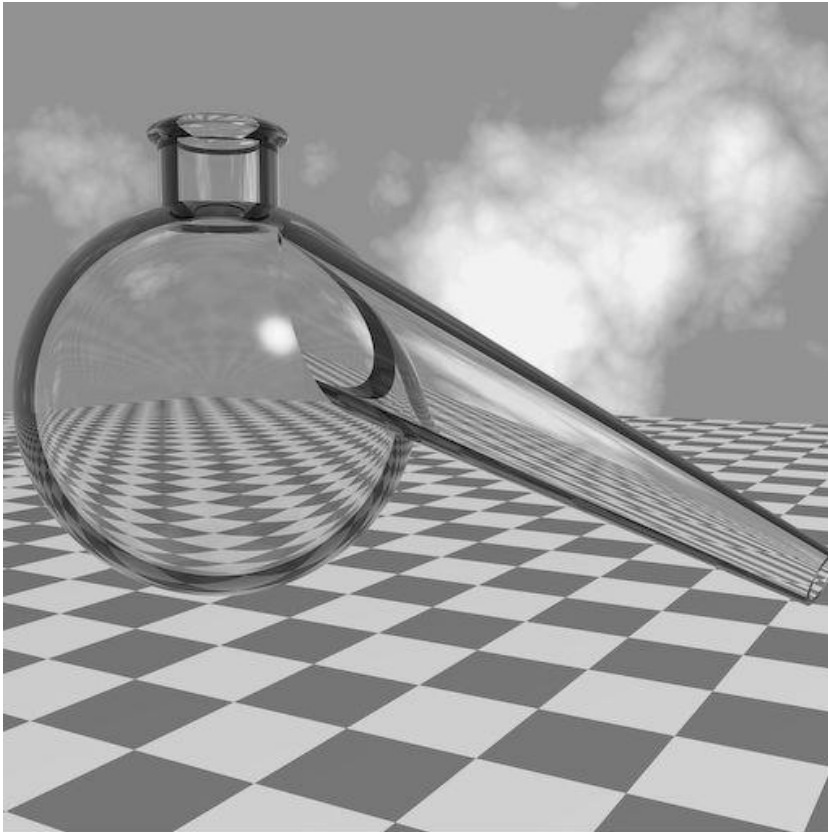
- Scanline of an image



(c)

2D Fourier Transform

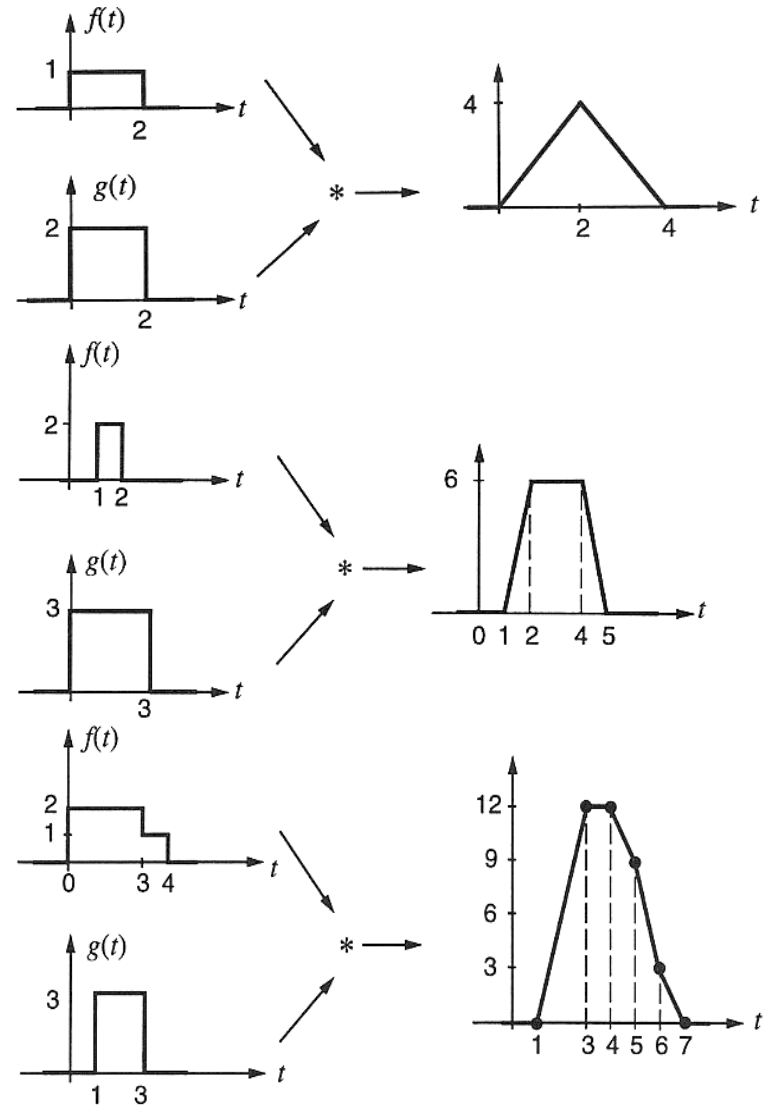
- 2 separate 1D Fourier transformations along x and y directions
- Discontinuities: orthogonal direction in Fourier domain !



Convolution

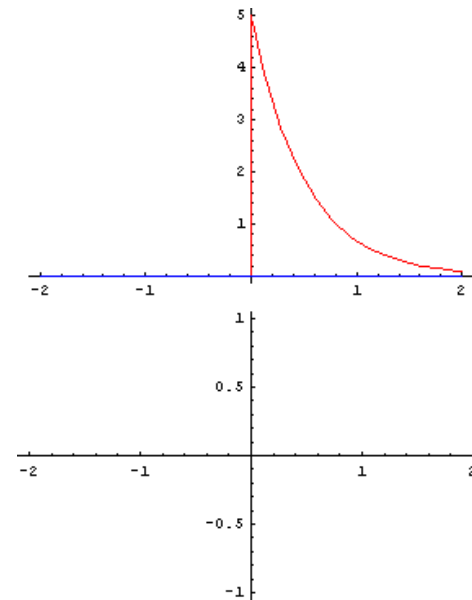
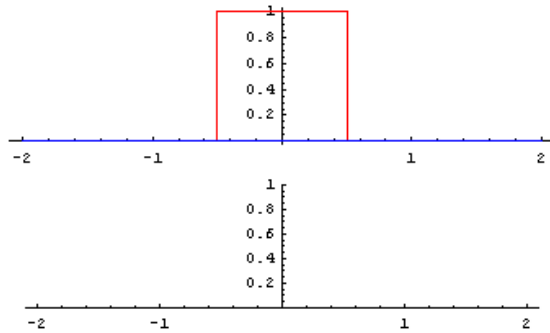
$$(f \otimes g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x - \tau)d\tau$$

- **Two functions f, g**
- **Shift one (reversed) function against the other by x**
- **Multiply function values**
- **Integrate overlapping region**
- **Numerical convolution: expensive operation**
 - For each x :
integrate over non-zero domain



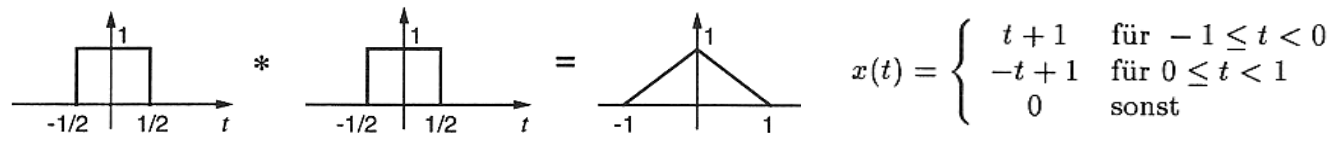
Convolution

- Examples

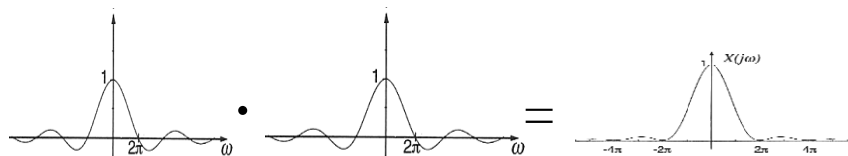


Convolution Theorem

- **Convolution in image domain**
→ Multiplication in Fourier domain
- **Convolution in Fourier domain**
→ Multiplication in image domain
- **Multiplication in transformed Fourier domain may be cheaper than direct convolution in image domain !**

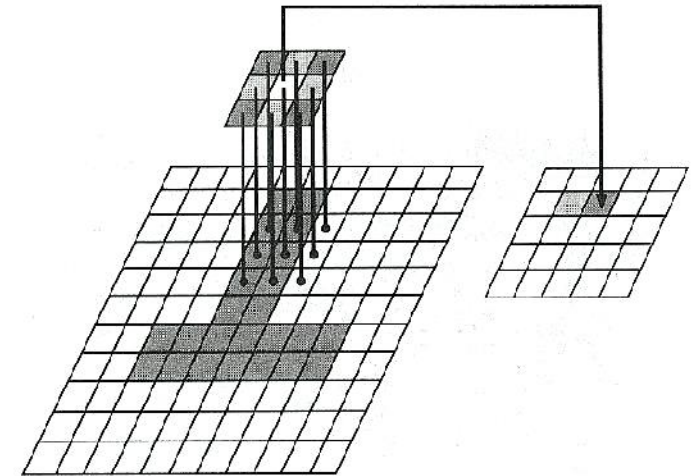
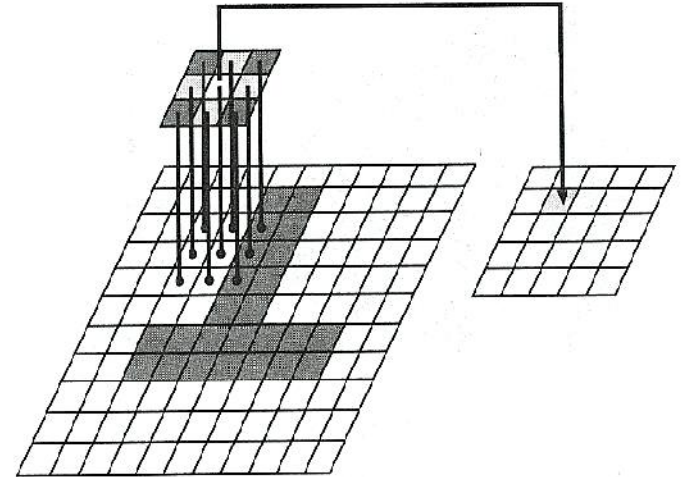


$$\begin{array}{ccc} \text{rect}(t) * \text{rect}(t) & = & x(t) \\ \text{si}\left(\frac{\omega}{2}\right) \cdot \text{si}\left(\frac{\omega}{2}\right) & = & X(j\omega) = \text{si}^2\left(\frac{\omega}{2}\right) \end{array}$$



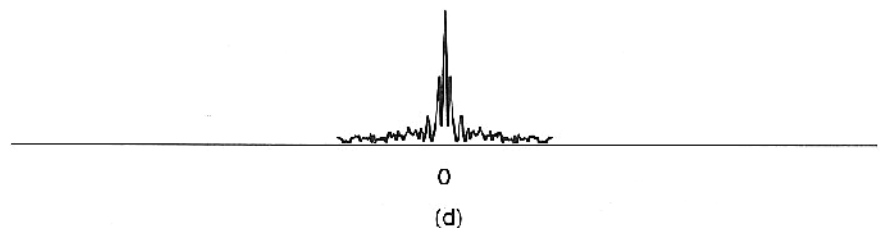
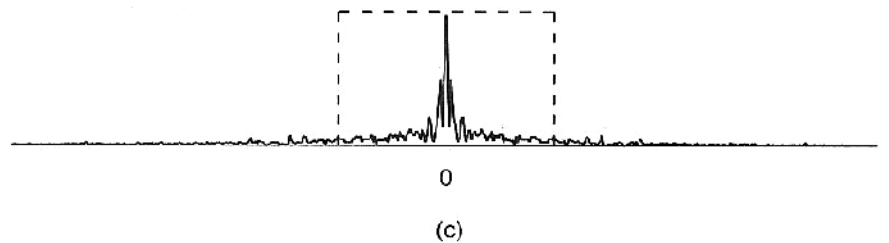
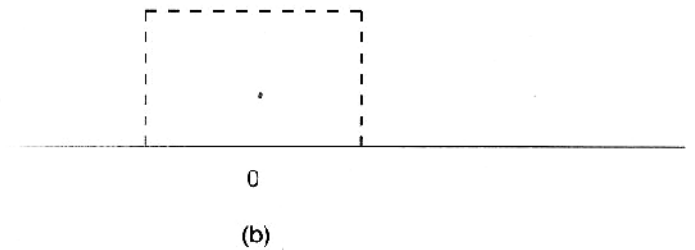
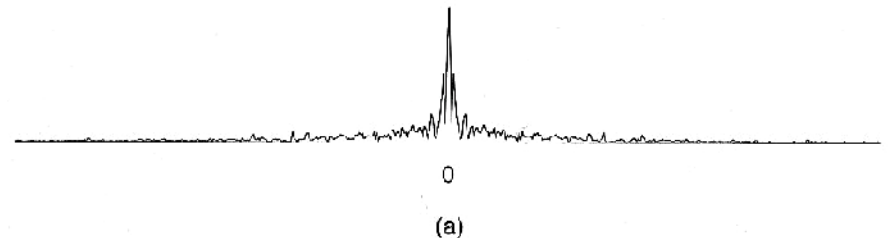
Convolution and Filtering

- **Technical realization**
 - In image domain
 - Pixel mask with weights
- **Problems (e.g. *sinc*)**
 - Large filter support
 - Large mask
 - A lot of computation
 - Negative weights
 - Negative light?



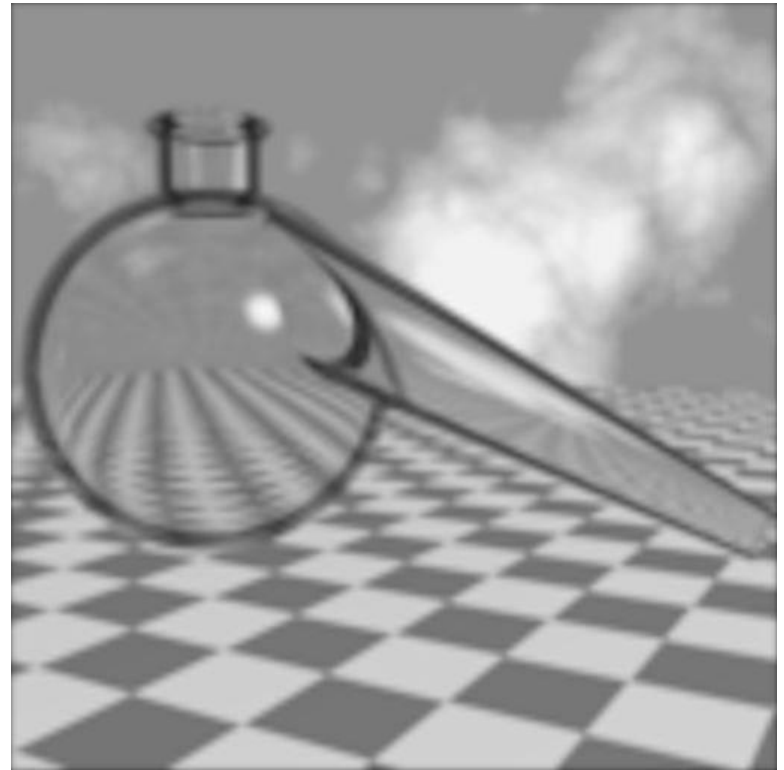
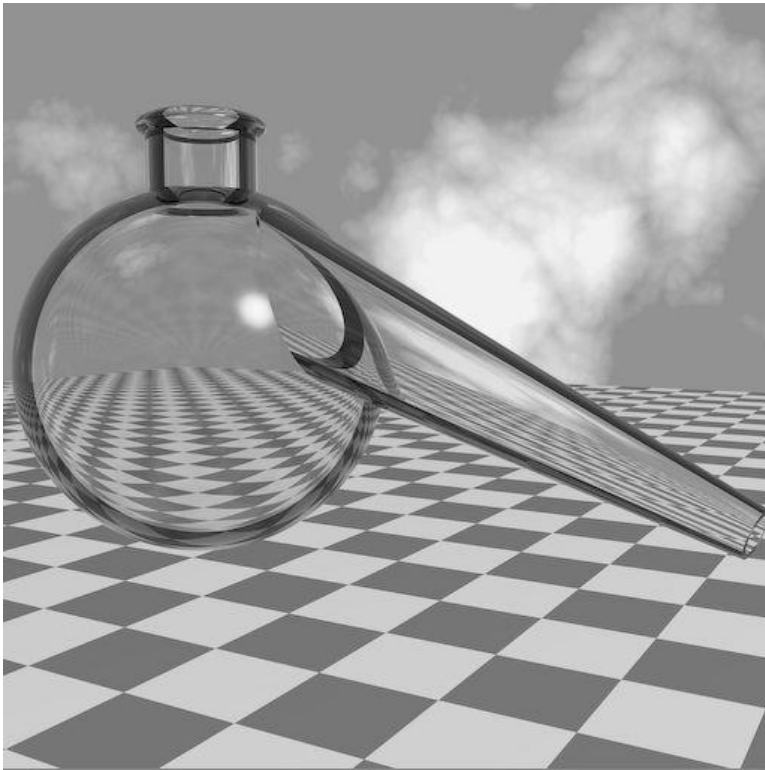
Filtering

- **Low-pass filtering**
 - Multiplication with box in frequency domain
 - Convolution with *sinc* in spatial domain
- **High-pass filtering**
 - Multiplication with $(1 - \text{box})$ in frequency domain
 - Only high frequencies
- **Band-pass filtering**
 - Only intermediate



Low-Pass Filtering

- “Blurring”



High-Pass Filtering

- **Enhances discontinuities in image**
 - Useful for edge detection

