Computer Graphics

- Splines -

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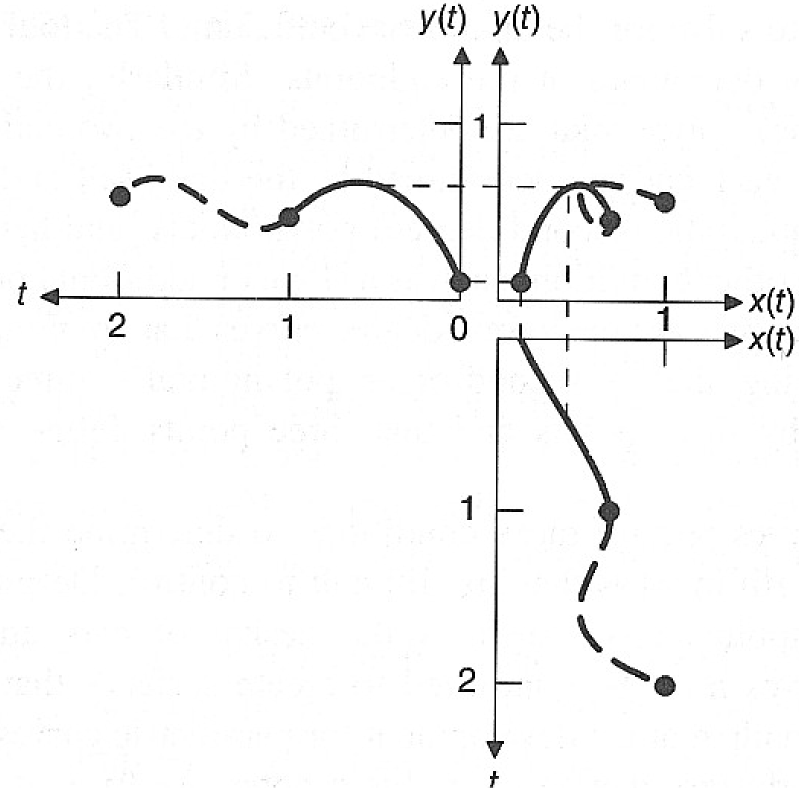
Curves

- **Curve descriptions**
  - Explicit functions
    - $y(x) = \pm \text{sqrt}(r^2 - x^2)$, restricted domain ($x \in [-1, 1]$)
  - Implicit functions
    - $x^2 + y^2 = r^2$ unknown solution set
  - Parametric functions
    - $x(t) = r \cos(t)$, $y(t) = r \sin(t)$, $t \in [0, 2\pi]$
    - Flexibility and ease of use
- **Typically, use of polynomials**
  - Avoids complicated functions (z.B. pow, exp, sin, sqrt)
  - Use simple polynomials, typically of low degree

Parametric curves

- **Separate function in each coordinate**
  - 3D: f(t)= (x(t), y(t), z(t))



Monomials

- **Monomial basis**
  - Simple basis: 1, t, t^2, ... (t usually in [0 .. 1])
- **Polynomial representation**

$$\underline{P}(t) = \begin{pmatrix} \underline{x}(t) & \underline{y}(t) & \underline{z}(t) \end{pmatrix} = \sum_{i=0}^{n} t^i \underline{A}_i$$

Degree (= Order – 1)

Coefficients ∈R³

Monomials

  - Coefficients can be determined from a sufficient number of constraints (e.g. interpolation of given points)
    - Given (n+1) parameter values t_i and points P_i
    - Solution of a linear system in the A_i − possible, but inconvenient
- **Matrix representation**

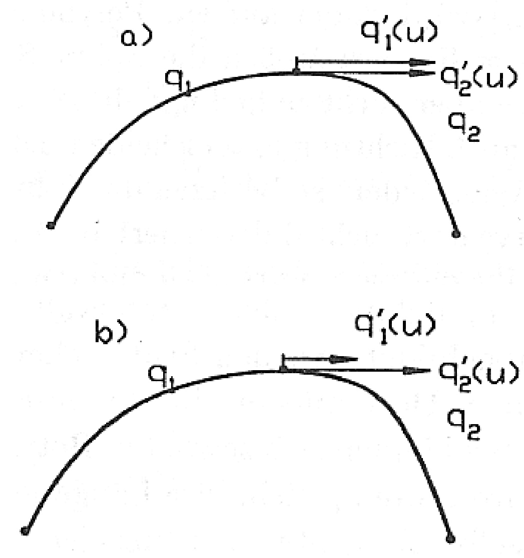
$$P(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix} = T(t)\,\mathbf{A} = \begin{bmatrix} t^n & t^{n-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} A_{x,n} & A_{y,n} & A_{z,n} \\ A_{x,n-1} & A_{y,n-1} & A_{z,n-1} \\ & \vdots & \\ A_{x,0} & A_{y,0} & A_{z,0} \end{bmatrix}$$

Derivatives

- **Derivative = tangent vector**
  - Polynomial of degree (n-1)

$$P'(t) = \begin{pmatrix} x'(t) & y'(t) & z'(t) \end{pmatrix} = T'(t)\,A = \begin{bmatrix} nt^{n-1} & (n-1)t^{n-2} & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} A_{x,n} & A'_{y,n} & A_{z,n} \\ A_{x,n-1} & A_{y,n-1} & A_{z,n-1} \\ & \vdots & \\ A_{x,0} & A_{y,0} & A_{z,0} \end{bmatrix}$$

- **Continuity and smoothness between parametric curves**
  - $C^0 = G^0$ = same point
  - *Parametric* continuity C^1
    - Tangent vectors are identical
  - *Geometric* continuity G^1
    - Same direction of tangent vectors
  - Similar for higher order derivatives



More on Continuity

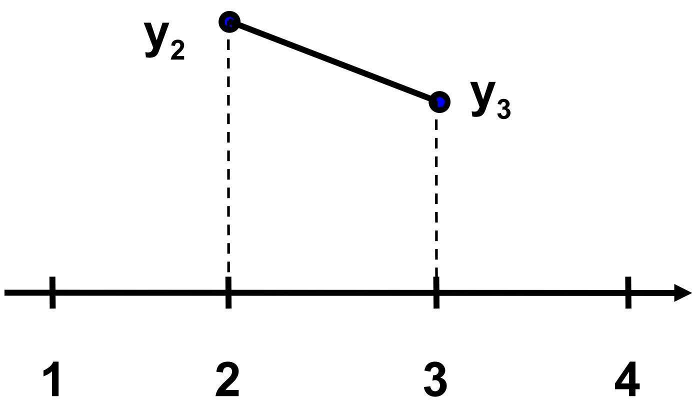
- **At one point:**

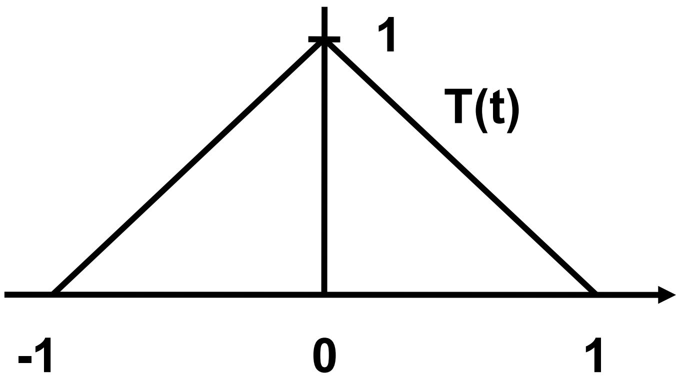
- **Geometric Continuity:**
  - G0: curves are joined together at that point
  - G1: first derivatives are proportional at joint point
    - Same direction but not necessarily same length
  - G2: first and second derivatives are proportional

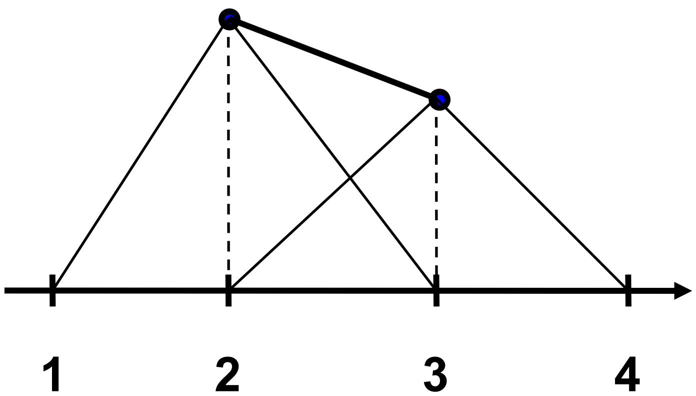
- **Parametric Continuity:**
  - C0: curves are joined
  - C1: first derivative equal
  - C2: first and second derivatives are equal.
    - If t is the time, this implies the acceleration is continuous.
  - Cn: all derivatives up to and including the nth are equal.

Linear Interpolation

- **Hat Functions and Linear Splines (C0/G0 continuity)**







$$T(\mathrm{t}) = \begin{cases} 0 & t < -1 \\ 1+t & -1 \le t < 0 \\ 1-t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

$$\mathrm{P(t)} = \sum \mathrm{P}_i \mathrm{T}_i(t) = \mathrm{y}_2\mathrm{T}_2(t) + \mathrm{y}_3\mathrm{T}_3(t)$$

$$T_i(t) = T(t - i)$$

Lagrange Interpolation

- **Interpolating basis functions**
  - Lagrange polynomials for a set of parameter values $T=\{t_0, ..., t_n\}$

$$L_i^n(t) = \prod_{\substack{j=o \\ i \neq j}}^{n} \frac{t - t_j}{t_i - t_j}, \quad \text{with} \quad L_i^n(t_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

- **Properties**
  - Good for interpolation at given parameter values
    - At each t_i: One basis function = 1, all others = 0
  - Polynomial of degree n (n factors linear in t)
    - Infinitely continuous derivatives everywhere
- **Lagrange Curves**
  - Use Lagrange Polynomials with point coefficients

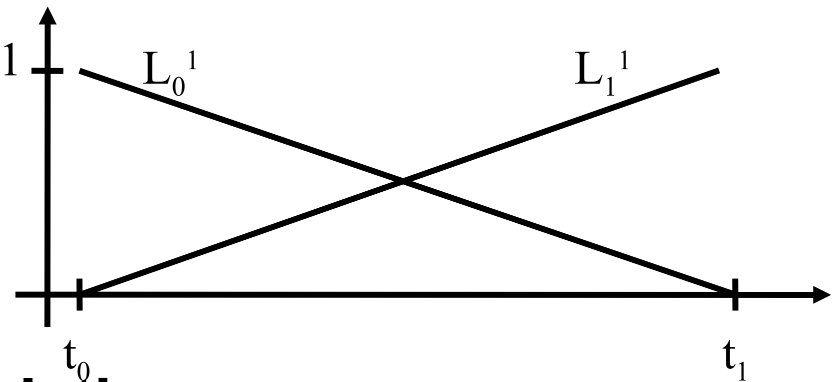
$$\underline{P}(t) = \sum_{i=0}^{n} L_i^n(t) \underline{P}_i$$

Lagrange Interpolation

- **Simple Linear Interpolation**
  - T={t_0, t_1}

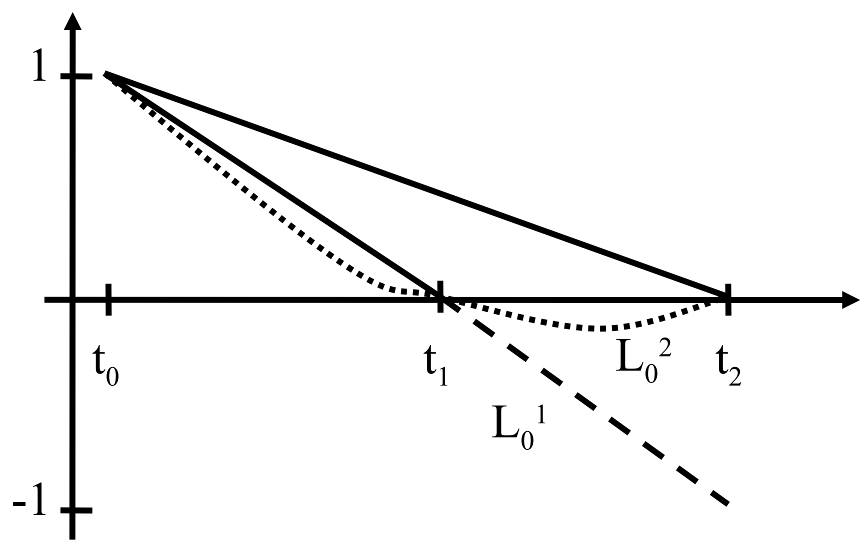
$$L_0^1(t) = \frac{t - t_1}{t_0 - t_1}$$

$$L_1^1(t) = \frac{t - t_0}{t_1 - t_0}$$



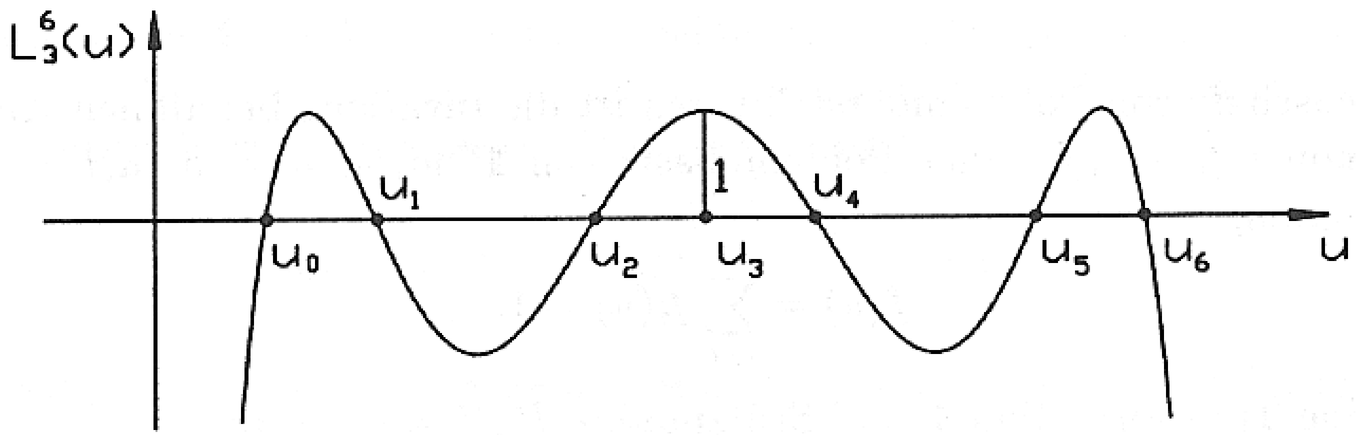
- **Simple Quadratic Interpolation**
  - T={t_0, t_1, t_2}

$$L_0^2(t) = \frac{t - t_1}{t_0 - t_1}\frac{t - t_2}{t_0 - t_2}$$



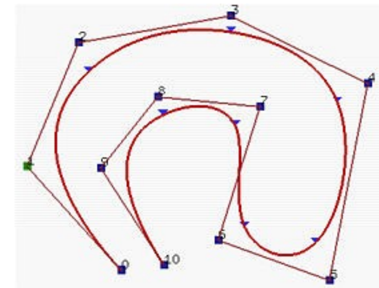
Problems

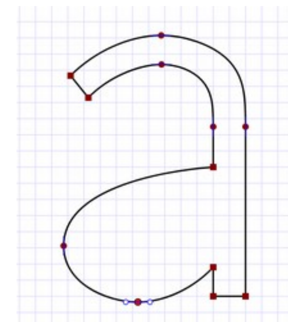
- **Problems with a single polynomial**
  - Degree depends on the number of interpolation constraints
  - Strong overshooting for high degree (n > 7)
  - Problems with smooth joints
  - Numerically unstable
  - No local changes

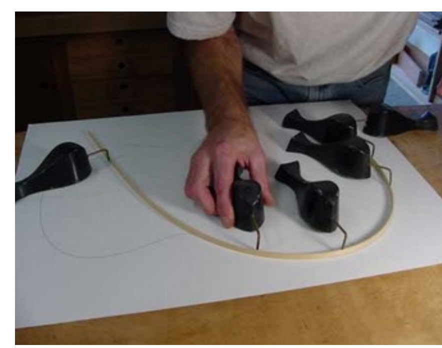


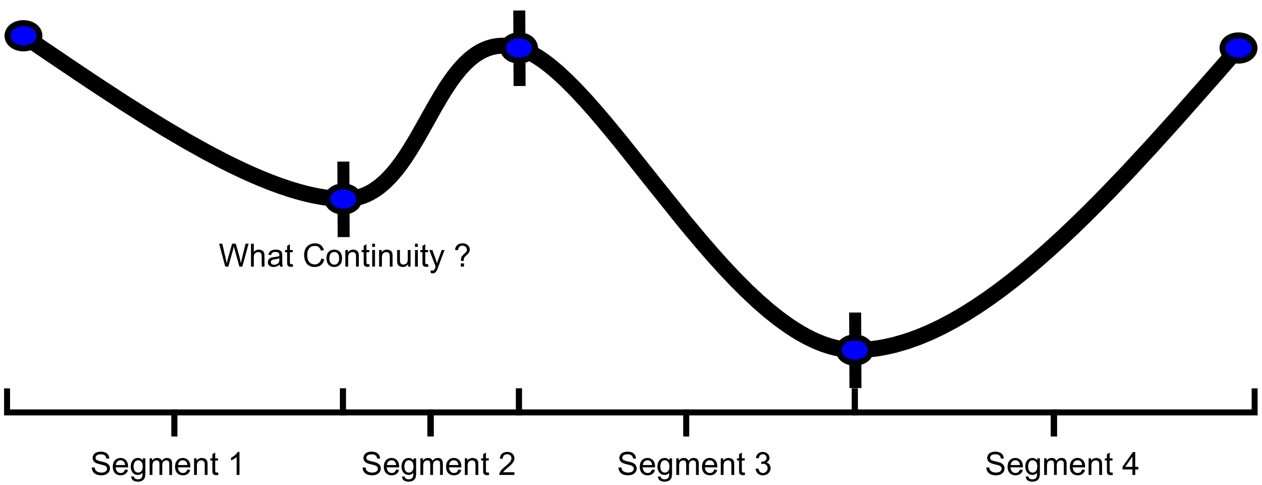
Splines

- **Functions for interpolation & approximation**
  - Standard curve and surface primitives in 3D modeling & fonts
  - Key frame and in-betweens in animations
  - Filtering and reconstruction of images
- **Historically**
  - Name for a tool in ship building
    - Flexible metal strip that tries to stay straight
  - Within computer graphics:
    - Piecewise polynomial function
    - Decouples continuity and degree of curve



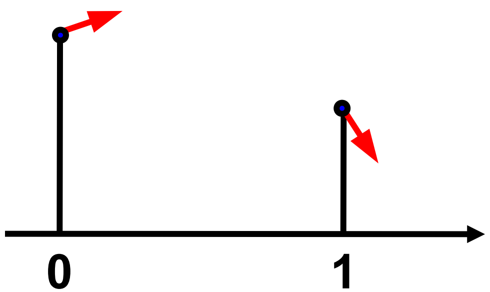


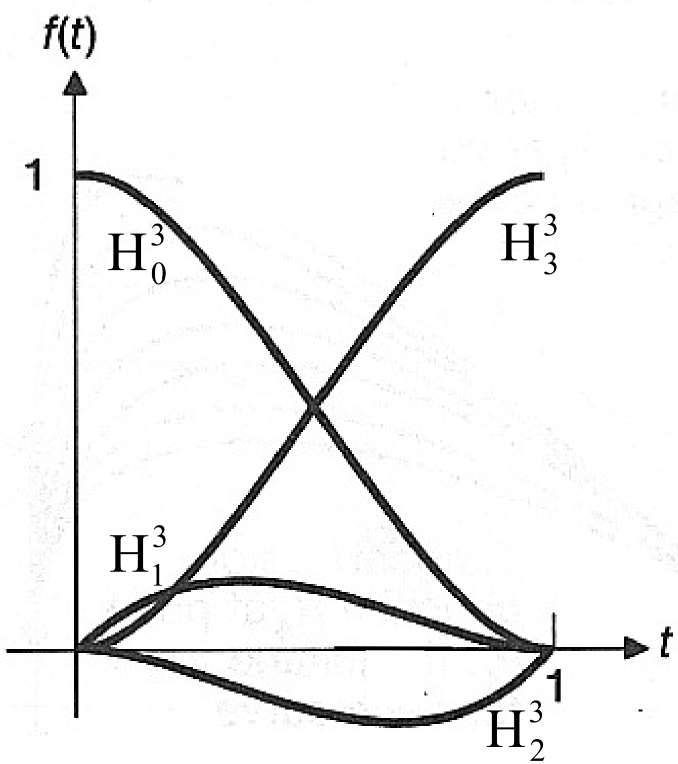




Hermite Interpolation

- **Hermite Basis (cubic)**
  - Interpolation of position P and tangent P´ information for t= {0, 1}
  - Very easy to piece together with G1/C1 continuity



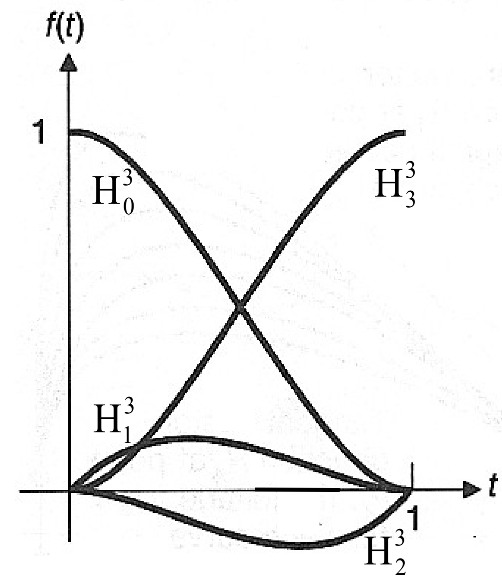


  - Basis functions

$$H_0^3(t) = (1 - t)^2(1 + 2t)$$
$$H_1^3(t) = t(1 - t)^2$$
$$H_2^3(t) = -t^2(1 - t)$$
$$H_3^3(t) = (3 - 2t)t^2$$

Hermite Interpolation

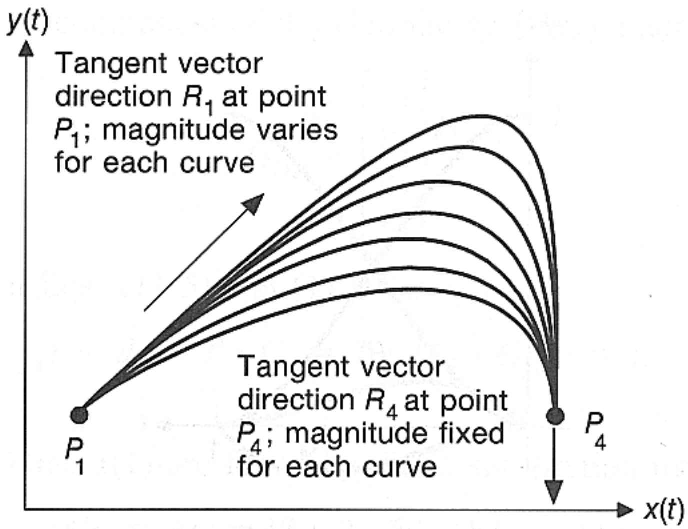
- **Properties of Hermite Basis Functions**
  - H_0 (H_3) interpolates smoothly from 1 to 0 (1 to 0)
  - H_0 and H_3 have zero derivative at t= 0 and t= 1
    - No contribution to derivative (H_1, H_2)
  - H_1 and H_2 are zero at t= 0 and t= 1
    - No contribution to position (H_0, H_3)
  - H_1 (H_2) has slope 1 at t= 0 (t= 1)
    - Unit factor for specified derivative vector

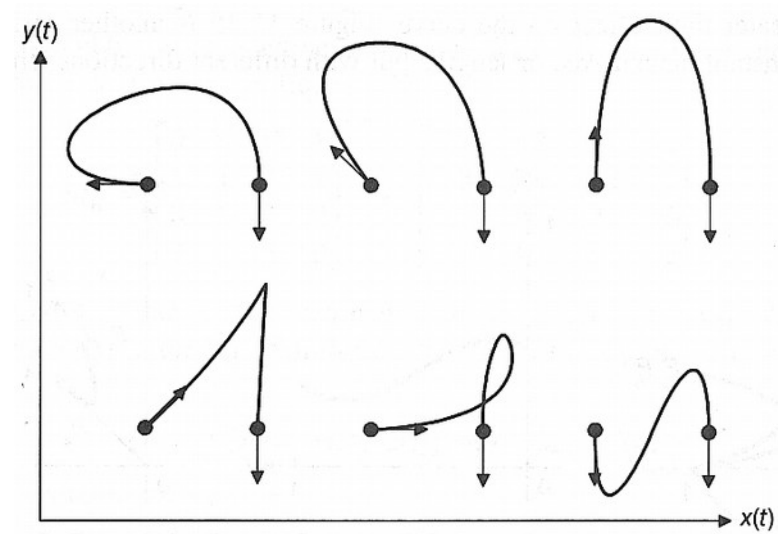


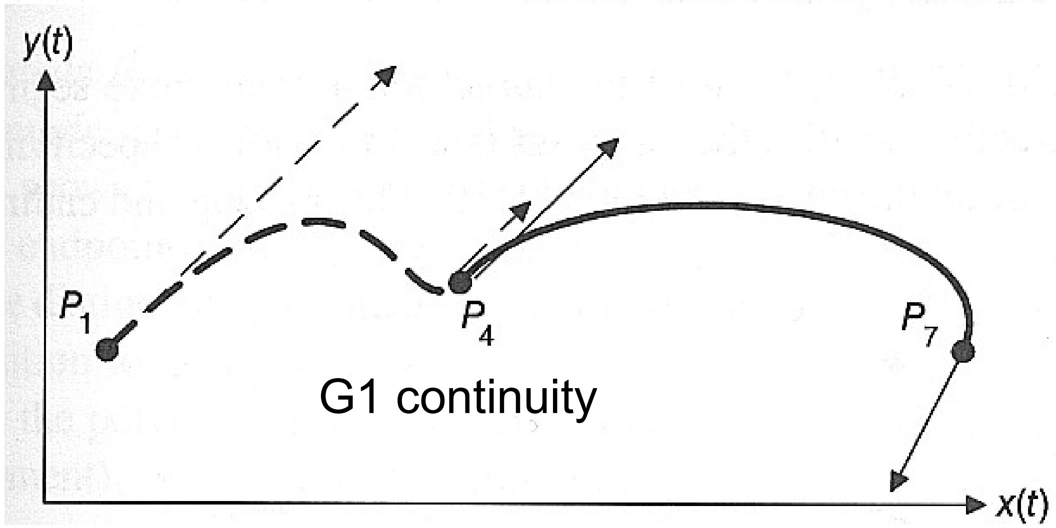
- **Hermite polynomials**
  - P_0, P_1 are positions $\in R^3$
  - $P`_0$, $P`_1$ are derivatives (tangent vectors) $\in R^3$

$$\underline{P}(t) = P_0 H_0^3(t) + P_0' H_1^3(t) + P_1' H_2^3(t) + P_1 H_3^3(t)$$

Examples: Hermite Interpolation







Matrix Representation

$$P(t) = [t^3 \quad t^2 \quad \cdots \quad 1] \begin{bmatrix} A_{x,n} & A'_{y,n} & A_{z,n} \\ A_{x,n-1} & A_{y,n-1} & A_{z,n-1} \\ & \vdots & \\ A_{x,0} & A_{y,0} & A_{z,0} \end{bmatrix} =$$

$$\underbrace{[t^3 \quad t^2 \quad \cdots \quad 1]}_{T} \underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} & \\ M_{21} & \ddots & & \\ & & & \\ & & & \end{bmatrix}}_{\text{Basis Matrix M (4x4)}} \underbrace{\begin{bmatrix} G_{x,3} & G_{y,3} & G_{z,3} \\ G_{x,2} & G_{y,2} & G_{z,2} \\ G_{x,1} & G_{y,1} & G_{y,1} \\ G_{x,0} & G_{y,0} & G_{z,0} \end{bmatrix}}_{\text{Geometry Matrix G (4x3)}} =$$

$$\underbrace{[t^3 \quad t^2 \quad \cdots \quad 1] \underbrace{\begin{bmatrix} M_{11} & M_{12} & M_{13} & \\ M_{21} & \ddots & & \\ & & & \\ & & & \end{bmatrix}}_{M_H}}_{\text{Basis Functions}} \underbrace{\begin{bmatrix} P_0^T \\ P_1^T \\ P'^T_0 \\ P'^T_1 \end{bmatrix}}_{G_H}$$

Matrix Representation

- **For cubic Hermite interpolation we obtain:**

$$
\begin{aligned}
P_0^T &= (0 \quad 0 \quad 0 \quad 1)\mathbf{M}_H \mathbf{G}_H \\
P_1^T &= (1 \quad 1 \quad 1 \quad 1)\mathbf{M}_H \mathbf{G}_H \\
P_0'^T &= (0 \quad 0 \quad 1 \quad 0)\mathbf{M}_H \mathbf{G}_H \\
P_1'^T &= (3 \quad 2 \quad 1 \quad 0)\mathbf{M}_H \mathbf{G}_H
\end{aligned}
$$

or

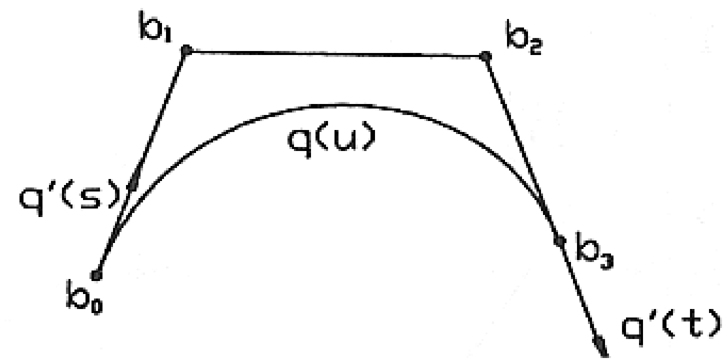
$$
\begin{pmatrix} P_0^T \\ P_1^T \\ P_0'^T \\ P_1'^T \end{pmatrix} = \mathbf{G}_H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} \mathbf{M}_H \mathbf{G}_H
$$

- **Solution:**
  - Two matrices must multiply to unit matrix

$$
\mathbf{M}_H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

Bézier

- **Bézier Basis [deCasteljau´59, Bézier´62]**
  - Different curve representation
  - Start and end point
  - 2 point that are approximated by the curve (cubics)
  - $P'_0 = 3(b_1 - b_0)$ and $P'_1 = 3(b_3 - b_2)$
    - Factor 3 due to derivative of t^3



$$G_H = \begin{bmatrix} P_0^T \\ P_1^T \\ P'^T_0 \\ P'^T_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} b_0^T \\ b_1^T \\ b_2^T \\ b_3^T \end{bmatrix} = M_{HB} G_B$$

Basis transformation

- **Transformation**
  - $P(t) = T\,M_H\,G_H = T\,M_H\,(M_{HB}\,G_B) = T\,(M_H M_{HB})\,G_B = T\,M_B\,G_B$

$$M_B = M_H M_{HB} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

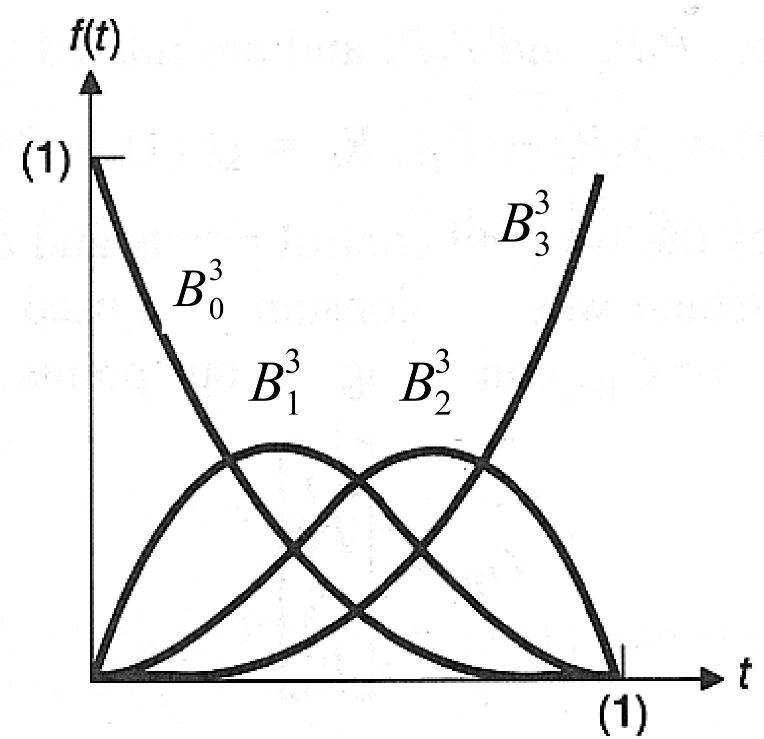
$$P(t) = \sum_{i=0}^{3} B_i^3(t) b_i =$$

$$(1-t)^3 b_0 + 3t(1-t)^2\, b_1 + 3t^2(1-t) b_2 + t^3 b_3$$

- **Bézier Curves & Basis Function**

$$P(t) = \sum_{i=0}^{n} B_i^n(t)\, b_i$$

$$\textit{with basis functions} \quad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

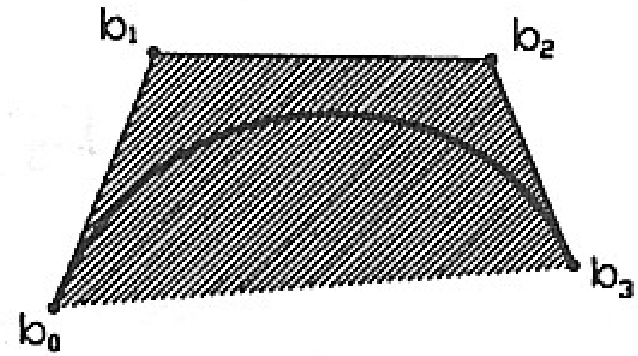


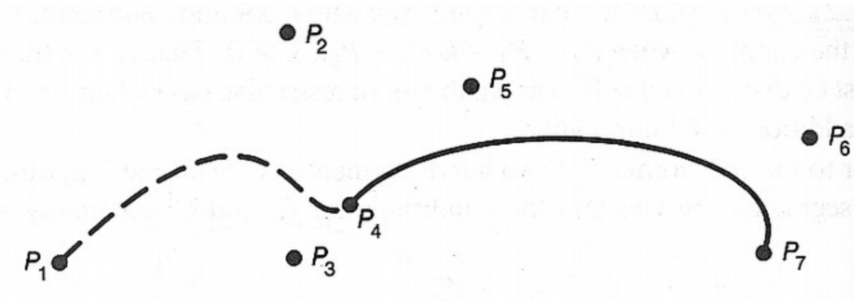
Bernstein-Polynomials

Properties: Bézier

- **Advantages:**
  - End point interpolation
  - Tangents explicitly specified
  - Smooth joints are simple
    - P_3, P_4, P_5 collinear → G^1 continuous
  - Geometric meaning of control points
  - Affine invariance

    $\forall \sum B_i(t) = 1$
  - Convex hull property
    - For 0<t<1: $B_i(t) \geq 0$
  - Symmetry: $B_i(t) = B_{n-i}(1-t)$
- **Disadvantages**
  - Smooth joints need to be maintained explicitly
    - Automatic in B-Splines (and NURBS)
    - See Geometric Modeling course





DeCasteljau Algorithm

- **Direct evaluation of the basis functions**
  - Simple but expensive
- **Use recursion**
  - Recursive definition of the basis functions

$$B_i^n(t) = tB_{i-1}^{n-1}(t) + (1-t)B_i^{n-1}(t)$$

  - Inserting this once yields:

$$P(t) = \sum_{i=0}^{n} b_i^0 B_i^n(t) = \sum_{i=0}^{n-1} b_i^1(t) B_i^{n-1}(t)$$

  - with the new Bézier points given by the recursion

$$b_i^k(t) = tb_{i+1}^{k-1}(t) + (1-t)b_i^{k-1}(t) \text{ and } b_i^0(t) = b_i$$

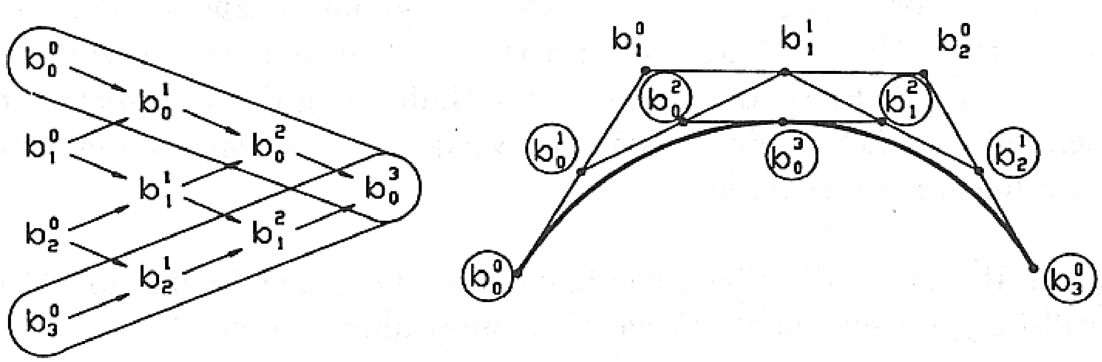
DeCasteljau Algorithm

- **DeCasteljau-Algorithm:**
  - Recursive degree reduction of the Bezier curve by using the recursion formula for the Bernstein polynomials

$$P(t) = \sum_{i=0}^{n} b_i^0 B_i^n(t) = \sum_{i=0}^{n-1} b_i^1(t) B_i^{n-1}(t) = \cdots = b_i^n(t) \cdot 1$$

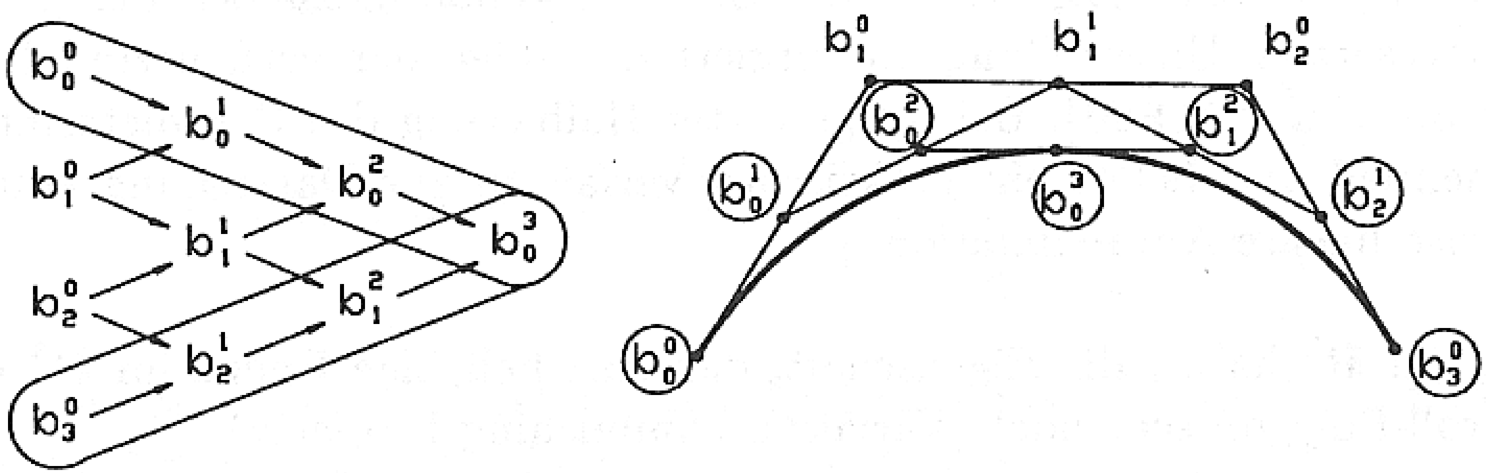
$$b_i^k(t) = t b_{i+1}^{k-1}(t) + (1 - t) b_i^{k-1}(t)$$

- **Example:**
  - t= 0.5



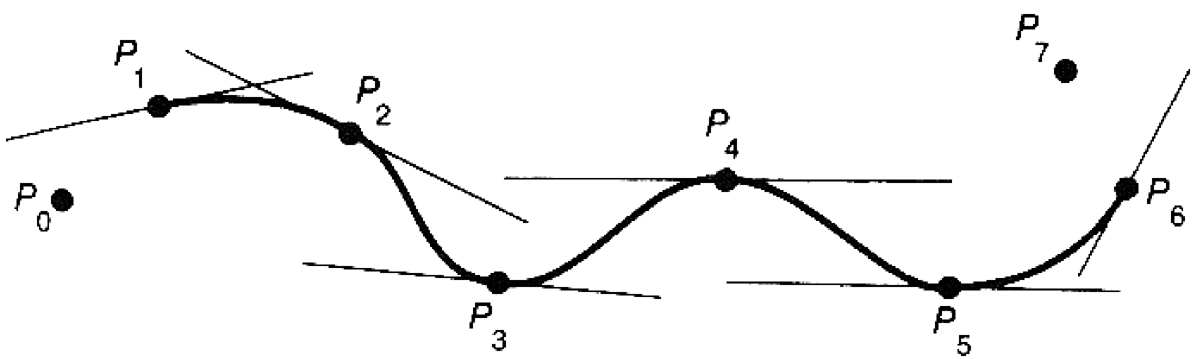
DeCasteljau Algorithm

- **Subdivision using the deCasteljau-Algorithm**
  - Take boundaries of the deCasteljau triangle as new control points for left/right portion of the curve
- **Extrapolation**
  - Backwards subdivision
    - Reconstruct triangle from one side



Catmull-Rom-Splines

- **Goal**
  - Smooth (C^1)-joints between (cubic) spline segments
- **Algorithm**
  - Tangents given by neighboring points P_{i-1} P_{i+1}
  - Construct (cubic) Hermite segments
- **Advantage**
  - Arbitrary number of control points
  - Interpolation without overshooting
  - Local control



Matrix Representation

- **Catmull-Rom-Spline**
  - Piecewise polynomial curve
  - Four control points per segment
  - For n control points we obtain (n-3) polynomial segments

$$\underline{P}^i(t) = T\mathbf{M}_{CR} G_{CR} = T\frac{1}{2}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} \underline{P}_i^T \\ \underline{P}_{i+1}^T \\ \underline{P}_{i+2}^T \\ \underline{P}_{i+3}^T \end{bmatrix}$$

- **Application**
  - Smooth interpolation of a given sequence of points
  - Key frame animation, camera movement, etc.
  - Only G^1-continuity
  - Control points should be equidistant in time

Choice of Parameterization

- **Problem**
  - Often only the control points are given
  - How to obtain a suitable parameterization t_i ?
- **Example: Chord-Length Parameterization**

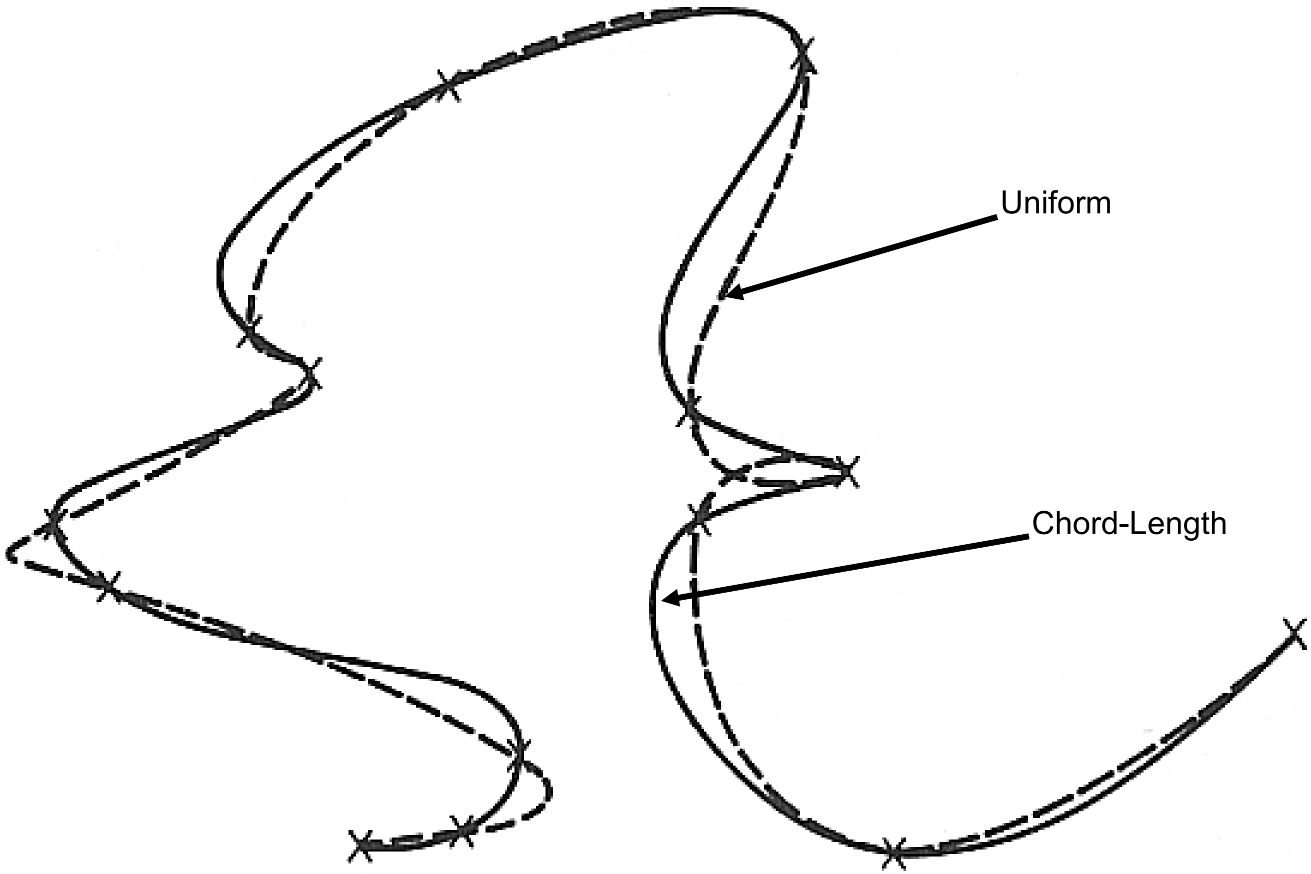
$$t_0 = 0$$

$$t_i = \sum_{j=1}^{i} dist(P_i - P_{i-1})$$

  - Arbitrary up to a constant factor
- **Warning**
  - Distances are not affine invariant !
  - Shape of curves changes under transformations !!

Parameterization

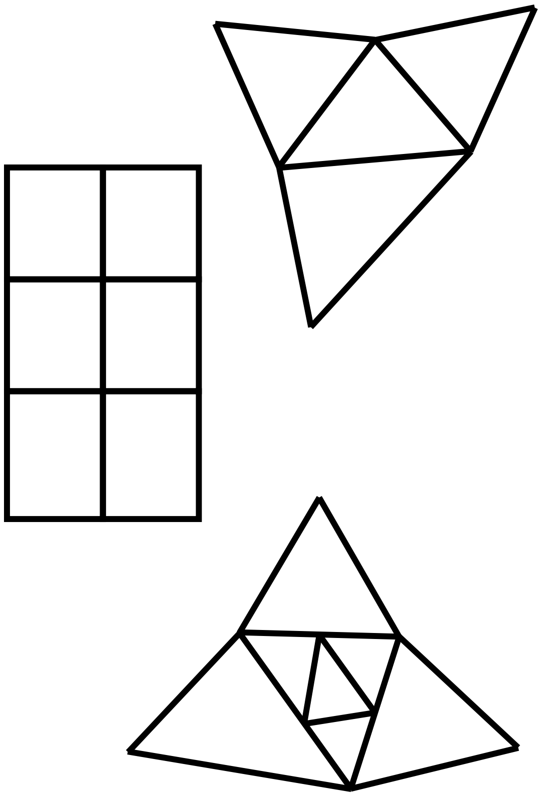
- **Chord-Length versus uniform Parameterization**
  - Analog: Think P(t) as a moving object with mass that may overshoot



Spline Surfaces

Parametric Surfaces

- **Same Idea as with Curves**
  - $\underline{P}: R^2 \rightarrow R^3$
  - $\underline{P}(u,v) = (x(u,v), y(u,v), z(u,v))^T \in R^3$ (also $P(R^4)$)
- **Different Approaches**
  - Tensor Product Surfaces
    - Separation into polynomials in u and in v
    - Discussed here (see Geometric Modeling course for others)
  - Subdivision Surfaces
    - Start with a triangular mesh in R^3
    - Subdivide mesh by inserting new vertices
      - Depending on local neighborhood
    - Only piecewise parameterization (in each triangle)
  - Triangular Splines
    - Single polynomial in (u,v) via barycentric coordinates with respect to a reference triangle (e.g. B-Patches)



Tensor Product Surfaces

- **Idea**
  - Create a “curve of curves"
- **Simplest case: Bilinear Patch**
  - Two lines in space

$$\underline{P}^1(v) = (1-v)\underline{P}_{00} + v\underline{P}_{10}$$

$$\underline{P}^2(v) = (1-v)\underline{P}_{01} + v\underline{P}_{11}$$

  - Connected by lines

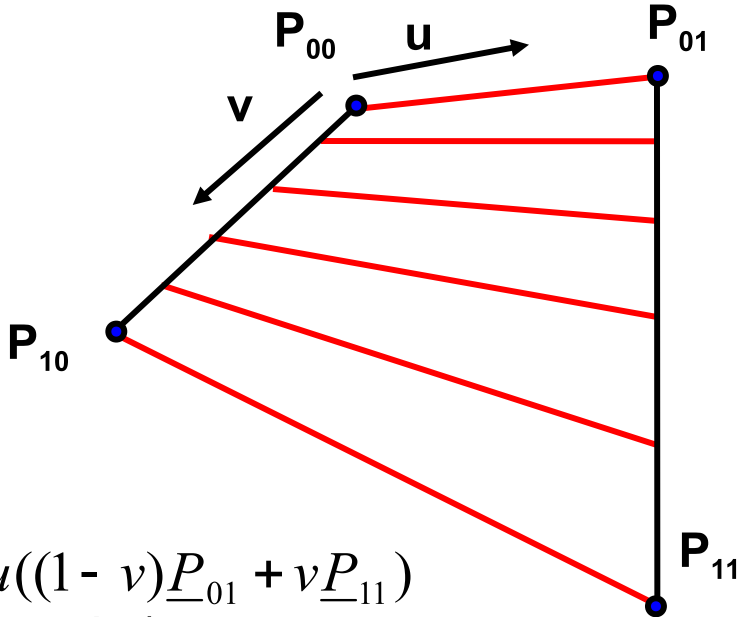
$$\underline{P}(u,v) = (1-u)\underline{P}^1(v) + u\underline{P}^2(v) =$$

$$(1-u)((1-v)\underline{P}_{00} + v\underline{P}_{10}) + u((1-v)\underline{P}_{01} + v\underline{P}_{11})$$

  - Bézier representation (symmetric in u and v)

$$\underline{P}(u,v) = \sum_{i,j=0}^{1} B_i^1(u) B_j^1(v) \underline{P}_{ij}$$

  - Control mesh P_{ij}



Tensor Product Surfaces

- **General Case**
  - Arbitrary basis functions in u and v
    - Tensor Product of the function space in u and v
  - Commonly same basis functions and same degree in u and v

$$\underline{P}(u,v)=\sum_{i=0}^{m}\sum_{j=0}^{n}B_i^m(u)B_j^n(v)\underline{P}_{ij}$$

- **Interpretation**
  - Curve defined by curves

$$\underline{P}(u,v)=\sum_{i=0}^{m}B_i'(u)\underbrace{\sum_{j=0}^{n}B_j(v)\underline{P}_{ij}}_{P_i'(v)}$$

  - Symmetric in u and v

Matrix Representation

- **Similar to Curves**
  - Geometry now in a „tensor“ (m x n x 3)

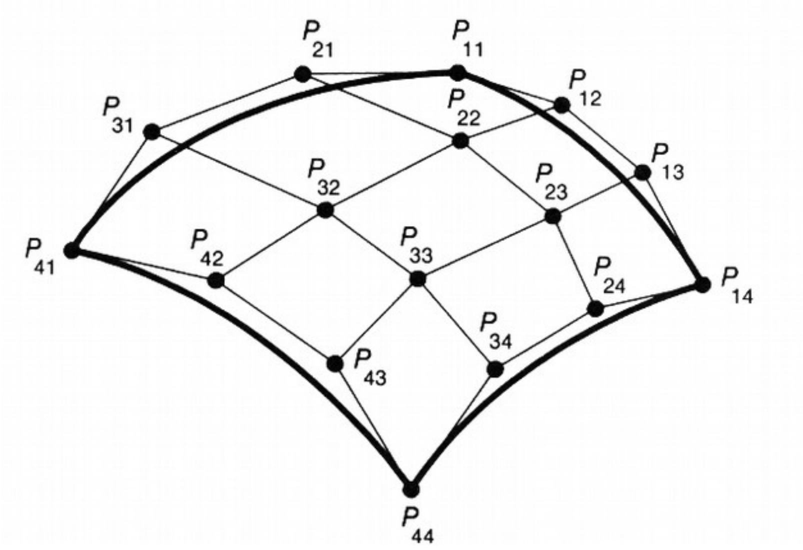
$$\underline{P}(u,v) = U\mathbf{G}_{monom}V^T = \begin{pmatrix} u^m & \cdots & u & 1 \end{pmatrix} \begin{pmatrix} G_{nn} & \cdots & G_{n0} \\ \vdots & \ddots & \vdots \\ G_{0n} & \cdots & G_{00} \end{pmatrix} \begin{pmatrix} v^n \\ \vdots \\ v \\ 1 \end{pmatrix} =$$

$$U\mathbf{B}'_U \mathbf{G}_{UV} \mathbf{B}_V^T V^T$$

  - Degree
    - u: m
    - v: n
    - Along the diagonal (u=v): m+n
      - Not nice → „Triangular Splines“

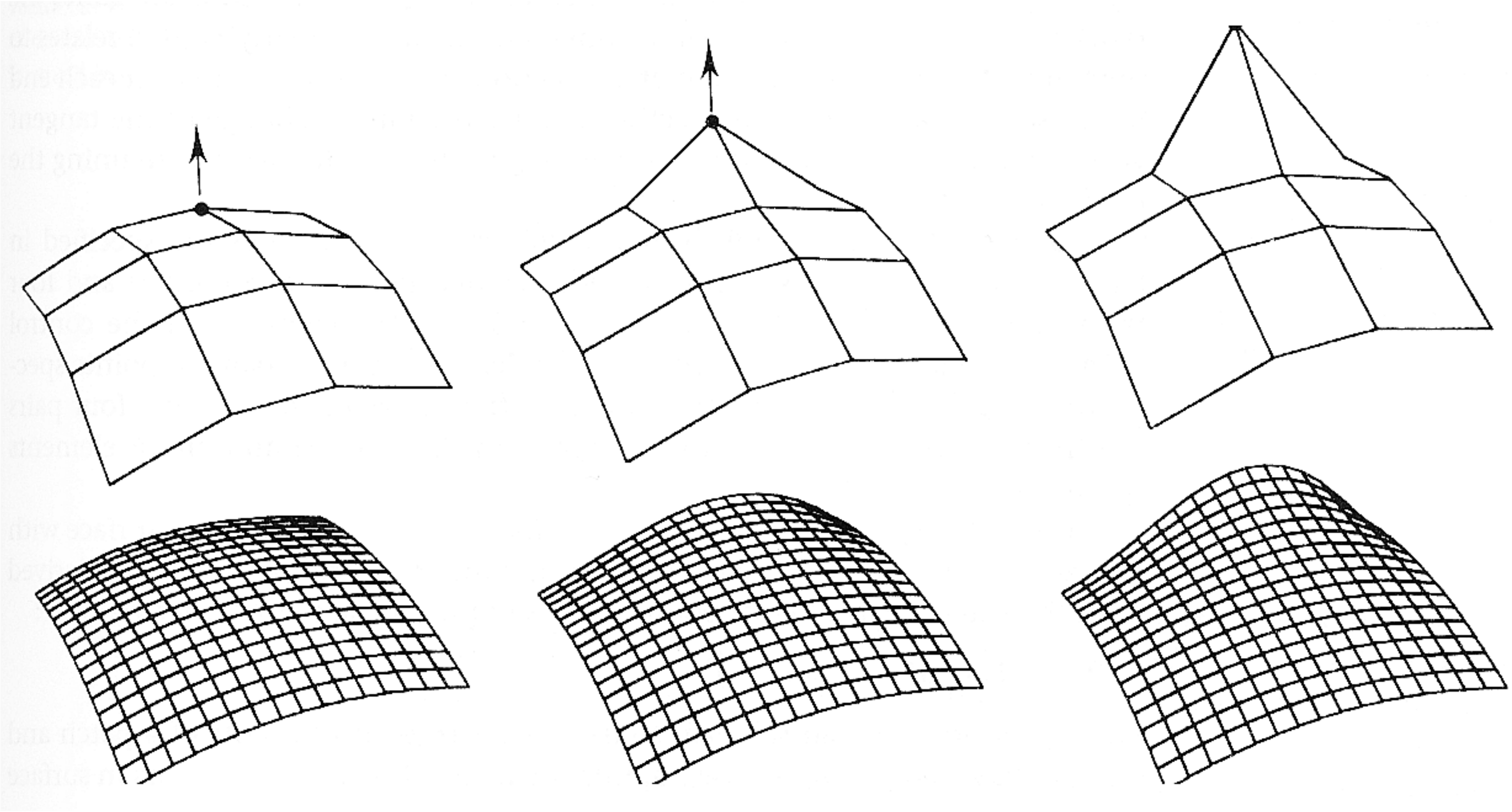
Tensor Product Surfaces

- **Properties Derived Directly From Curves**
- **Bézier Surface:**
  - Surface interpolates corner vertices of mesh
  - Vertices at edges of mesh define boundary curves
  - Convex hull property holds
  - Simple computation of derivatives
  - Direct neighbors of corners vertices define tangent plane
- **Similar for Other Basis Functions**



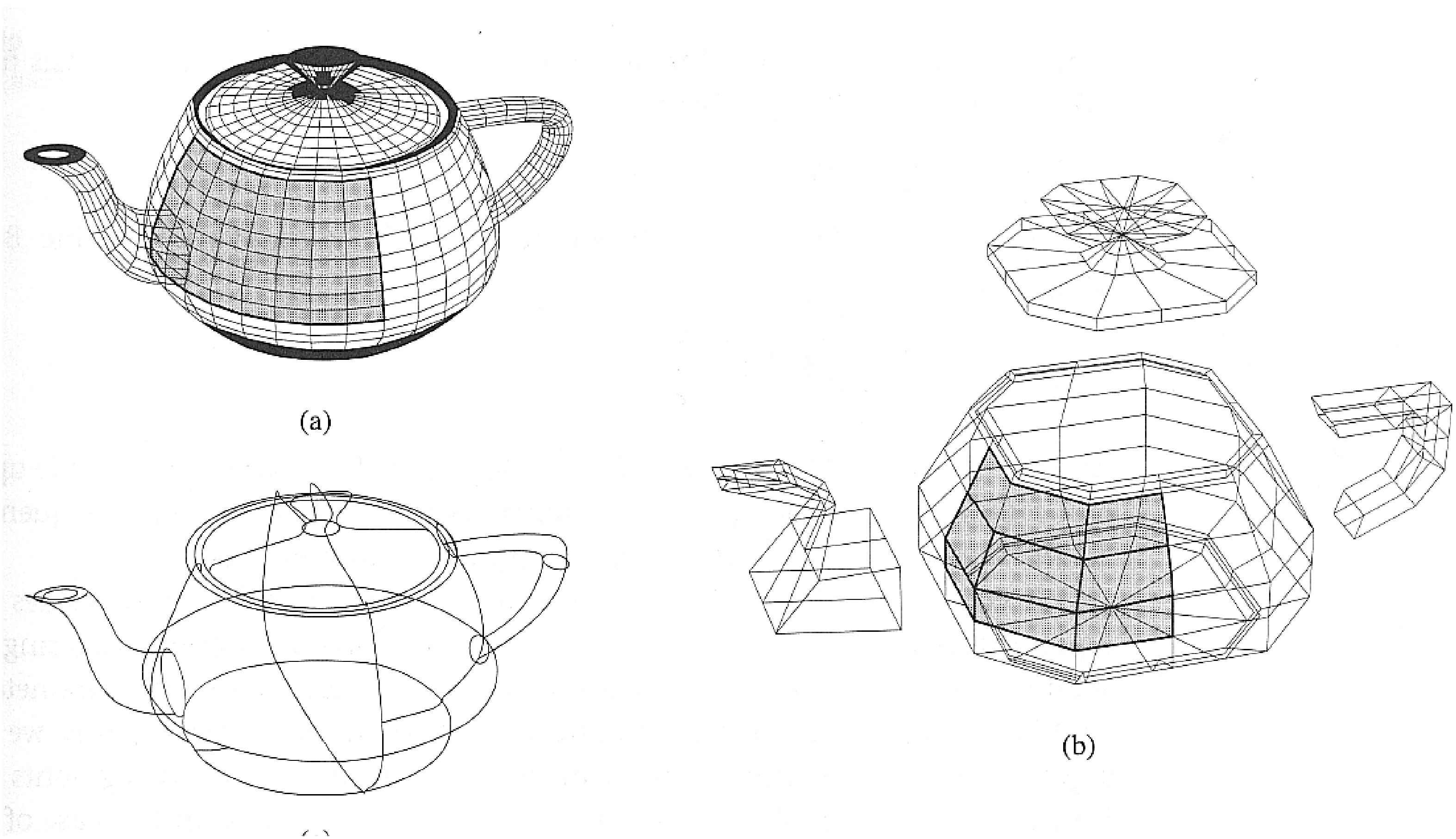
Tensor Product Surfaces

- **Modifying a Bézier Surface**



Tensor Product Surfaces

- **Representing the Utah Teapot as a set continuous Bézier patches**
  - http://www.holmes3d.net/graphics/teapot/

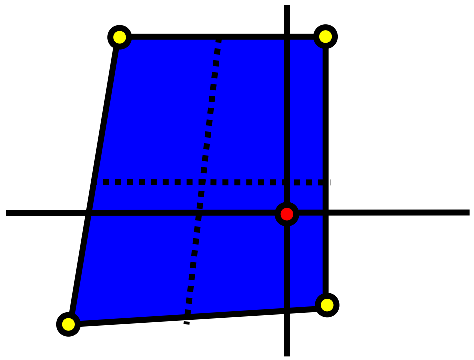


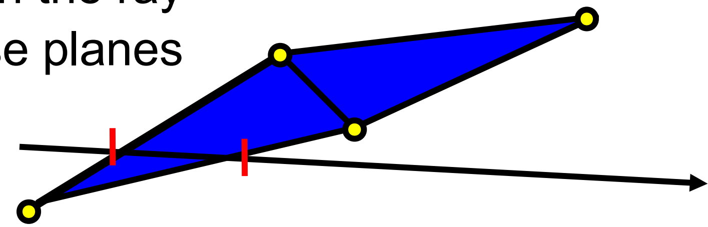
Operations on Surfaces

- **deCausteljau/deBoor Algorithm**
  - Once for u in each column
  - Once for v in the resulting row
  - Due to symmetry also in other order
- **Similarly we can derive the related algorithms**
  - Subdivision
  - Extrapolation
  - Display
  - ...

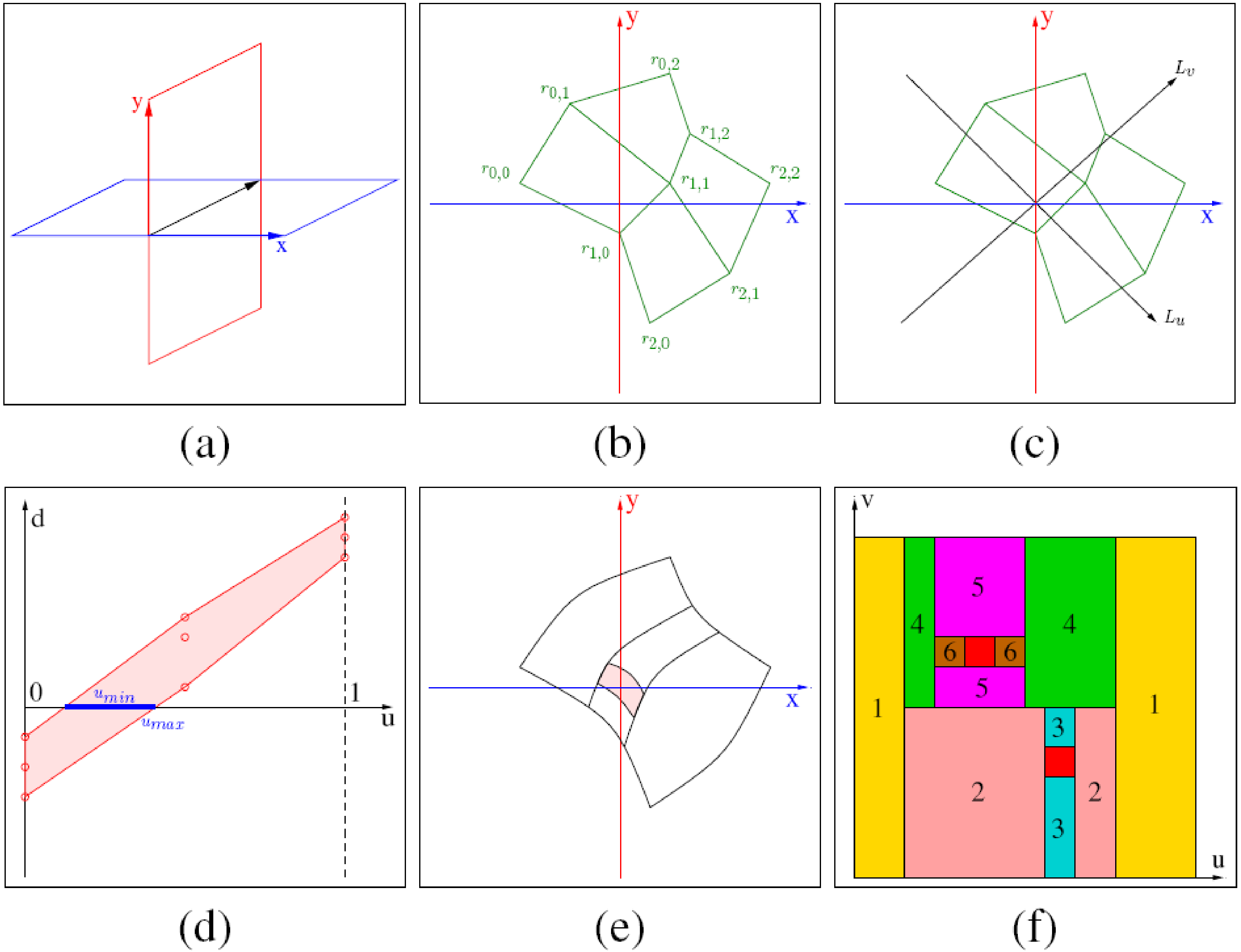
Ray Tracing of Spline Surfaces

- **Several approaches**
  - Tessellate into many triangles (using deCasteljau or deBoor)
    - Often the fasted method
    - May need enormous amounts of memory
  - Recursive subdivision
    - Simply subdivide patch recursively
    - Delete parts that do not intersect ray (Pruning)
    - Fixed depth ensures crack-free surface
    - May cache intermediate results for next rays
  - Bézier Clipping [Sederberg et al.]
    - Find two orthogonal planes that intersect in the ray
    - Project the surface control points into these planes
    - Intersection must have distance zero
      - ➔ Root finding
      - ➔ Can eliminate parts of the surface where convex hull does not intersect ray
    - Must deal with many special cases – rather slow

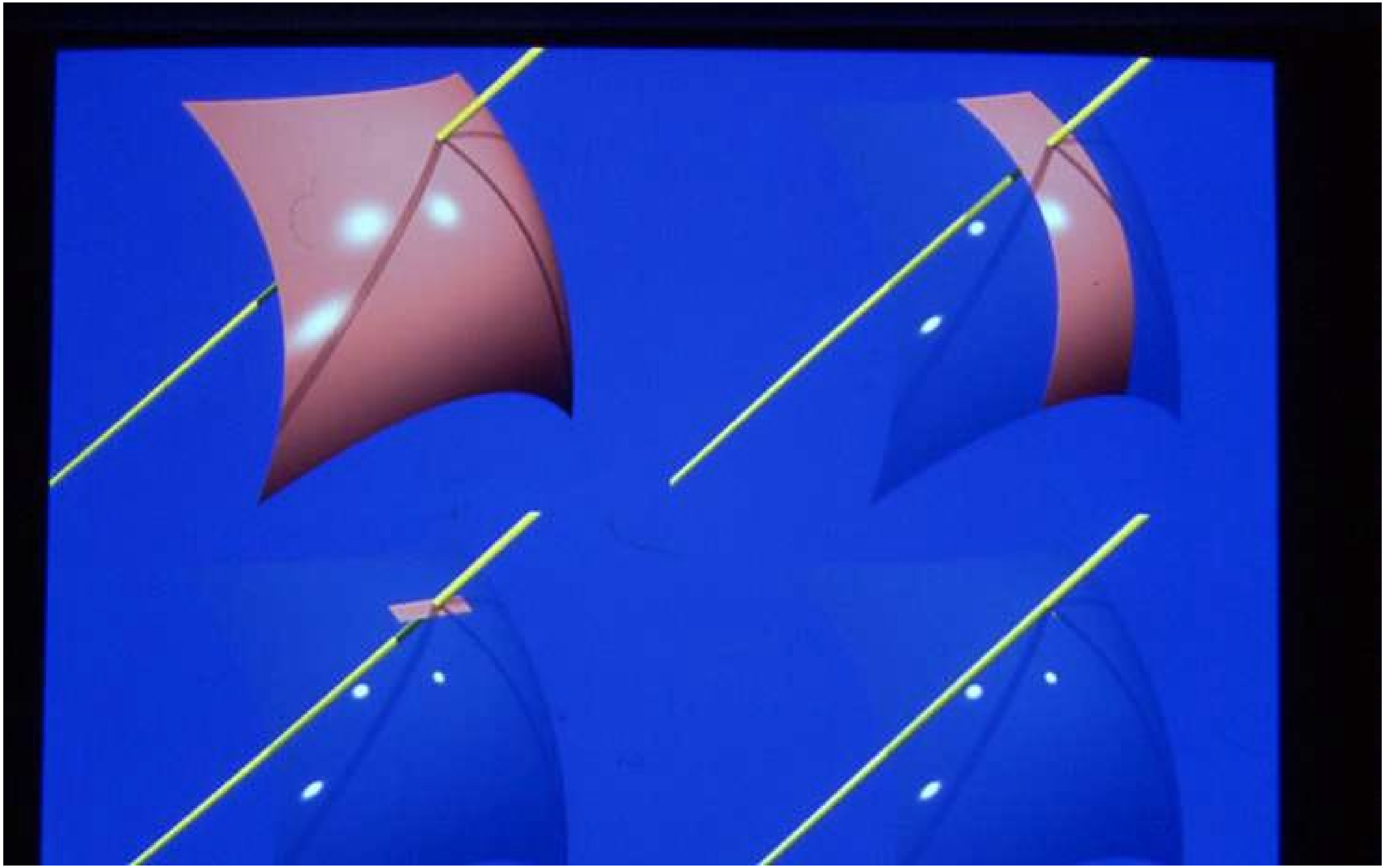




Bézier Clipping

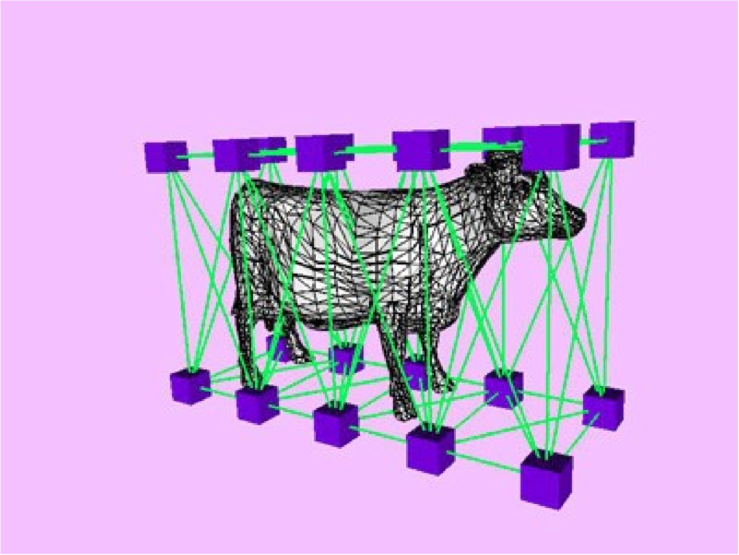


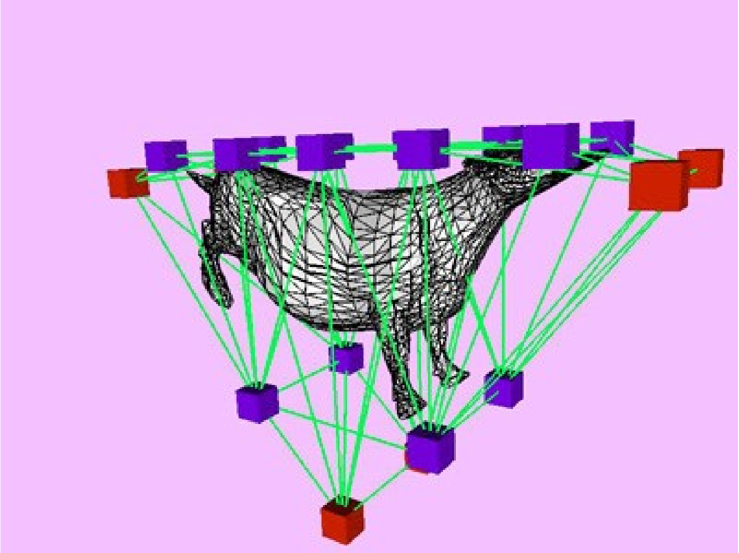
Bézier Clipping



Higher Dimensions

- **Volumes**
  - Spline: $R^3 \rightarrow R$
    - Volume density
    - Rarely used
  - Spline: $R^3 \rightarrow R^3$
    - Modifications of points in 3D
    - Displacement mapping
    - Free Form Deformations (FFD)





FFD