Probability:
Theory and practice

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Administrative updates

- Please register for the exams (in HISPOS for Computer Science).
- Withdrawal deadline is **one week** before the main exam (or re-exam).
- For Seminars, withdrawal is allowed **within three weeks** after topic assignment

A la Carte

- σ- algebra and measure

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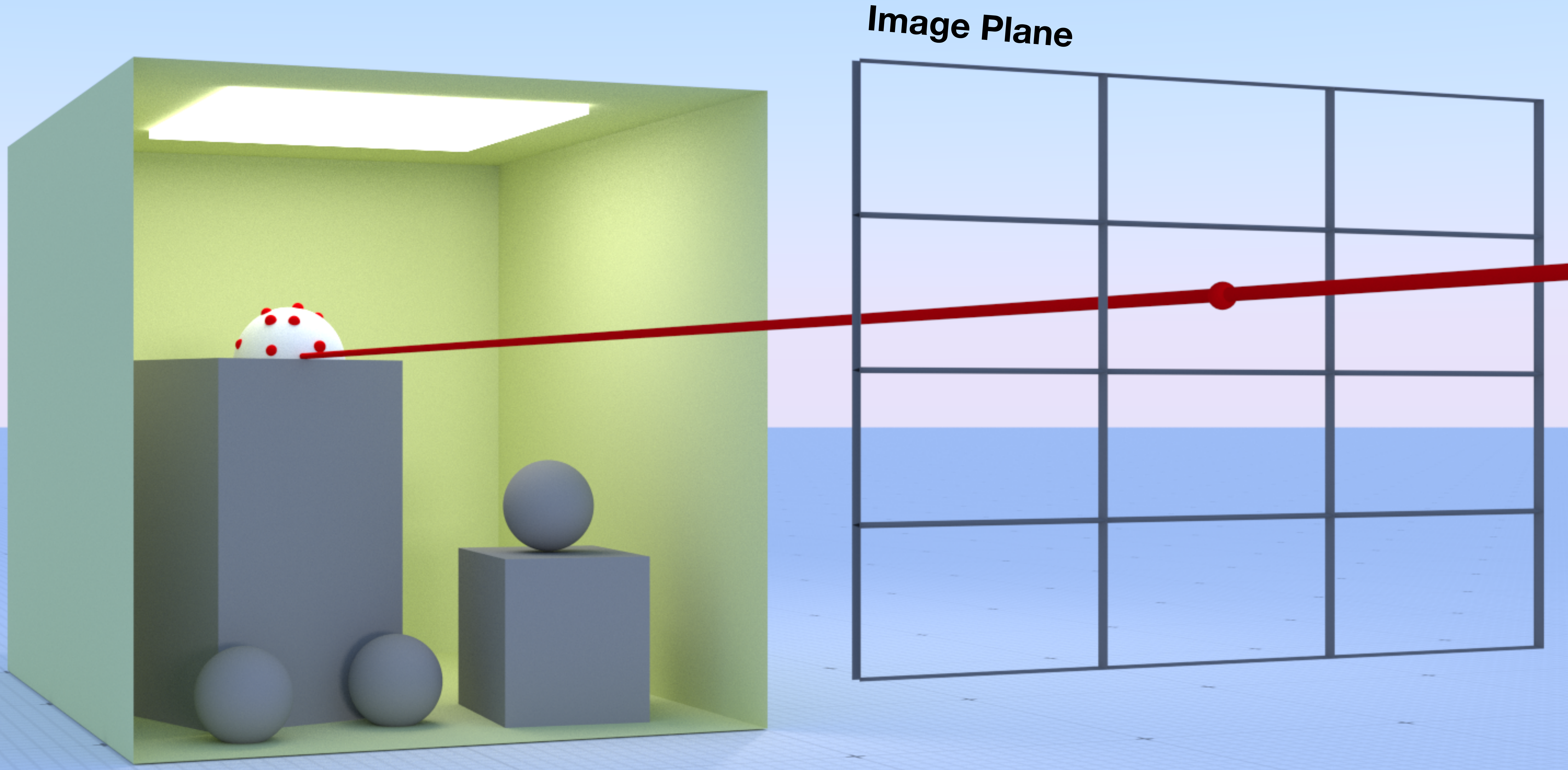
- σ- algebra and measure
- Random Variables
- Probability distribution functions (PDFs and PMFs)
- Conditional and Marginal PDFs
- Expected value and Variance of a random variable

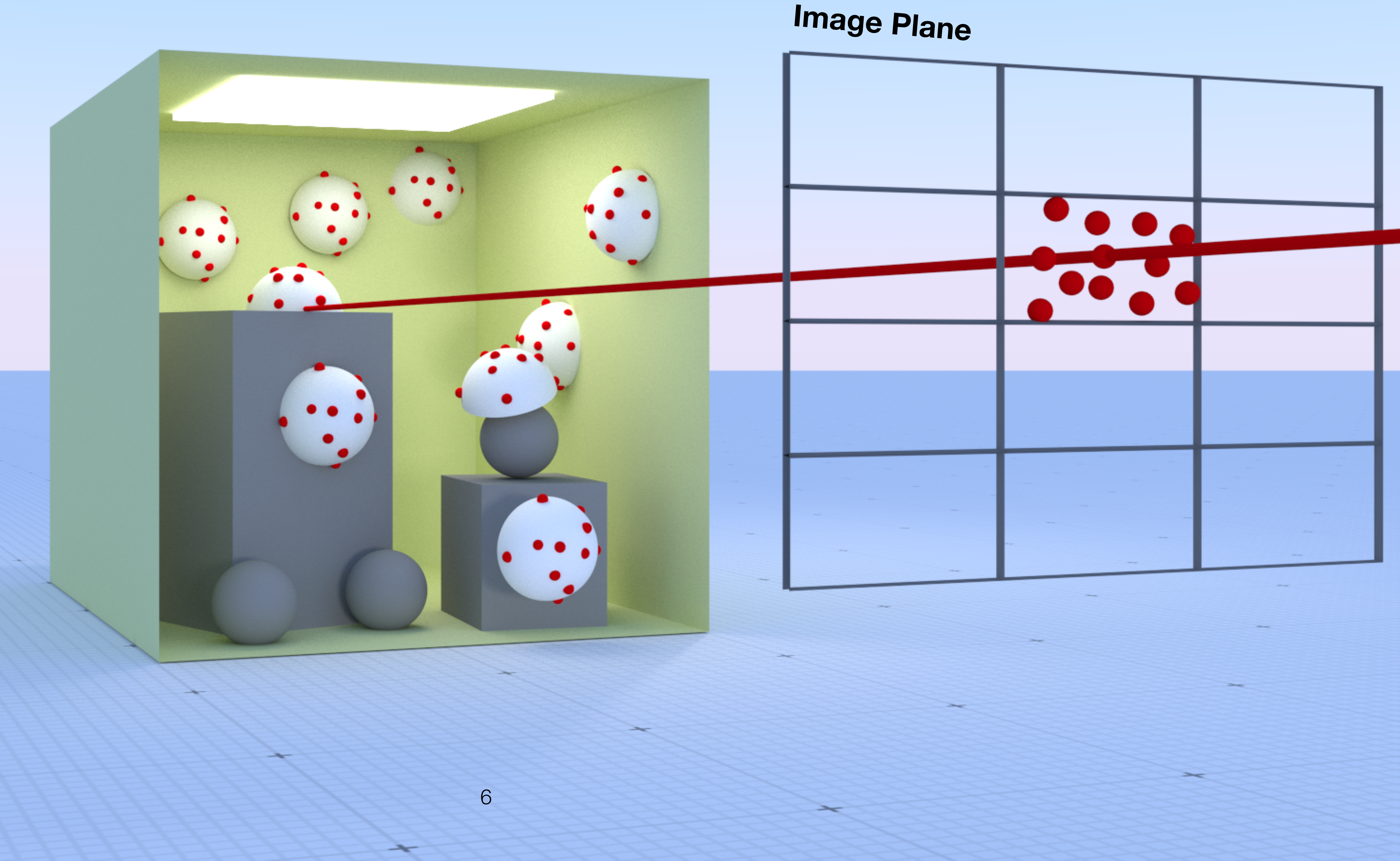
Motivation: Ray Tracing

Ray Tracing



Scene designed by David Coeurjolly

Ray Tracing



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Direct Illumination

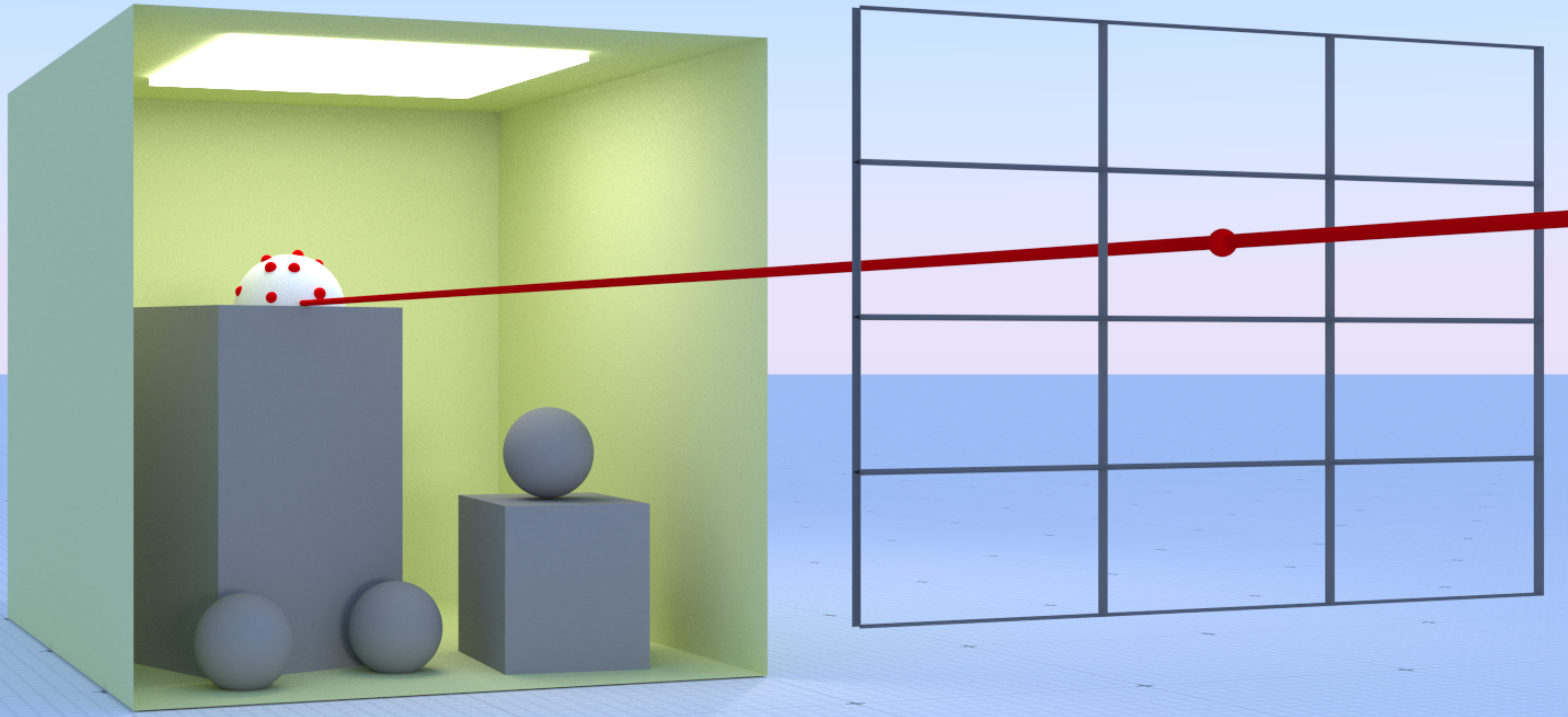
Direct Illumination

Direct and Indirect Illumination

4096 spp

Image rendered using PBRT

Path Tracing



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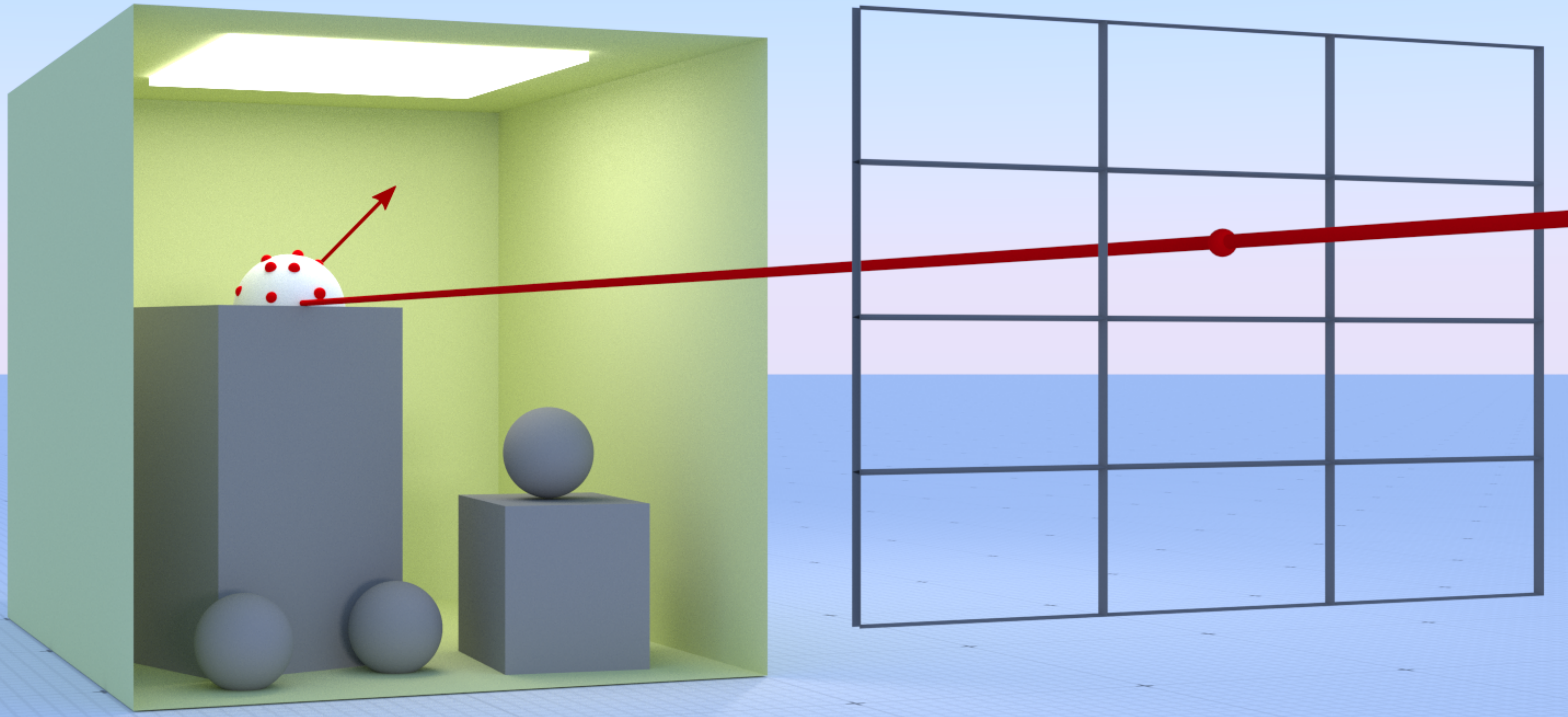

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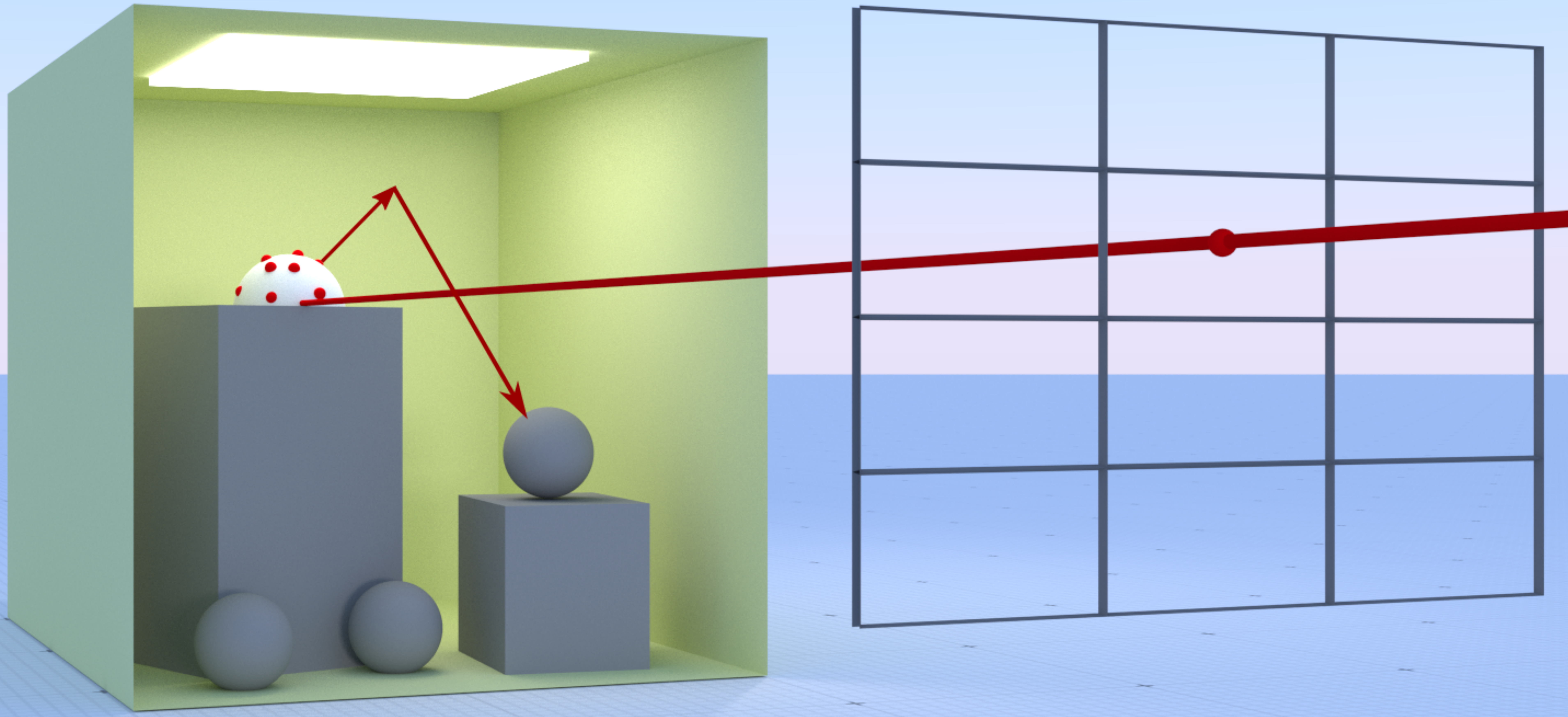

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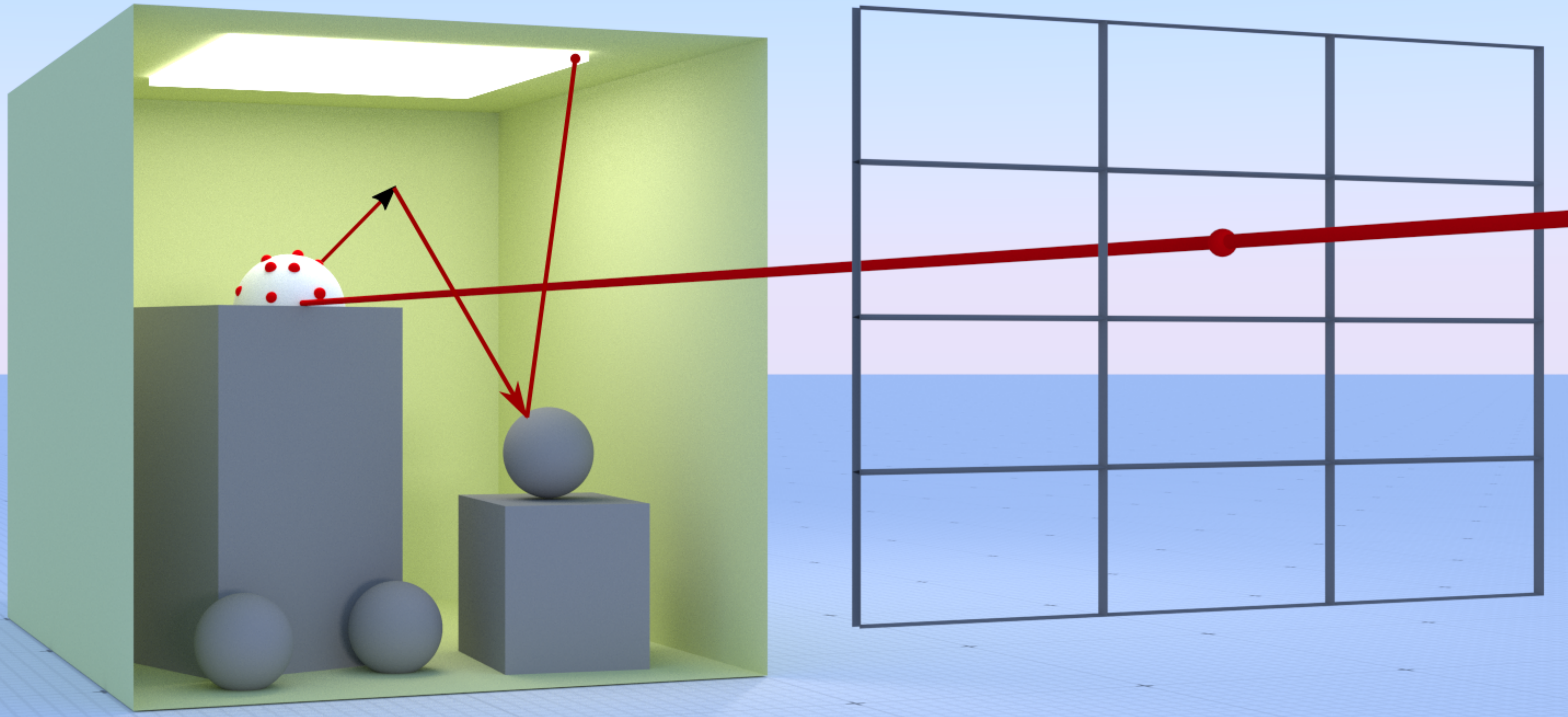

Direct and Indirect Illumination

How can we analyze the noise present in the images ?

Probability Theory
and/or
Number Theory

Probability Theory

- Discrete Probability Space
- Continuous Probability Space

Rolling a fair dice

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Finite outcomes: **discrete** random experiment

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- Finite outcomes: **discrete** random experiment
- Can ask the outcome is a number: 1 or 6
- Can ask the outcome is a subset, e.g. all prime numbers: $\{2, 3, 5\}$

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The second to the mathematical construct of a measure

Random number in [0,1]

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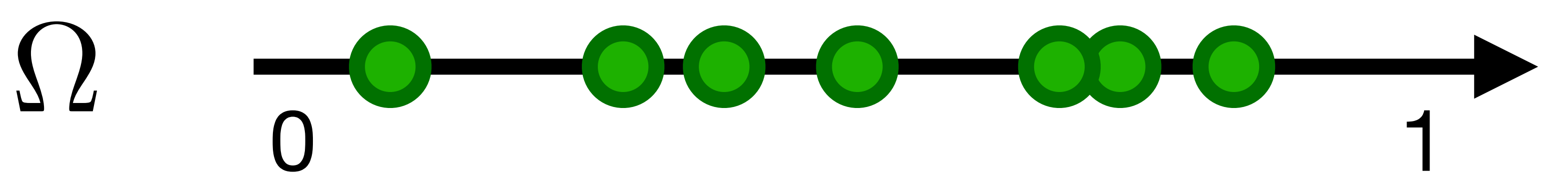

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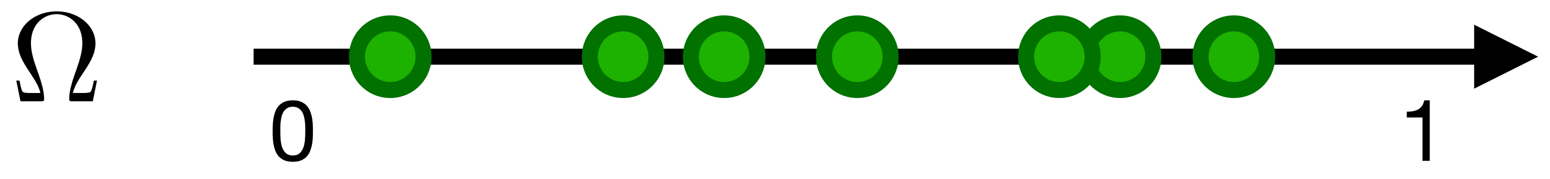

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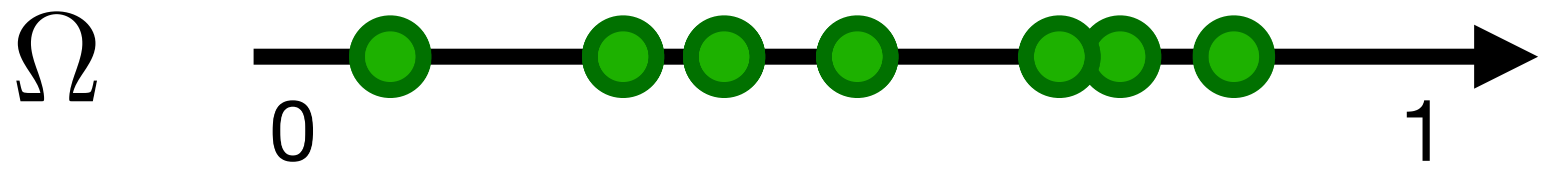

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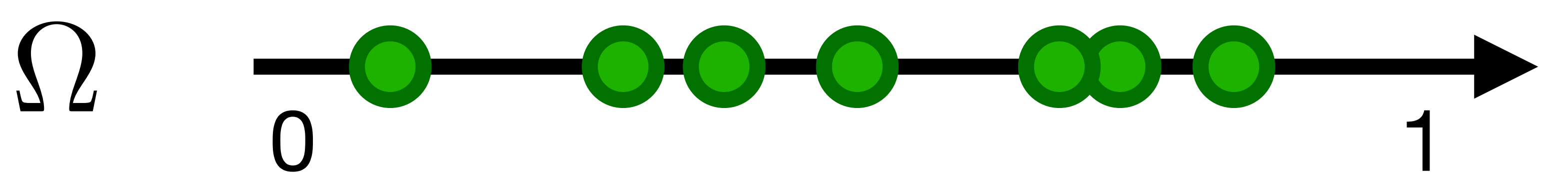
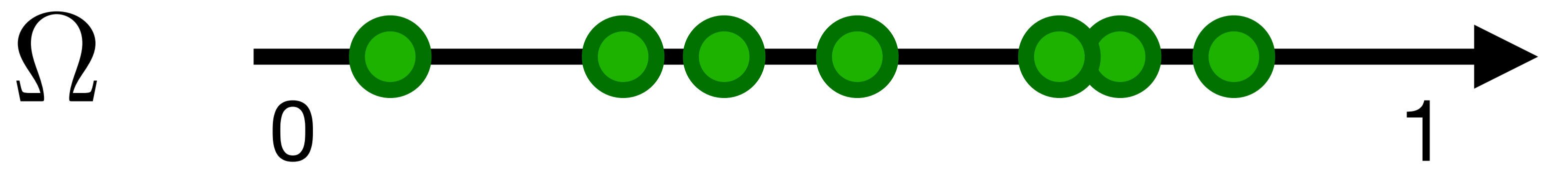


- Uncountably infinite outcomes: **continuous** random experiment
- Does not make sense to ask for one number as output, e.g. 0.245
- We need to ask for the probability of a region, e.g. [0.2,0.4] or [0.36,0.89]

Random number in [0,1]



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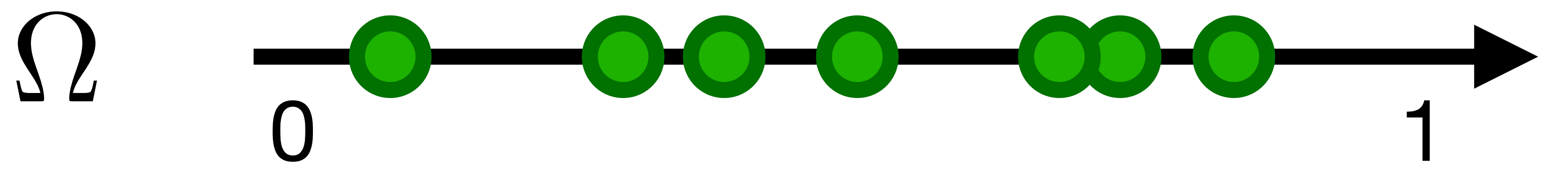

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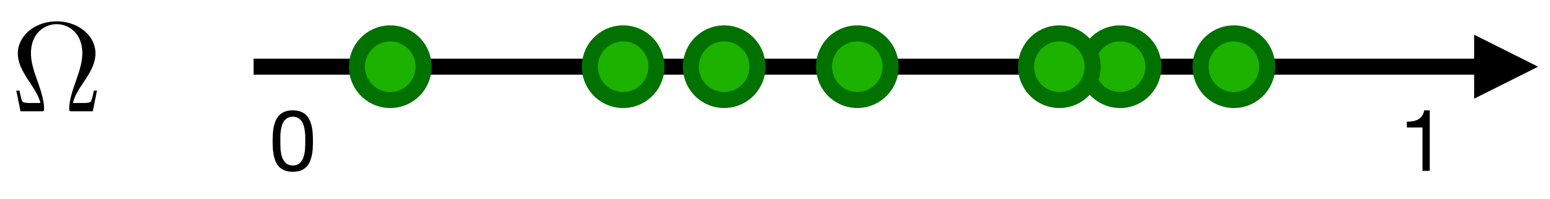

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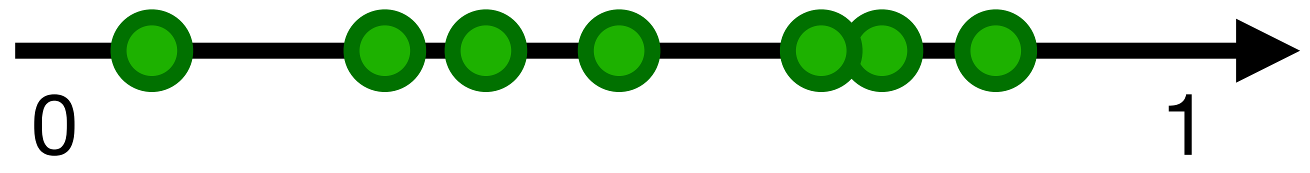
Ω



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    1. Take on the role of system of events in probability theory
- Simply spoken: Collection of subsets of a given set Ω
    A. A non-empty collection of subsets of Ω that is **closed** under the set theoretical operations of: countable unions, countable intersections, and complement

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$$\Omega = \{a, b, c, d\}$$

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Here, $\mathbb{T}$ is not a σ-algebra because, generally speaking, neither the union nor the difference of two half-intervals is a half-interval.

σ-Algebra

It is the mathematical construct that allows defining a measure

Measure

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- A real-valued set function defined on a sigma-algebra that assigns each subset of a sigma-algebra a non-negative real number.
- A sigma-additive set function: i.e., the measure of the union of disjoint sets is equal to the sum of the measures of the individual sets

Lebesgue Measure

- Standard way of assigning measure to subsets of n-dimensional Euclidean space.

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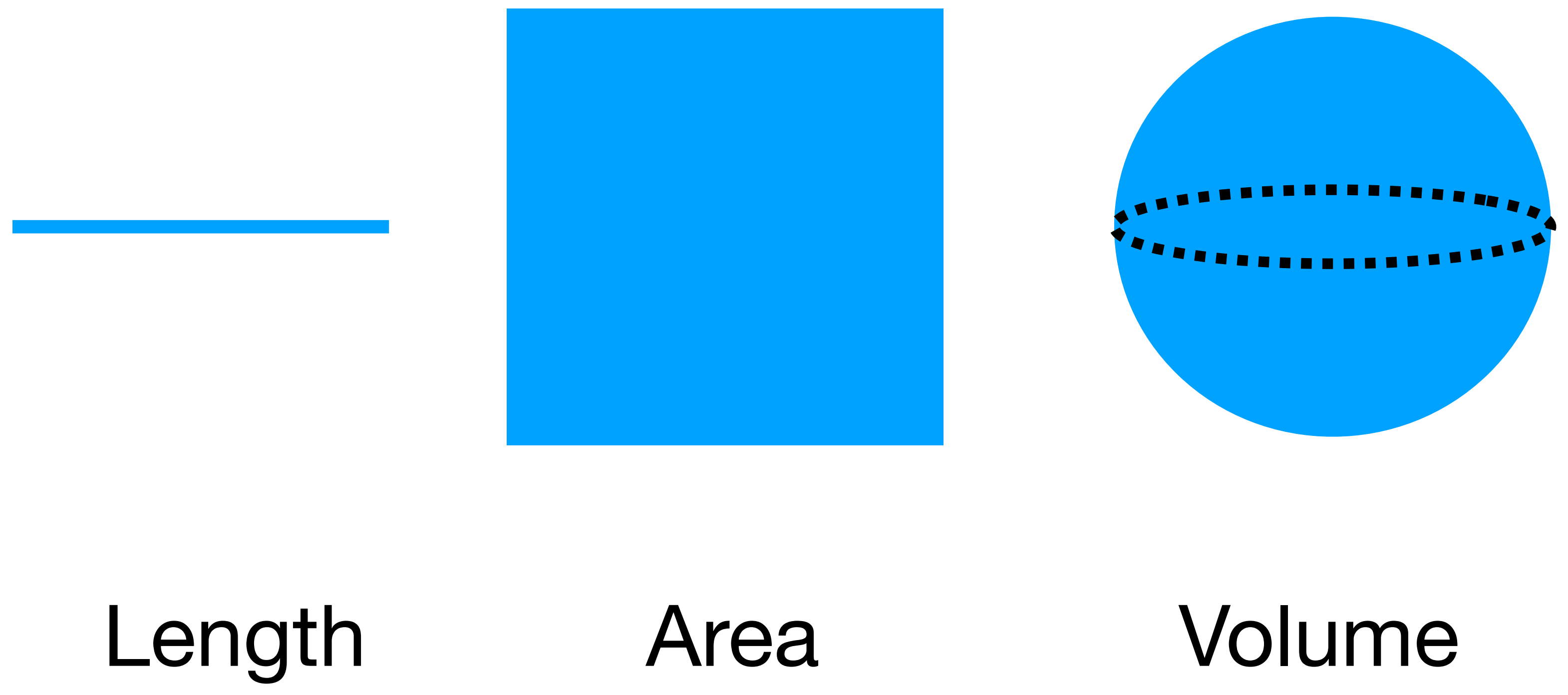
- Standard way of assigning measure to subsets of n-dimensional Euclidean space.
- For n = 1,2 or 3, it coincides with the standard measure of length, area or volume, respectively.



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- Enables to construct a simpler probability space from a rather complex one
- Correspond to a measurable function defined on a σ-algebra that assigns each element to a real number

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- A random variable X is a value chosen by some random process
- Random variables are always drawn from a domain: discrete (e.g., a fixed set of probabilities) or continuous (e.g., real numbers)
- Applying a function f to a random variable X results in a new random variable $Y = f(X)$

Discrete Probability Space

Discrete Random Variable

- Random variable (RV):

$$X : \Omega \to E \qquad\qquad \Omega = \{x_1, x_2, \dots, x_n\}$$

- Probabilities:

$$\{p_1, p_2, \dots, p_n\}$$

$$\sum_{i=1}^{N} p_i = 1$$

Discrete Random Variable

- Example: Rolling a Die

$$x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5, x_6 = 6$$

- Probability of each event:

$$p_i = 1/6 \quad \text{for} \;\; i = 1, \ldots, 6$$

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Discrete Random Variable

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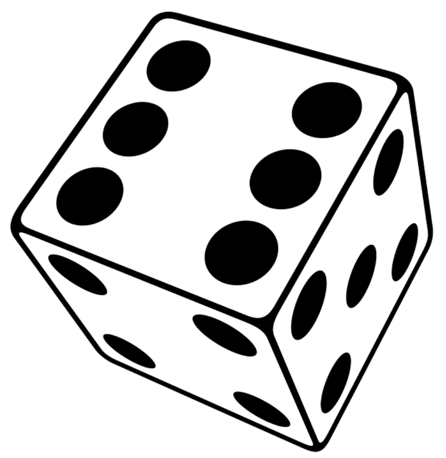
Discrete Random Variable



$$P(2 \leq X \leq 4) = \sum_{i=2}^{4} P(X = i)$$

$$= \sum_{i=2}^{4} \frac{1}{6} = \frac{1}{2}$$

Probability mass function

- PMF is a function that gives the probability that a discrete RV is exactly equal to some value.

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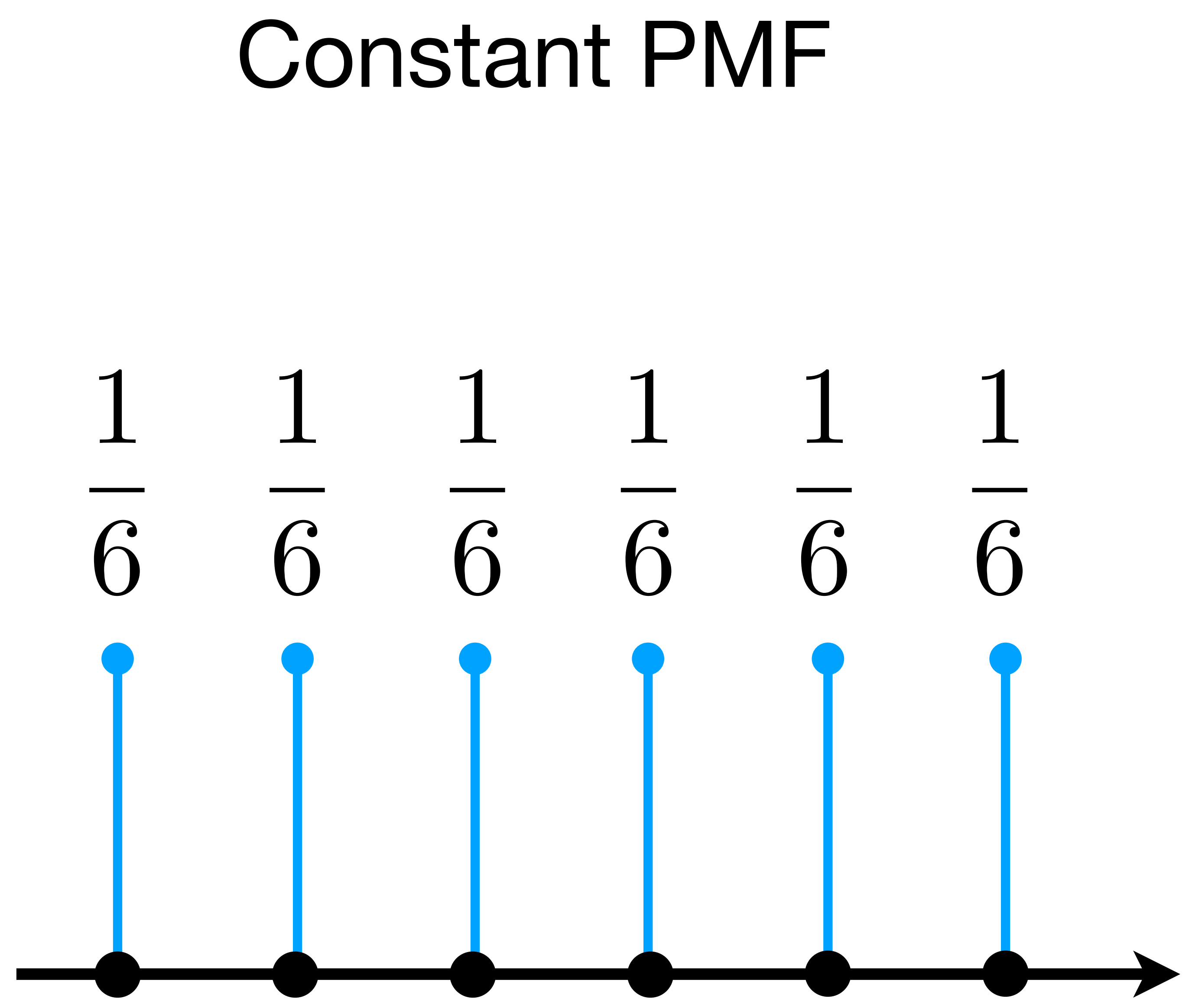
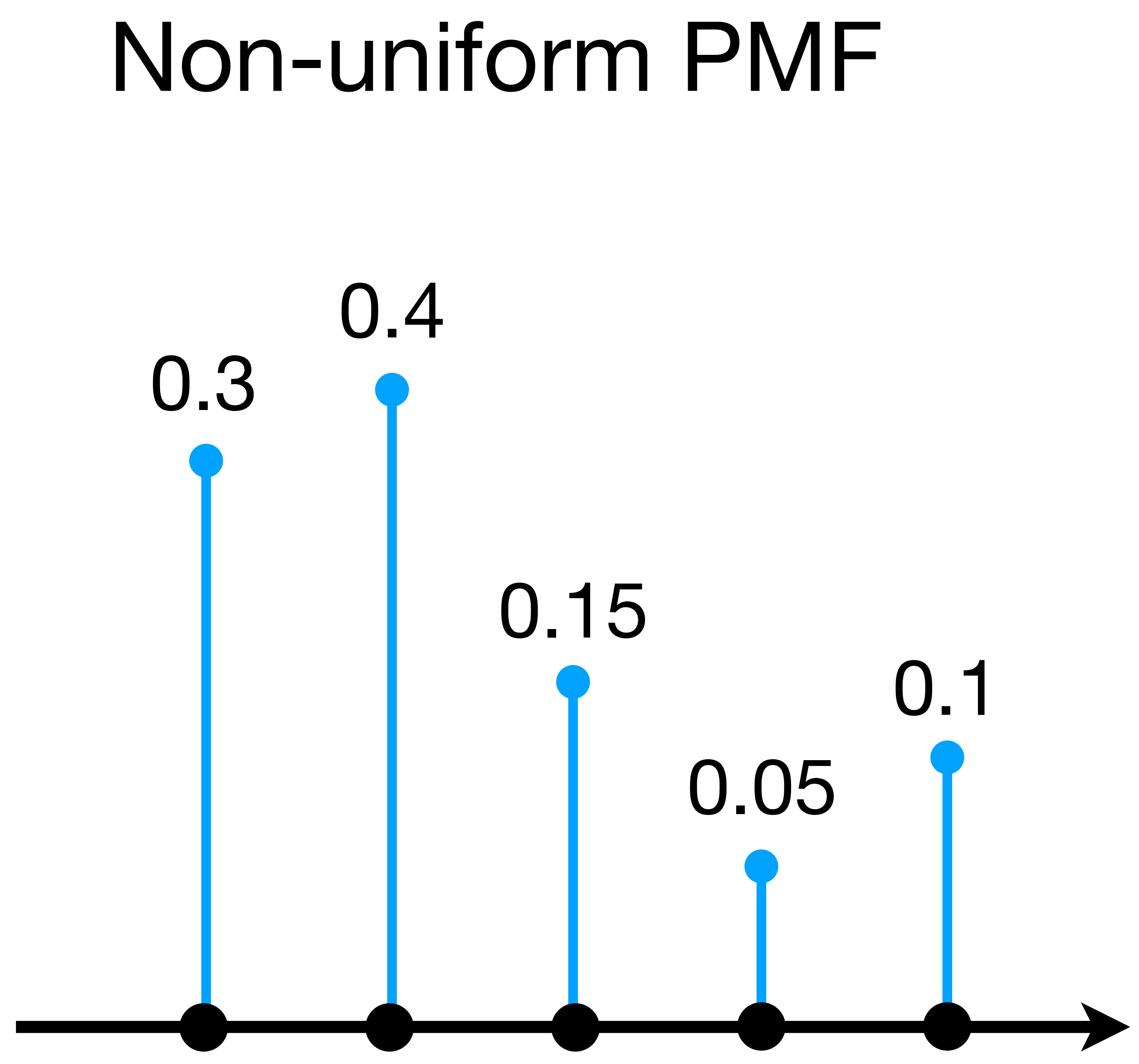
- PMF is different from PDF (probability density function) which is for continuous RVs.

Probability mass function





Continuous Probability Space

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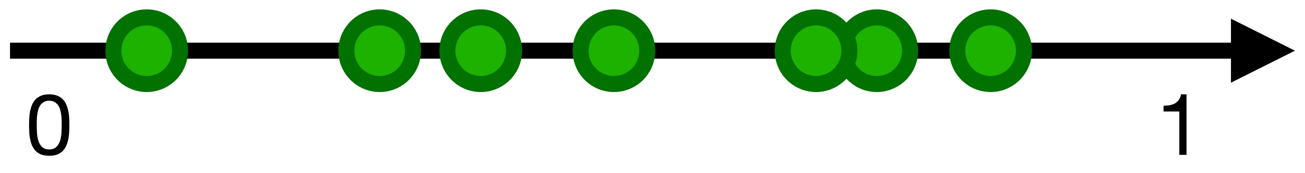
- In rendering, discrete random variables are less common than continuous random variables
- Continuous random variables take on values that ranges of continuous domains (e.g. real numbers or directions on the unit sphere)
- A particularly important random variable is the *canonical uniform random variable, which we write as* ξ

Continuous Random Variable

$$\xi \in [0, 1)$$

Continuous Random Variable



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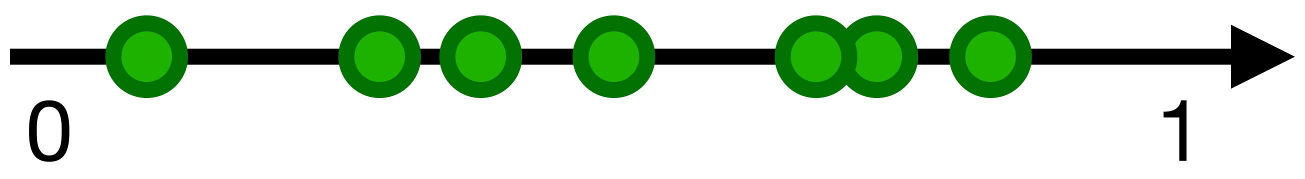
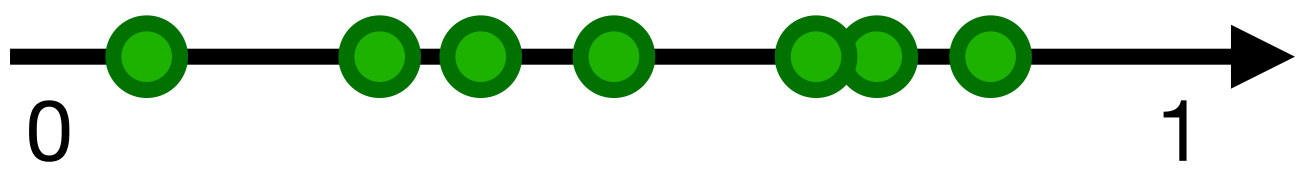


- We can take a continuous, uniformly distributed random variable $\xi \in [0, 1)$ and map to a discrete random variable, choosing X_i if:

Continuous Random Variable



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$$\sum_{j=1}^{i-1} p_j < \xi \leq \sum_{j=1}^{i} p_j$$

$$X_i = \{1, 2, 3, 4, 5, 6\}$$

Questions ?

Image rendered using PBRT

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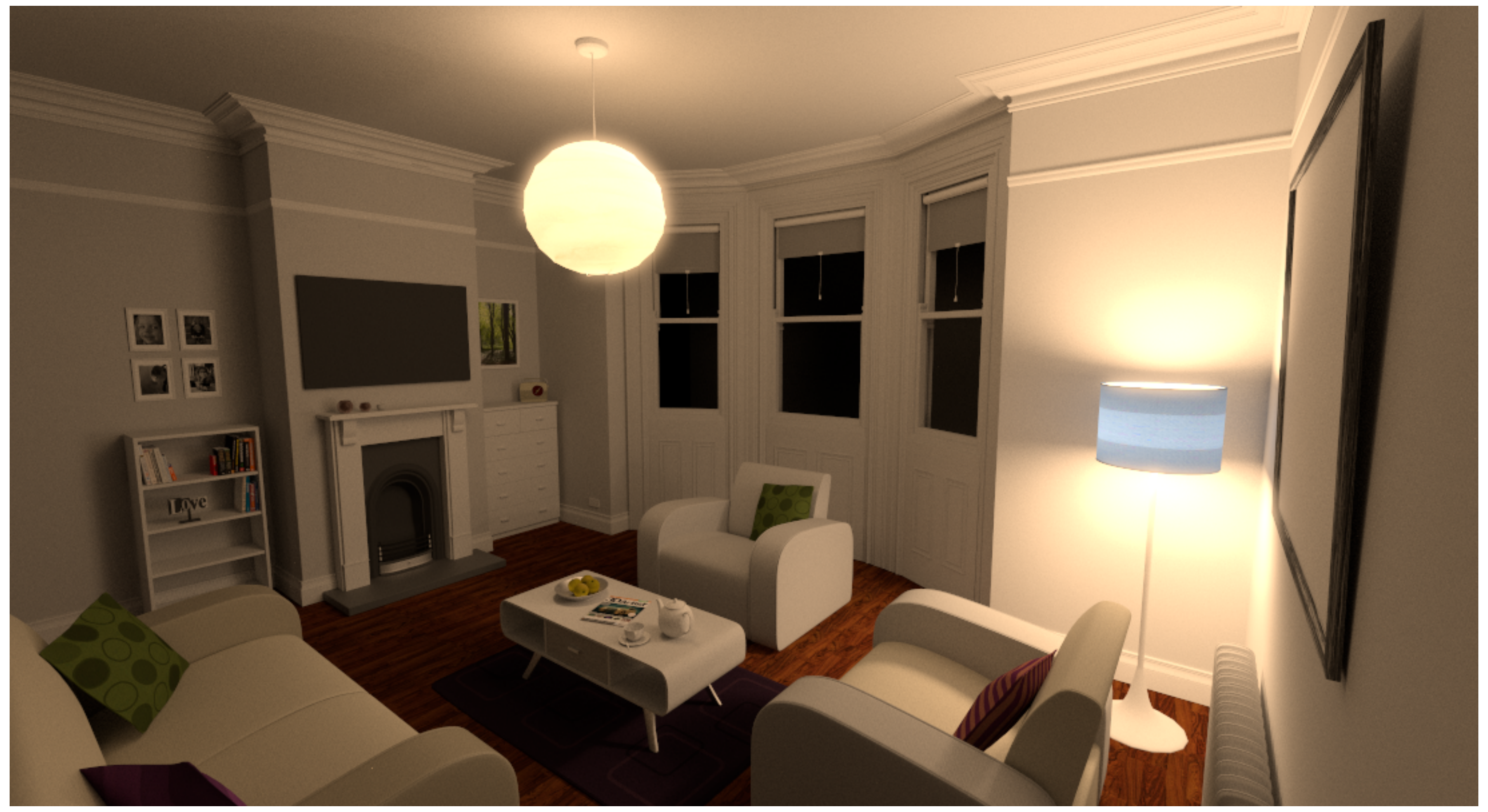
Continuous Random Variable

- For lighting application, we might want to define probability of sampling illumination from each light source in the scene based on its power Φ_i

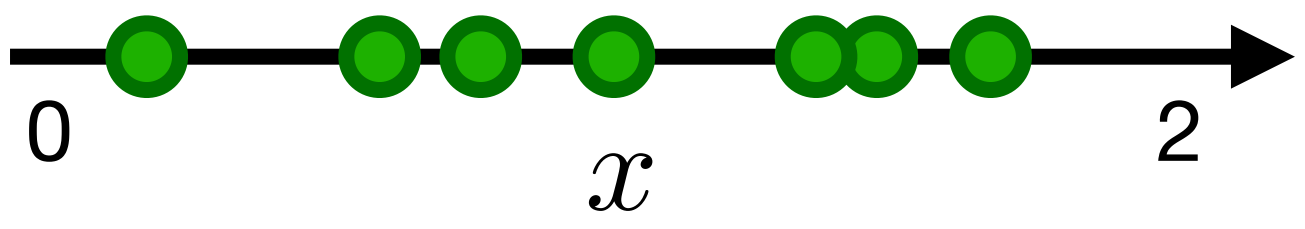
$$p_i = \frac{\Phi_i}{\sum_j \Phi_j}$$

Here, the probability is relative to the total power



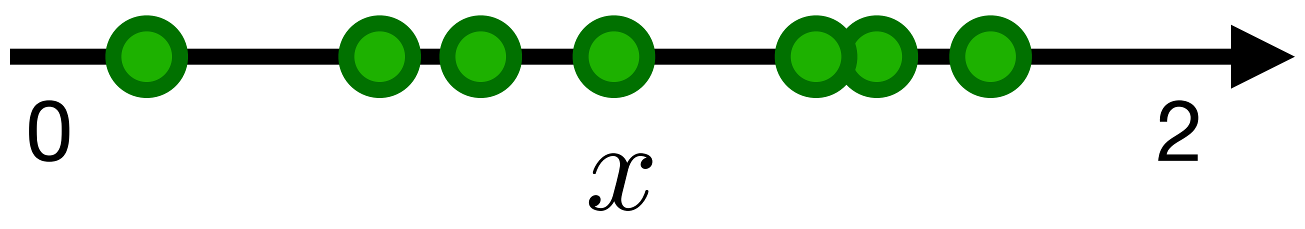
Probability Density Functions

Probability density function



- Consider a continuous RV that ranges over real numbers: $[0, 2)$, where the probability of taking on any particular value x is **proportional** to the value $2 - x$

Probability density function



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- It is twice as likely for this random variable to take on a value around 0 as it is to take around 1, and so forth.

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- The probability density function (PDF) formalizes this idea: it describes the relative probability of a RV taking on a particular value.

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- Unlike PMF, the values of the PDFs are not the probabilities as such: a PDF must be integrated over an interval to yield a probability

Probability density function

For uniform random variables:

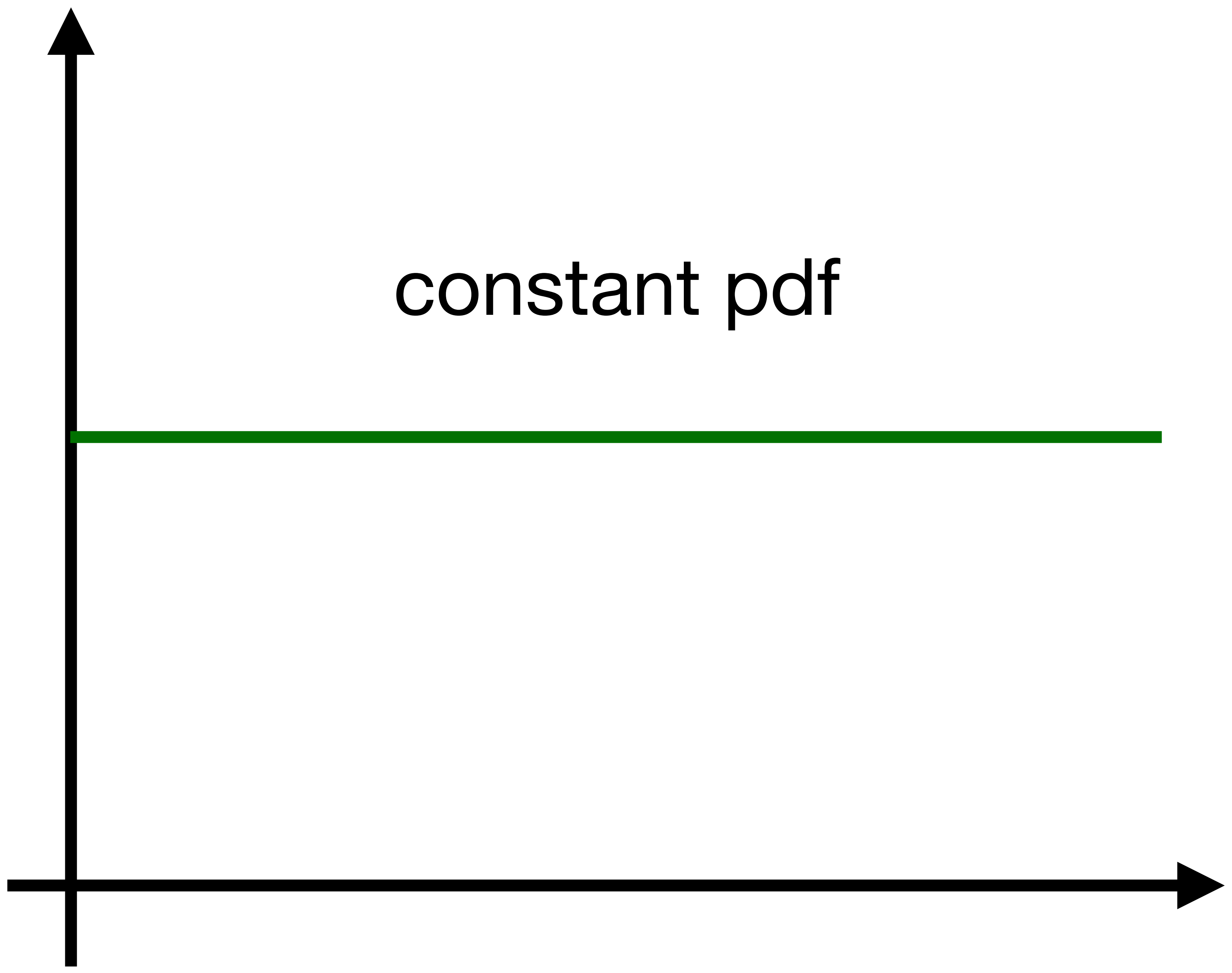
$$p(x) = \begin{cases} 1 & x \in [0, 1) \\ 0 & \text{otherwise} \end{cases}$$

For non-uniform random variables:

$p(x)$ could be any function

Probability density function

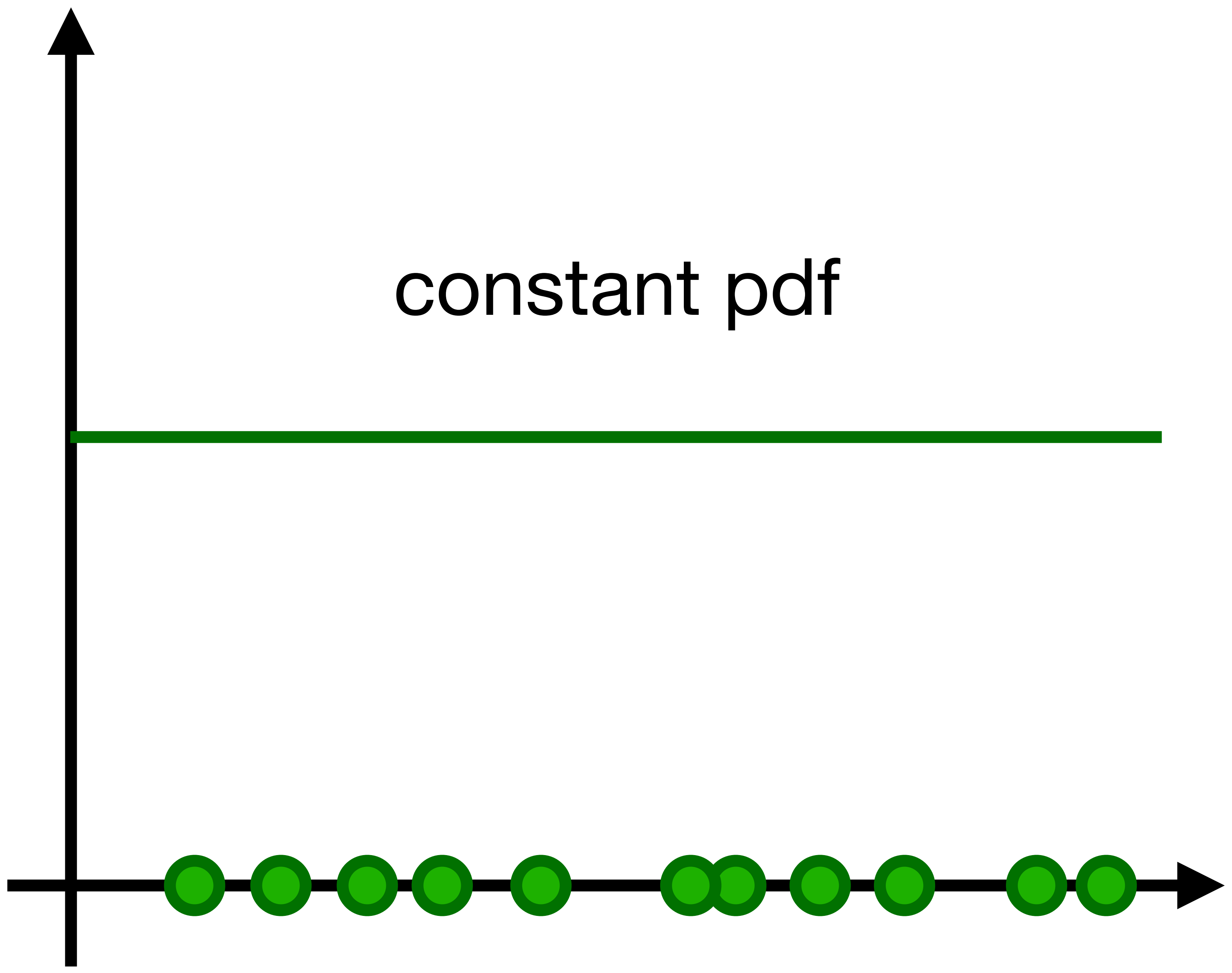
Uniform distribution



Non-uniform distribution

Probability density function

Uniform distribution



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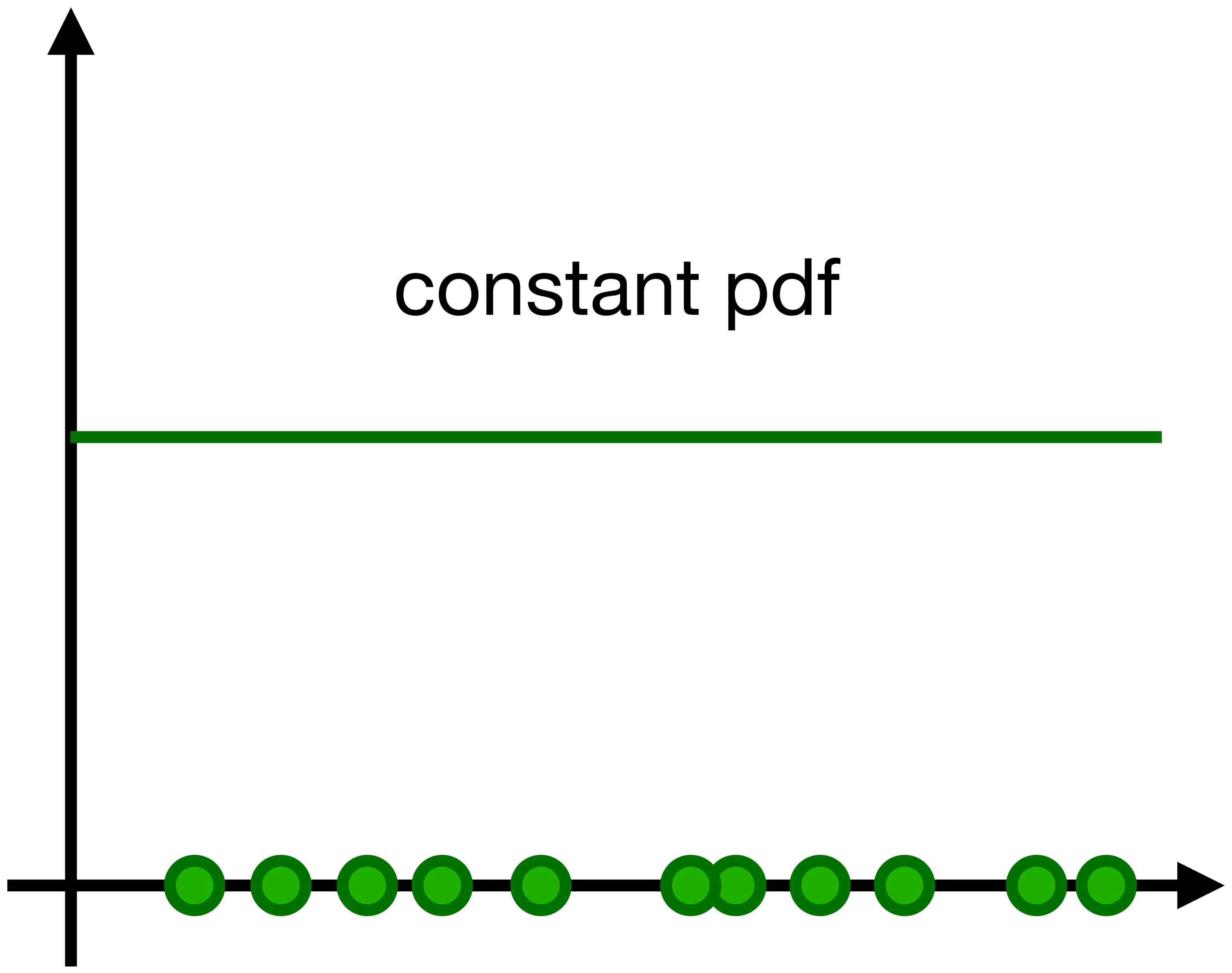

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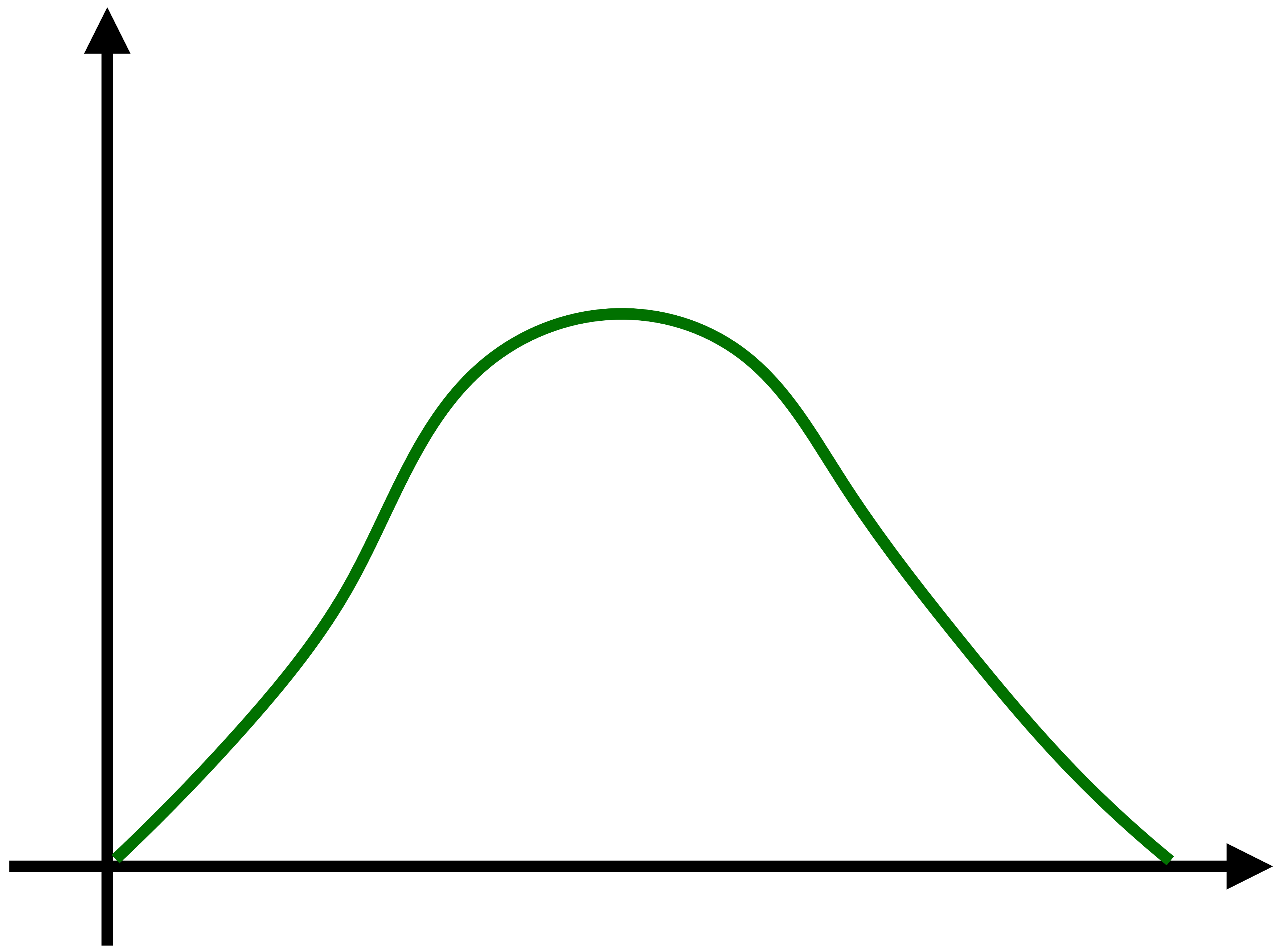
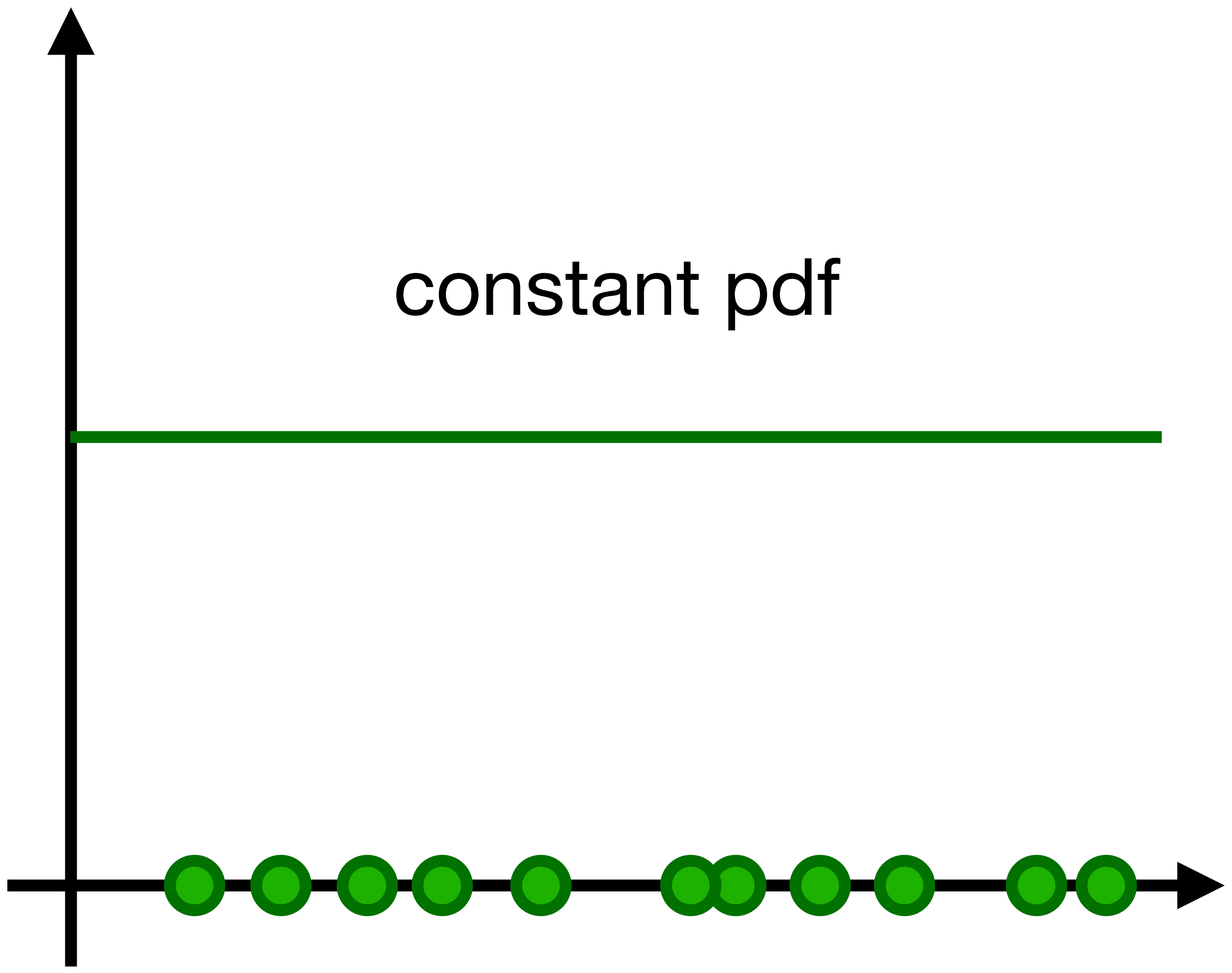


Probability density function

Uniform distribution



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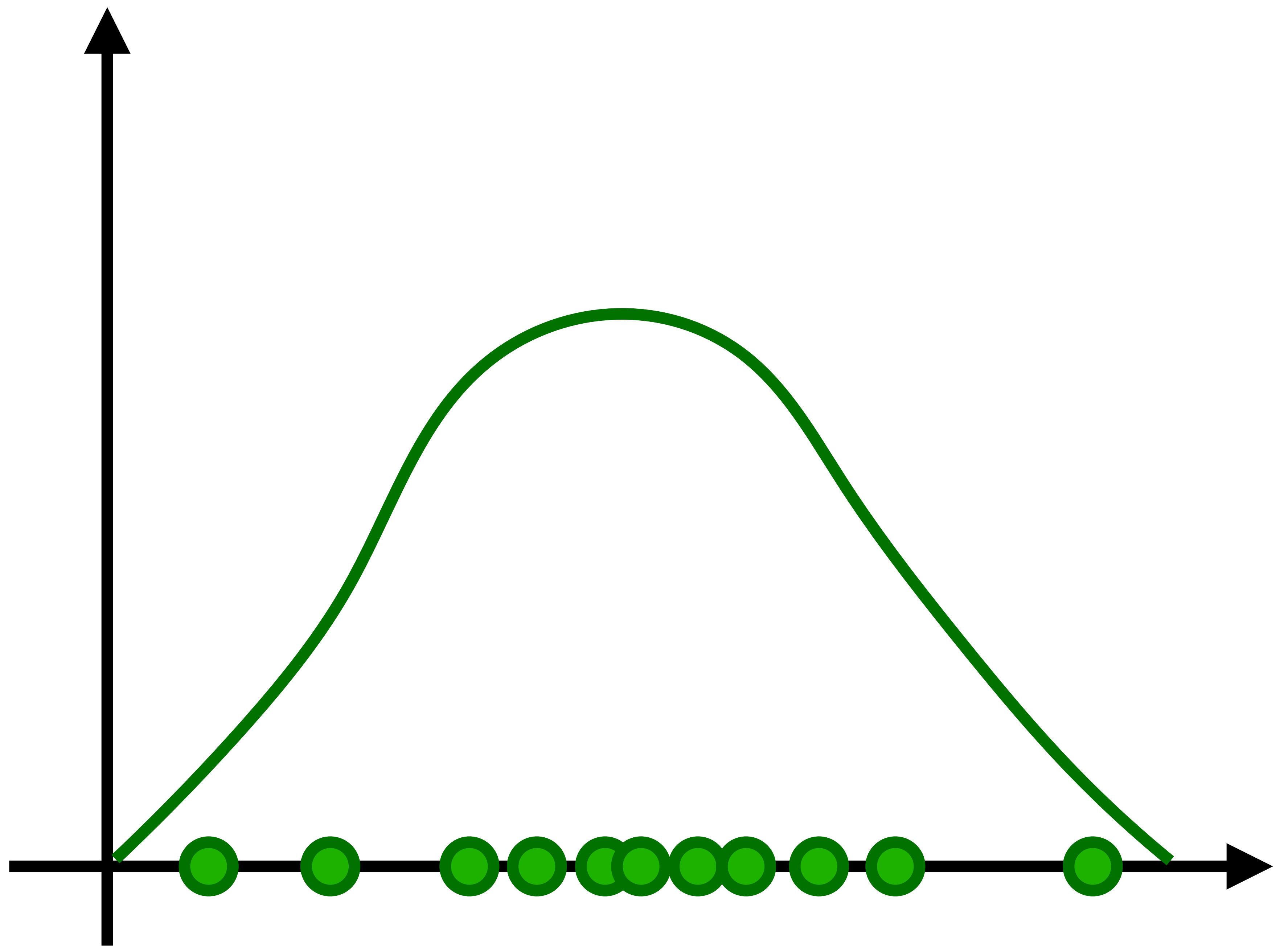
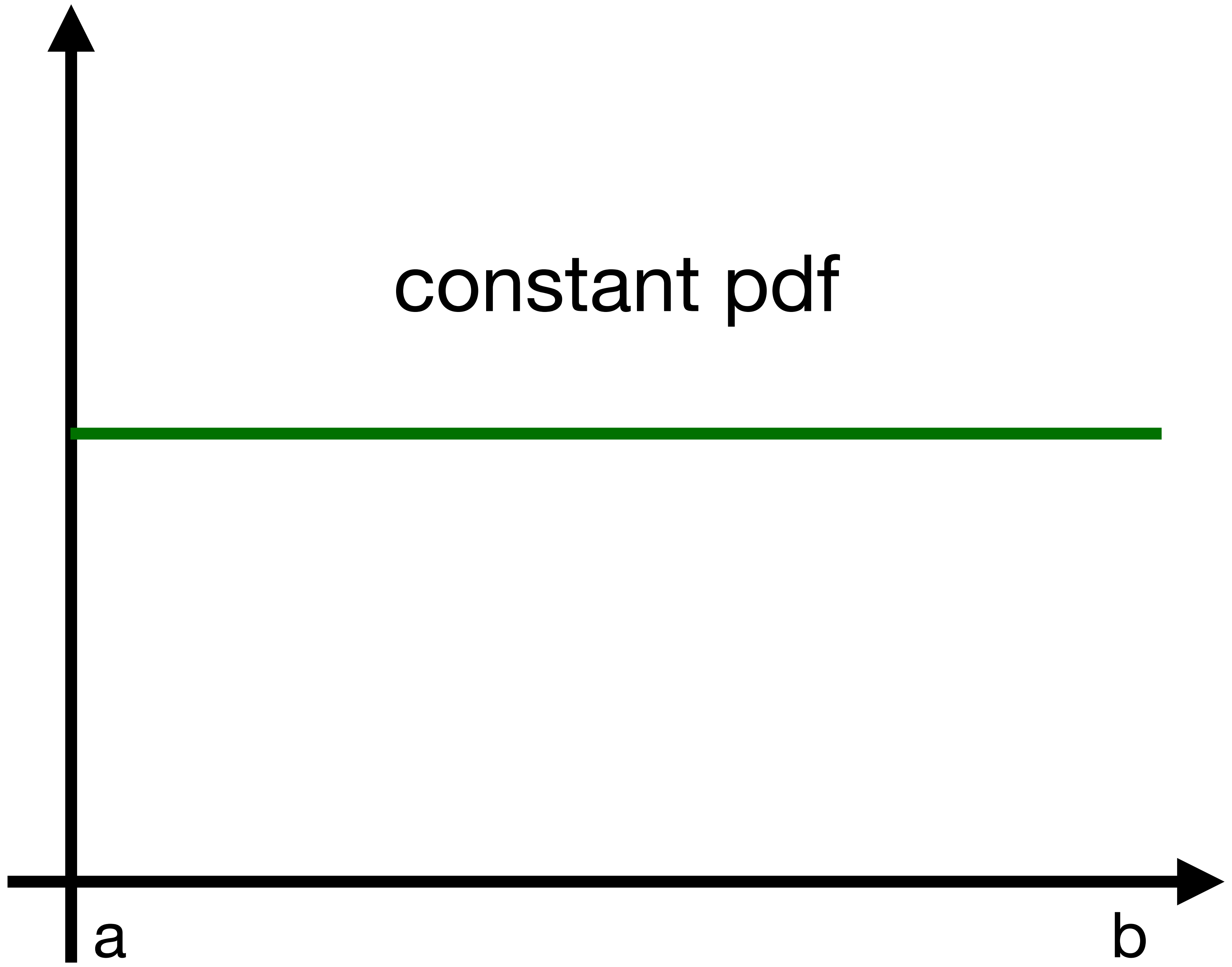


Probability density function

Some properties of PDFs: $p(x) > 0$

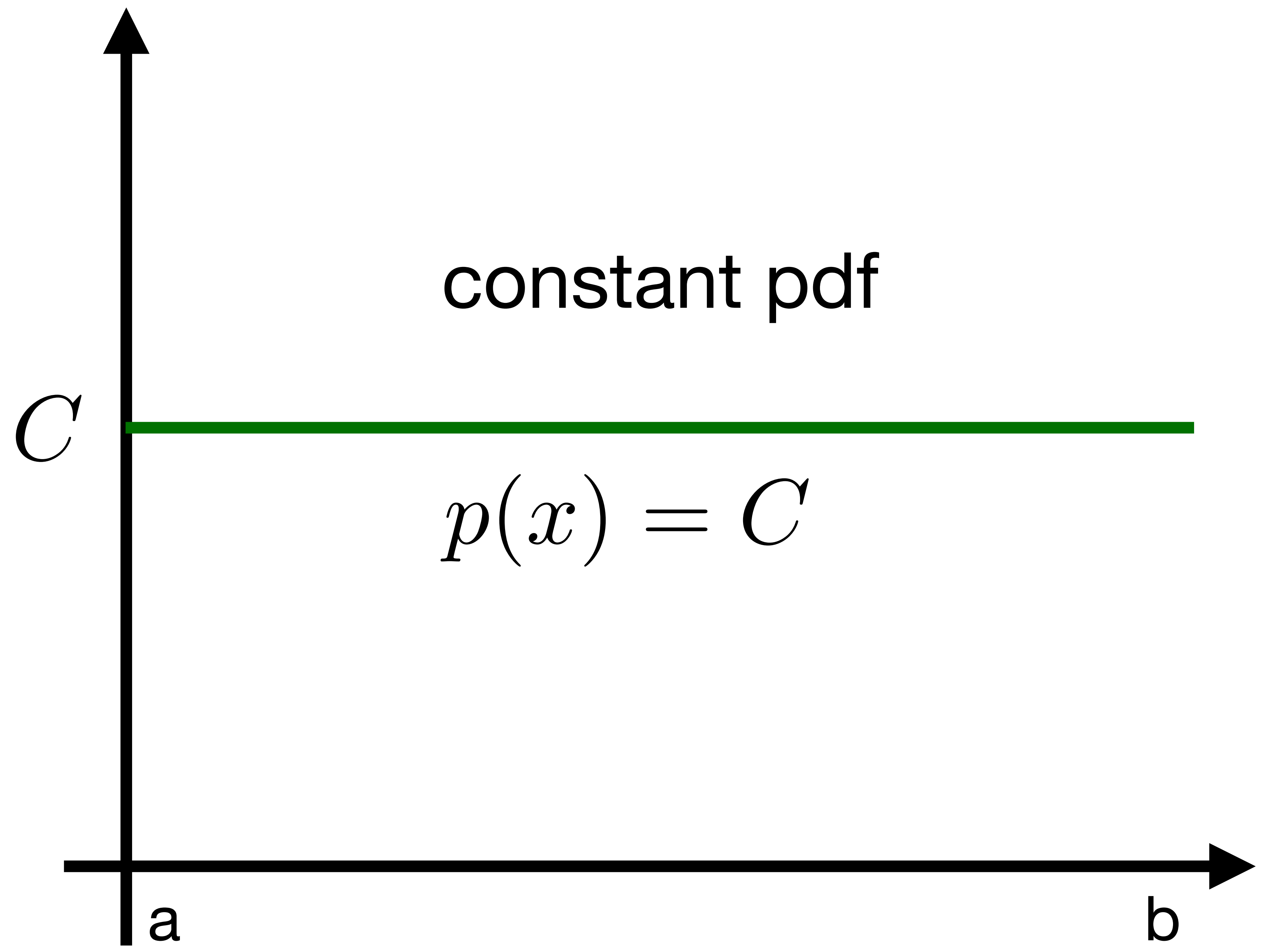
$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Probability density function



$$\int_a^b p(x)dx = 1 \quad x \in [a, b)$$

Probability density function



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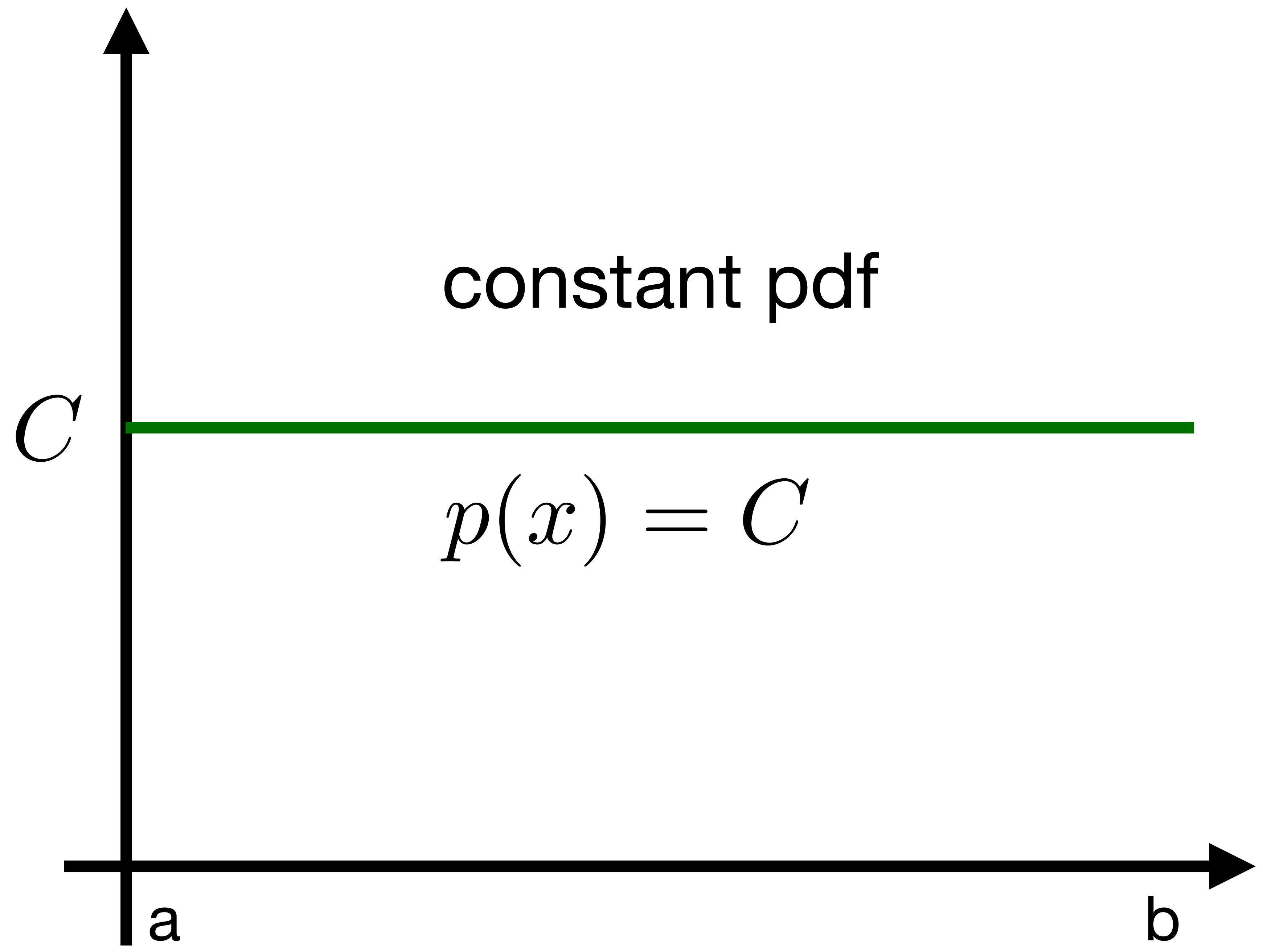

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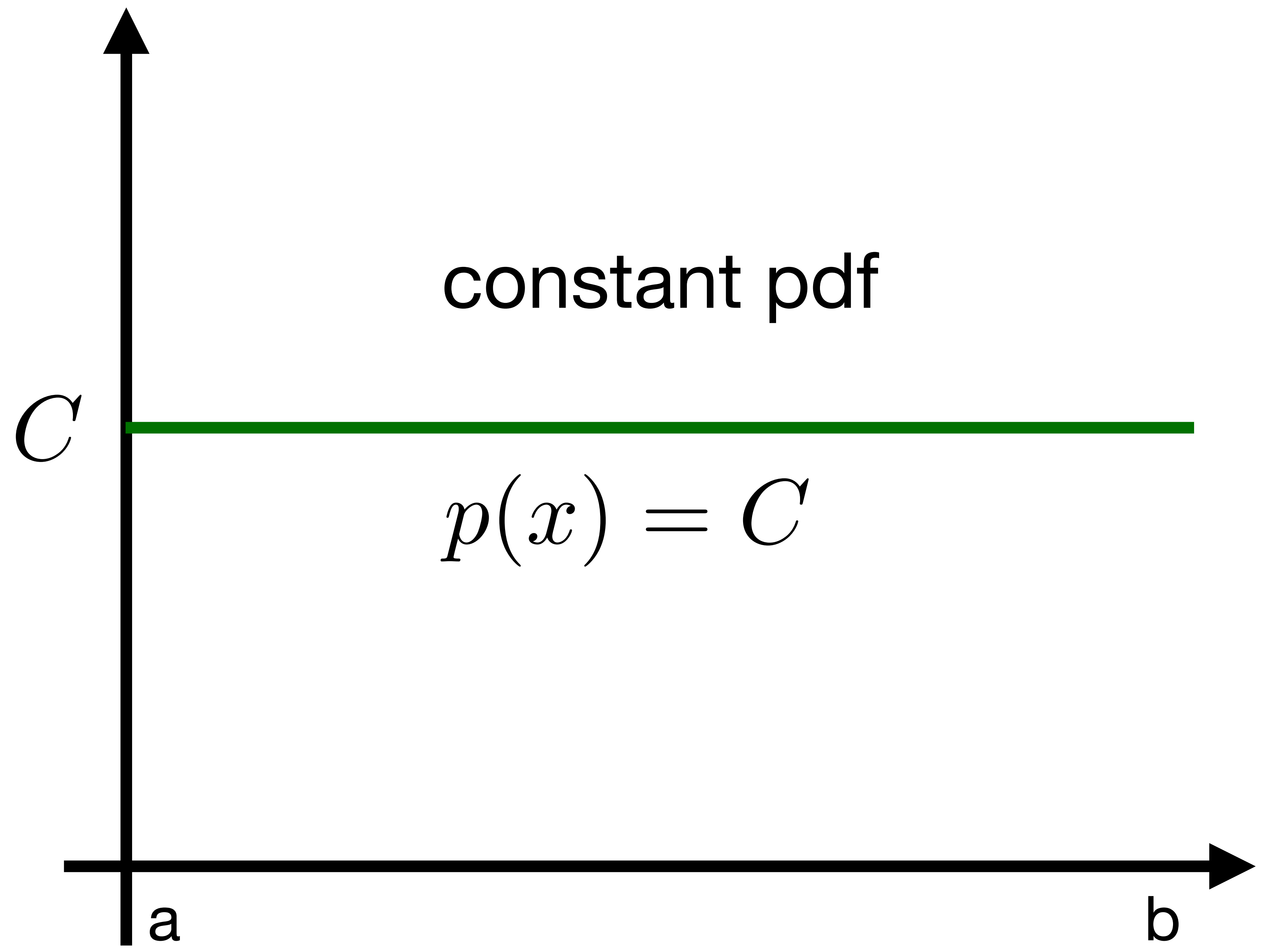

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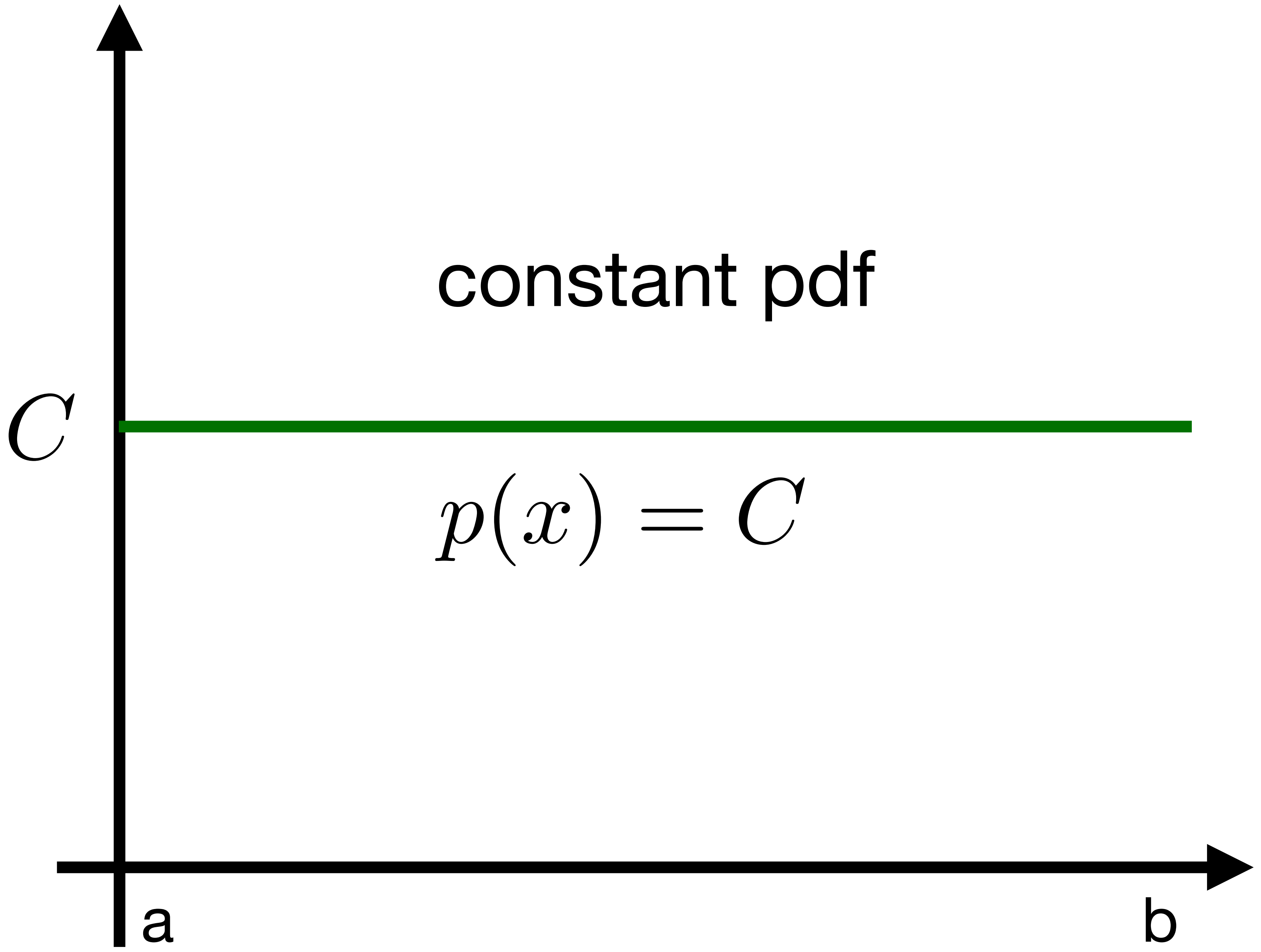

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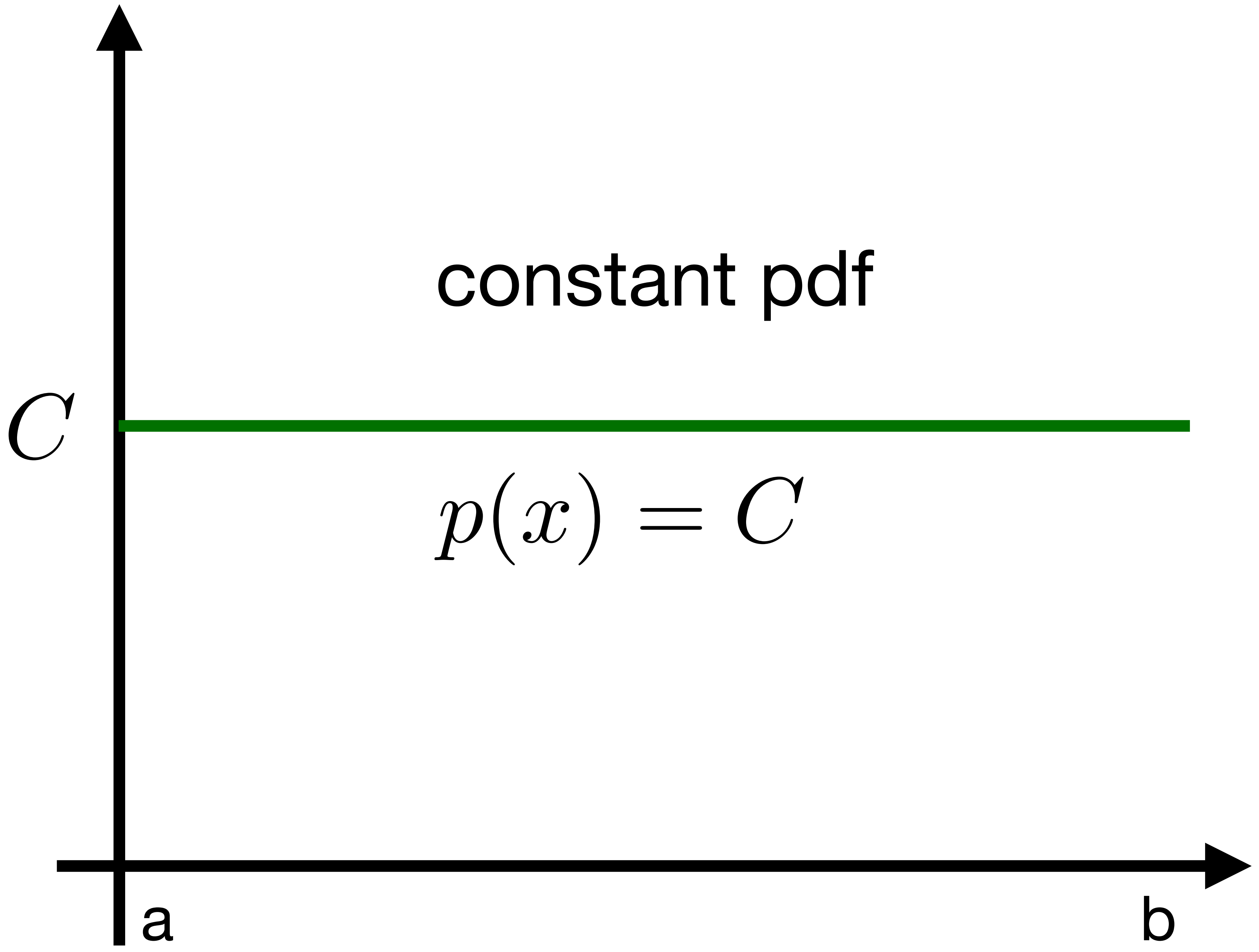

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$$C = \frac{1}{b - a}$$

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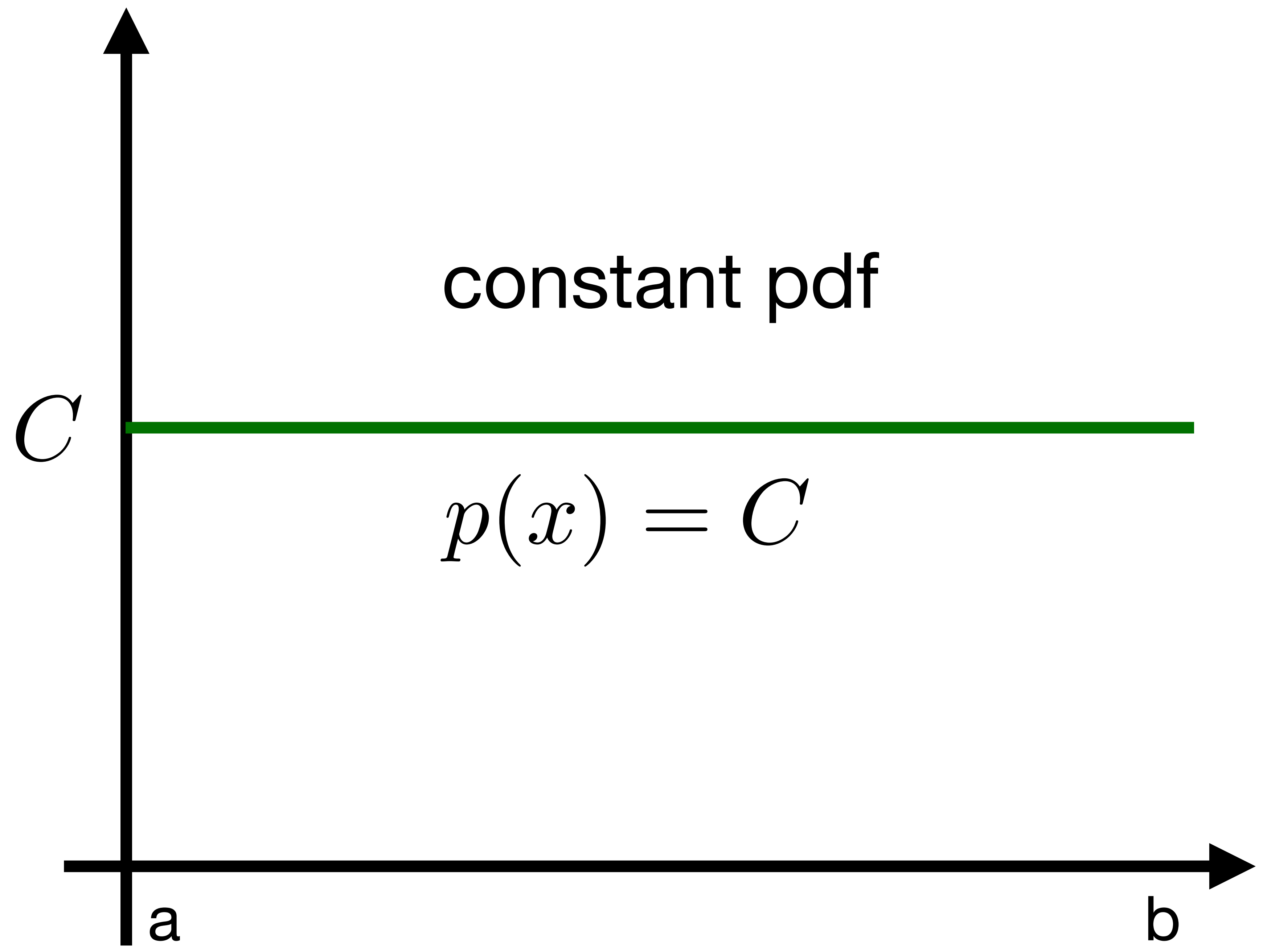

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$$C(b-a) = 1$$

$$C = \frac{1}{b-a}$$

$$\boxed{p(x) = \frac{1}{b-a}}$$

Cumulative distribution function

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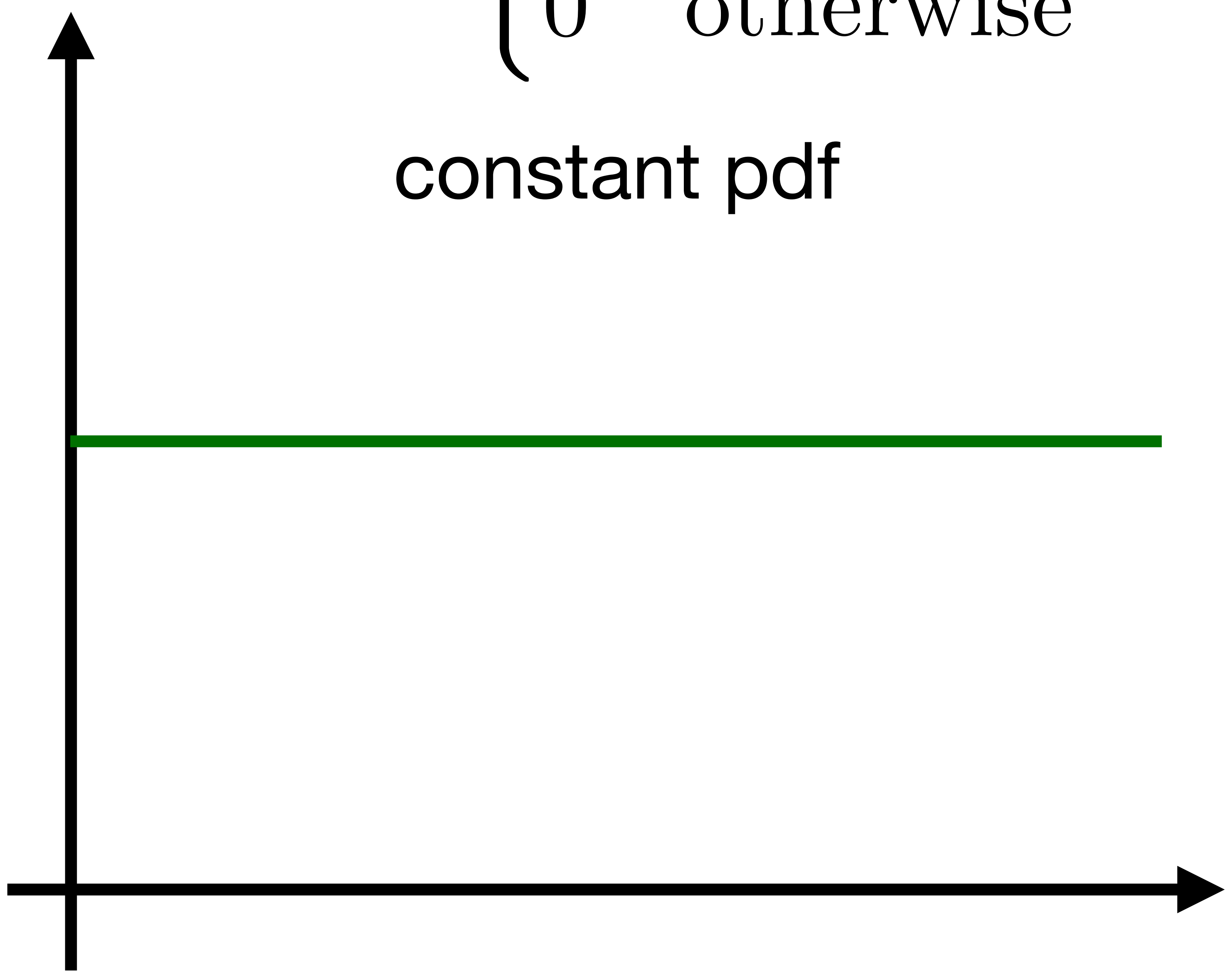
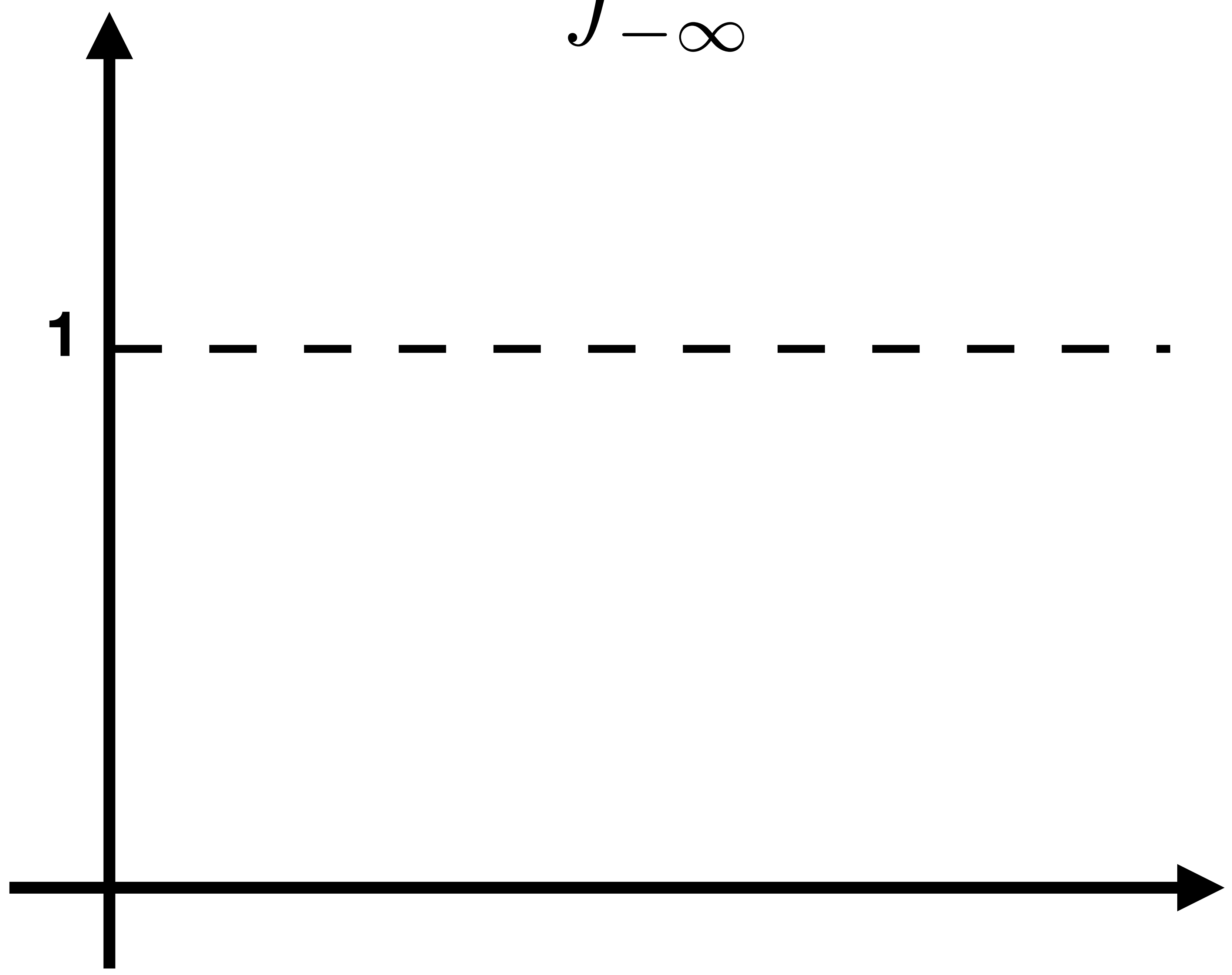
$$p(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & \text{otherwise} \end{cases}$$

constant pdf



$$P(x) = \int_{-\infty}^{x} p(x)dx$$



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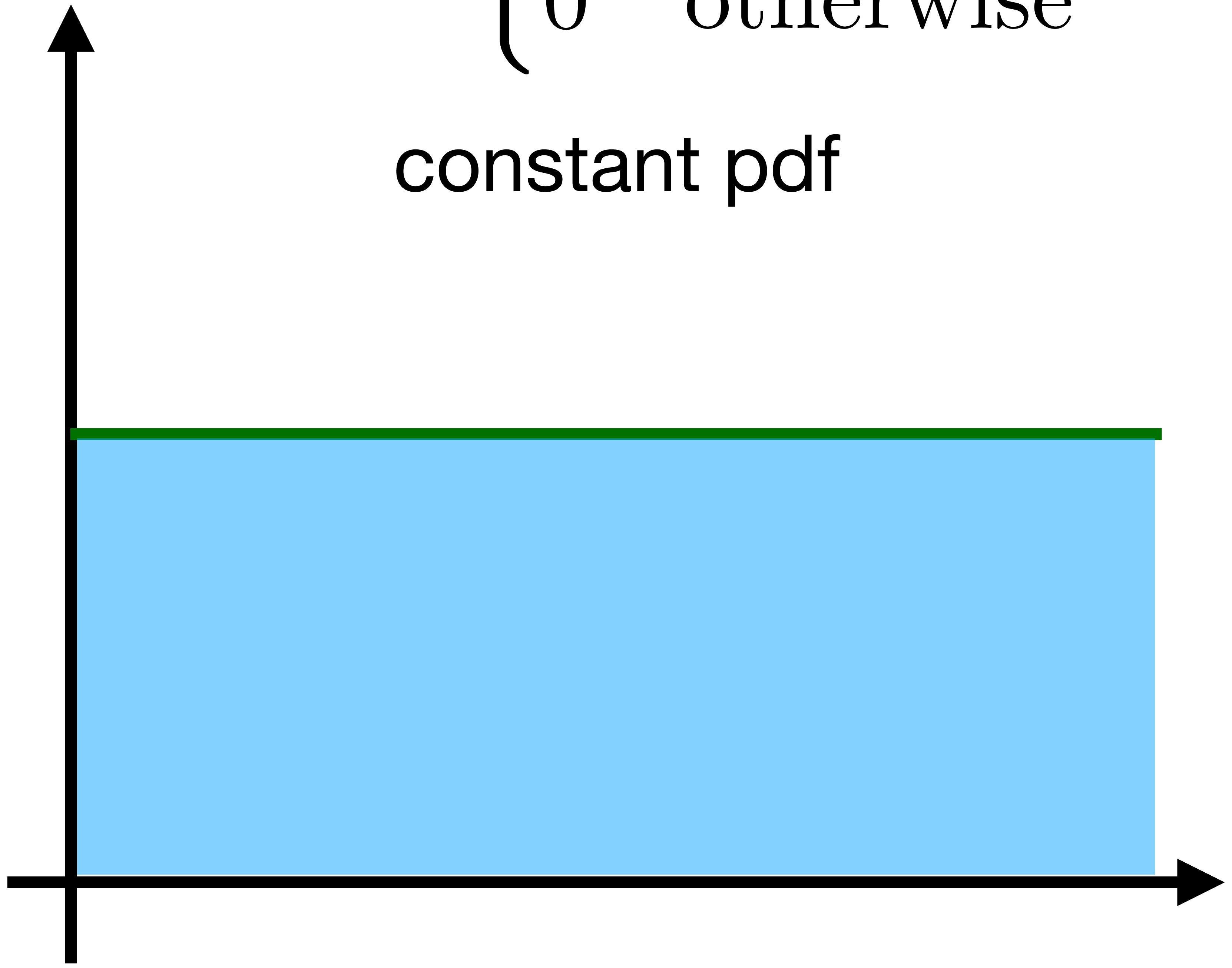
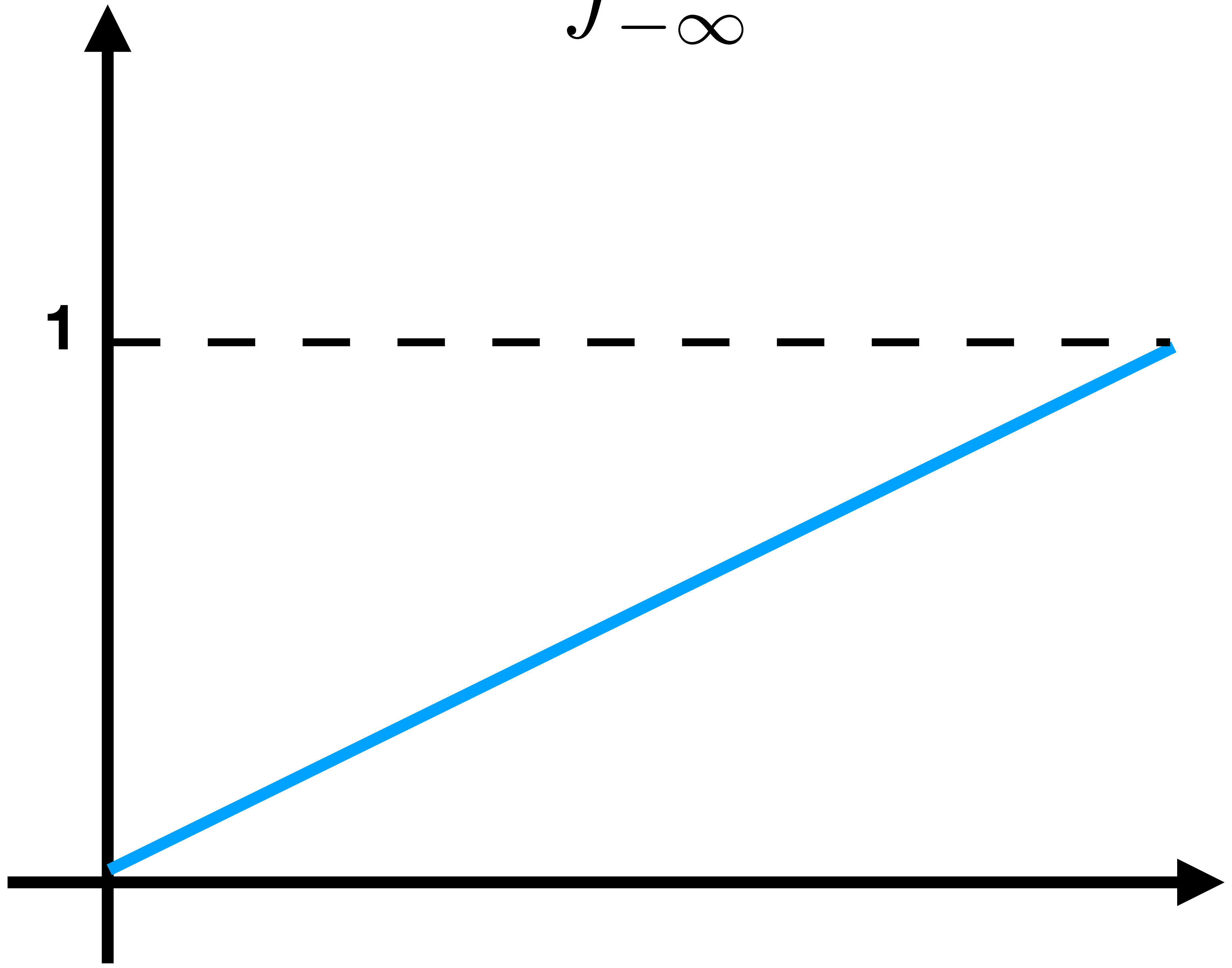
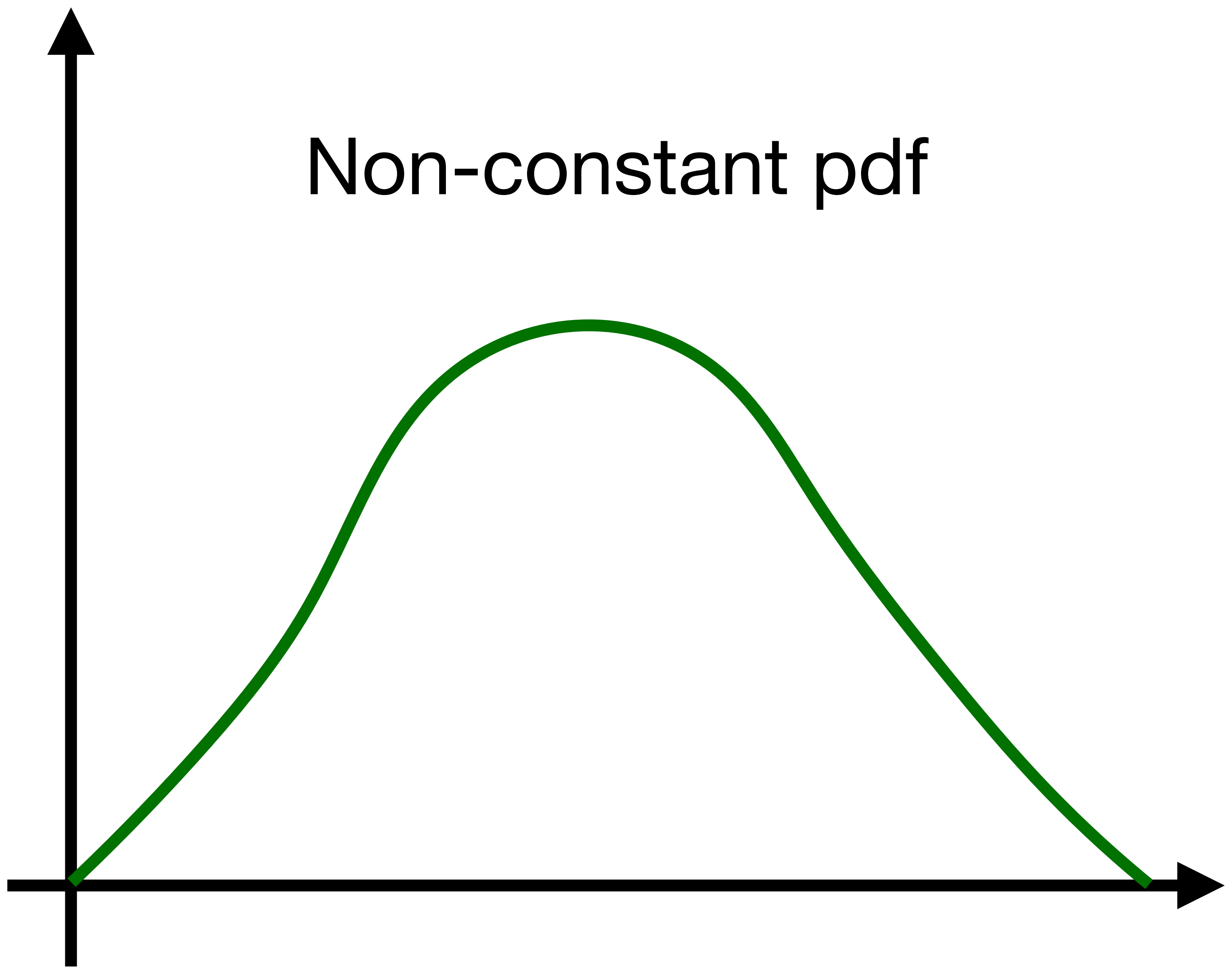




Cumulative distribution function

$$p(x)$$



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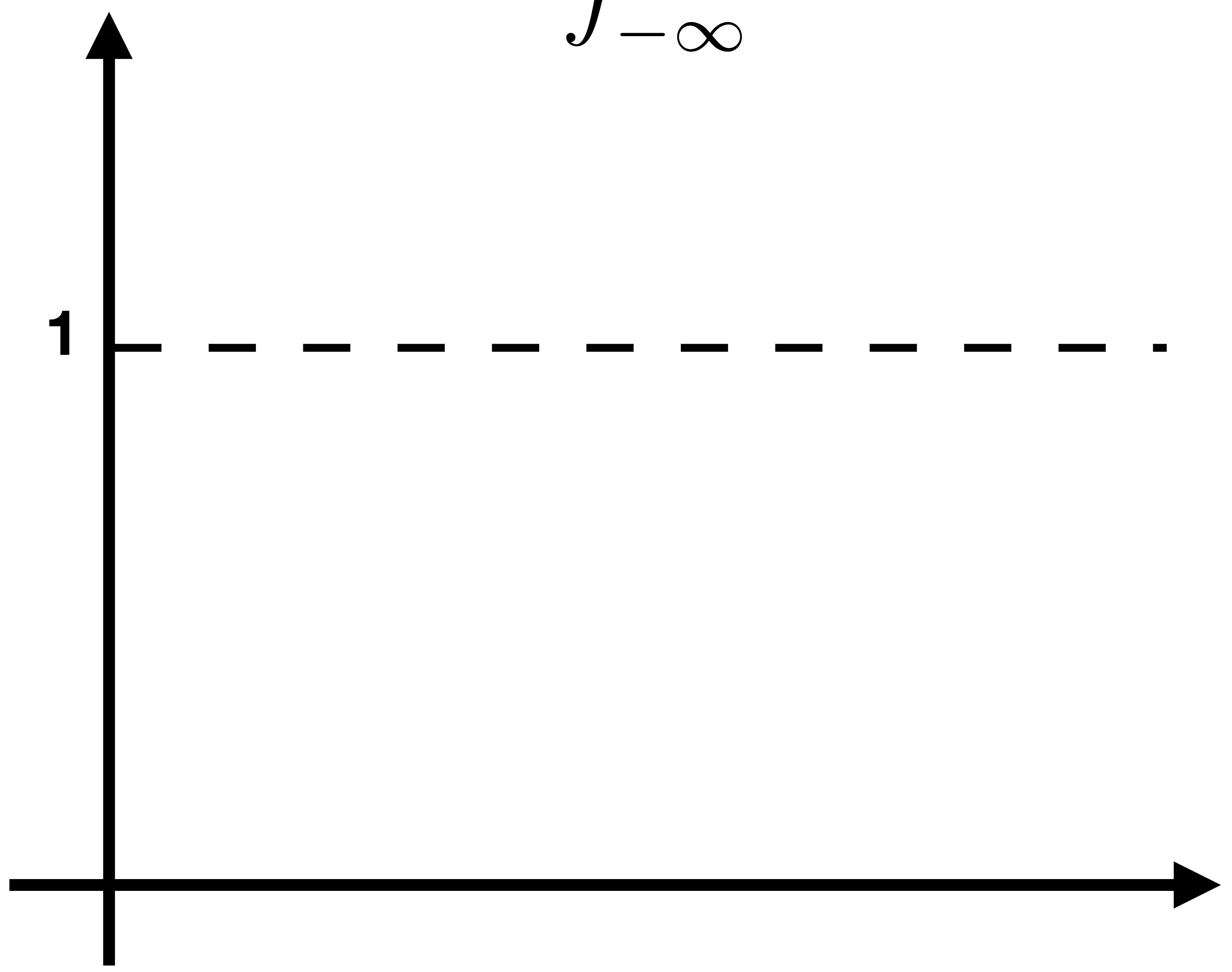

Cumulative distribution function

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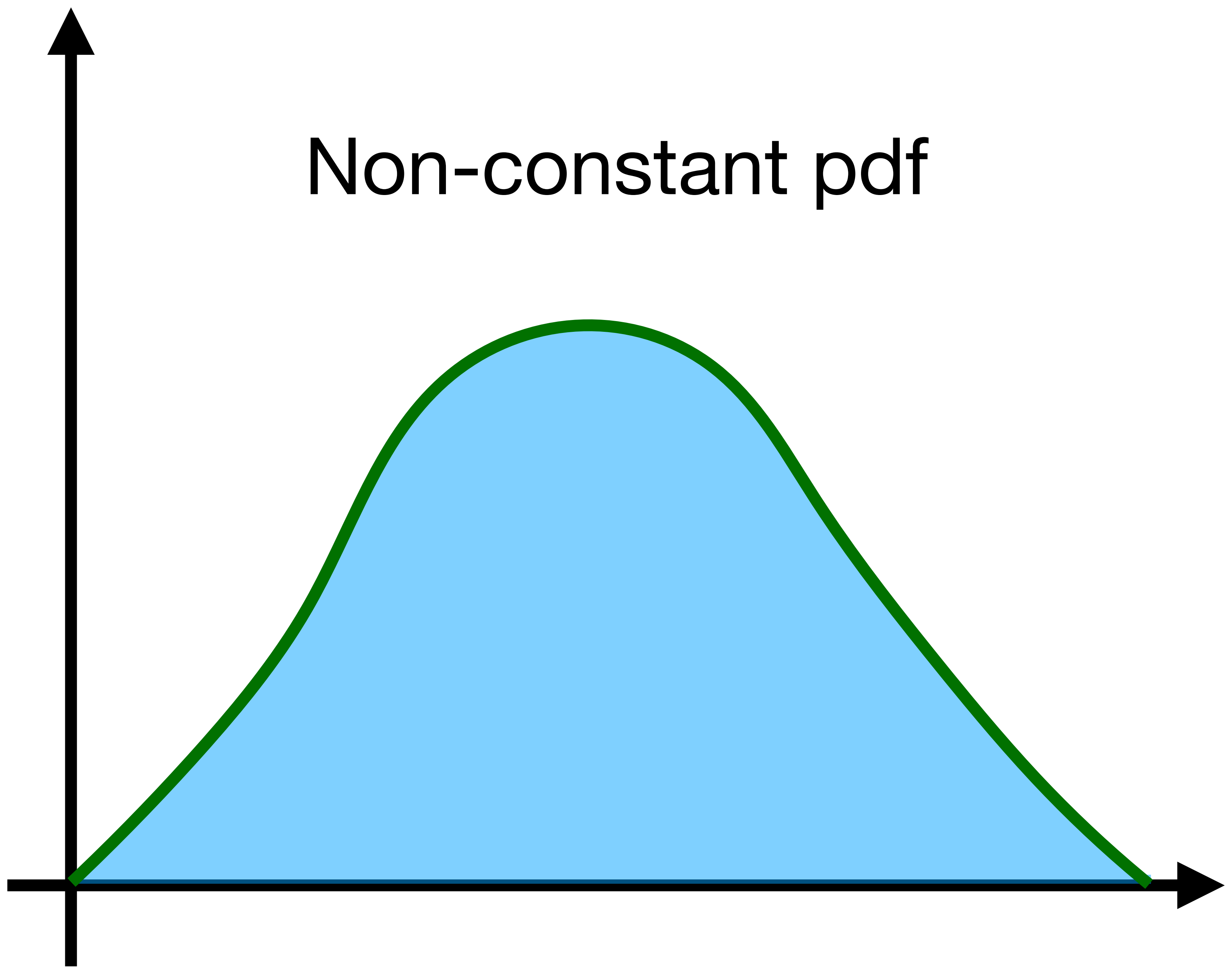

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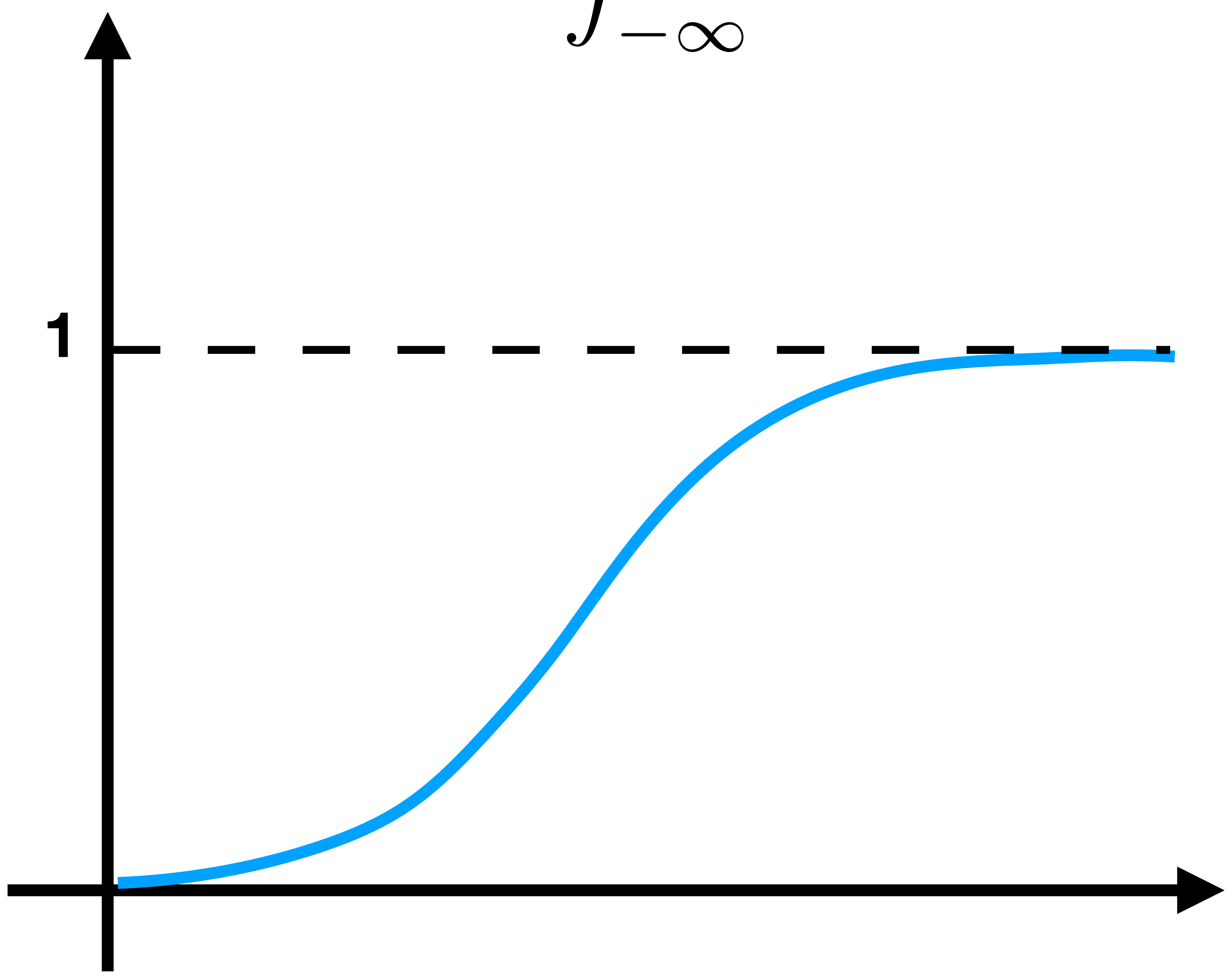


Questions ?
Image rendered using PBRT

Questions ?

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Probability: Integral of PDF

- Given the arbitrary interval $[a, b]$ in the domain, integrating the PDF gives the probability that a RV lies inside that interval:

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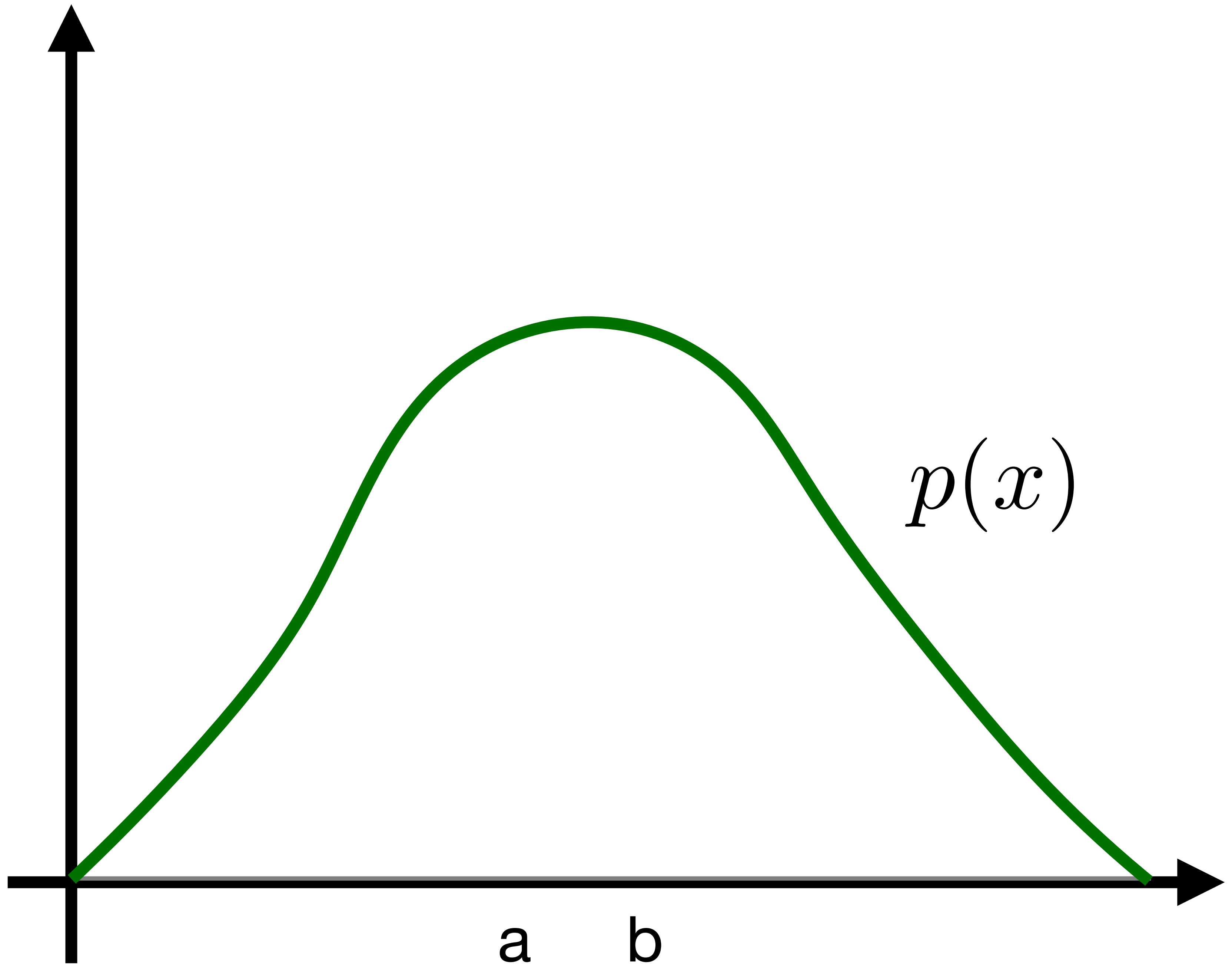

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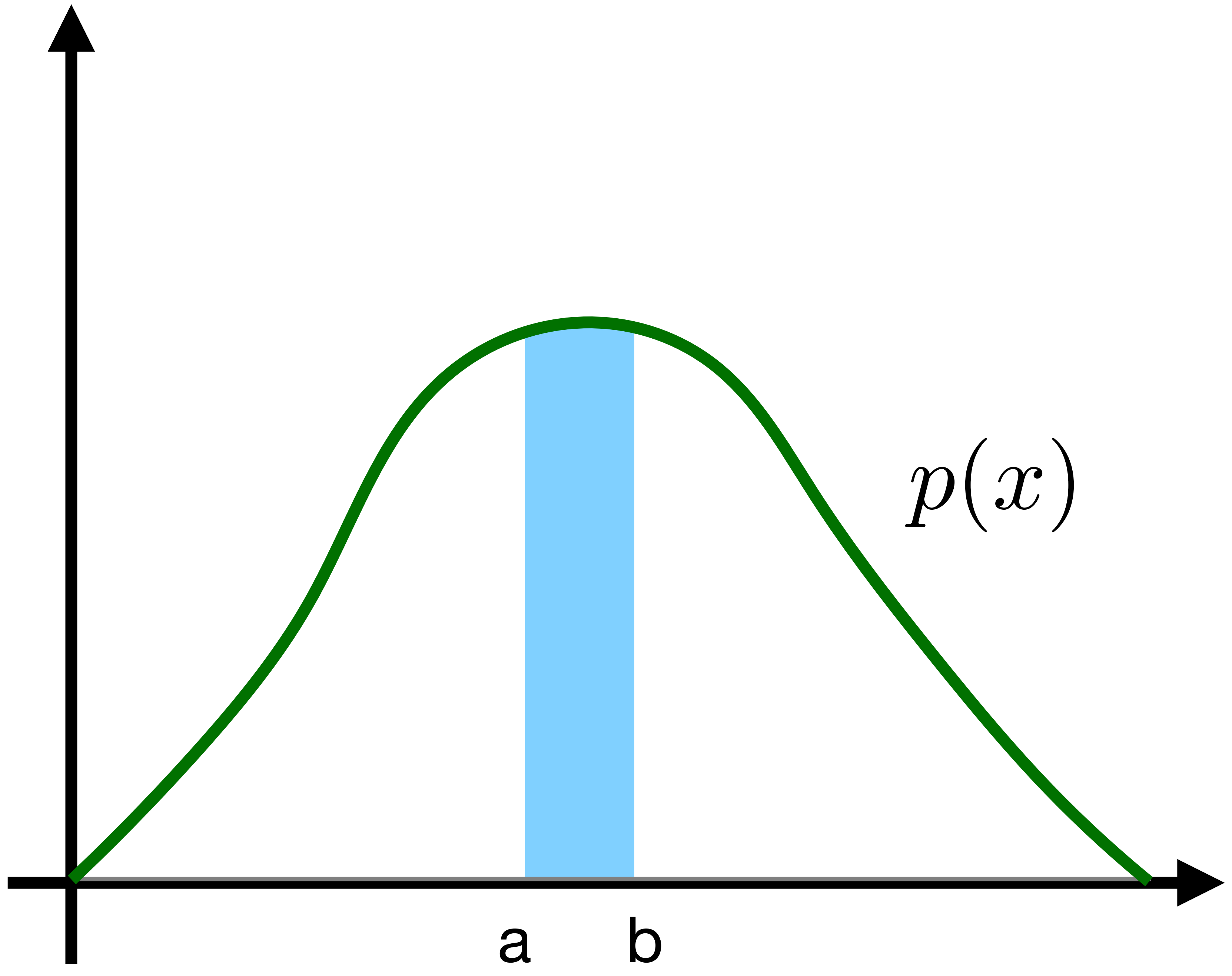
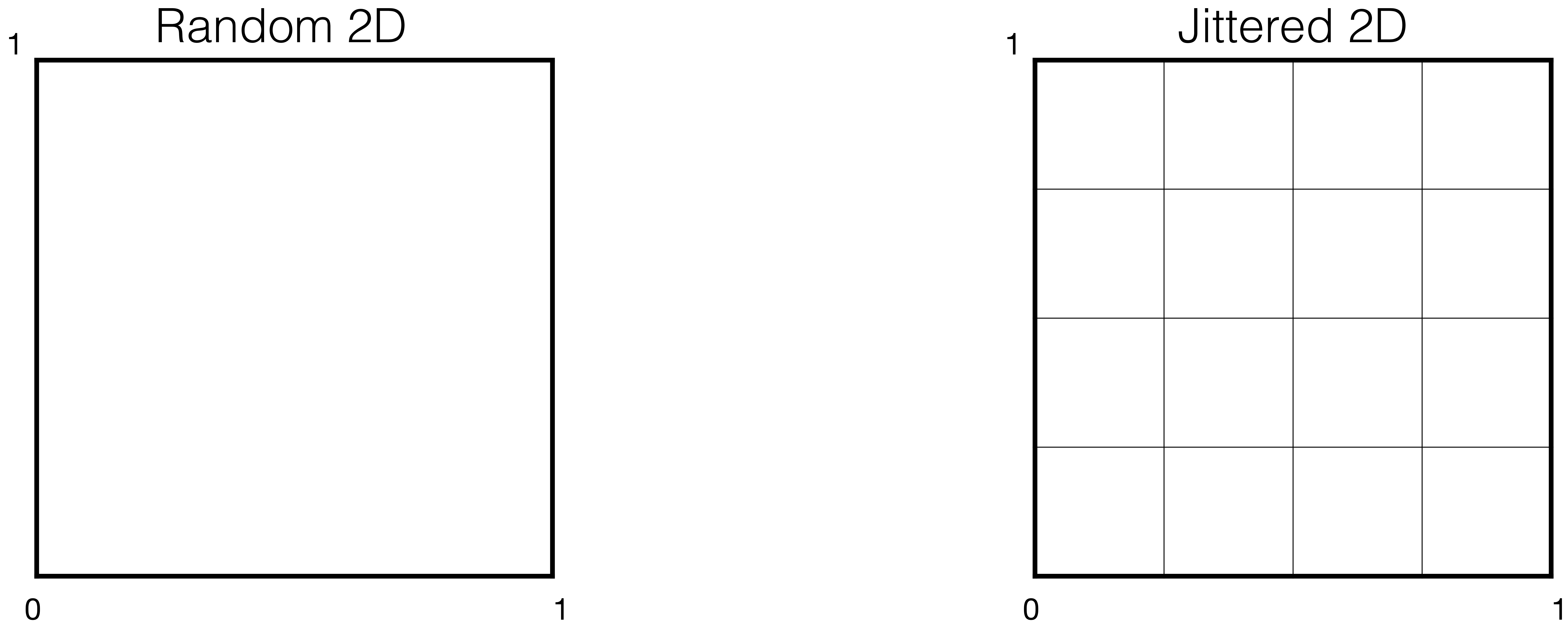


Examples: Sampling PDFs

Constant Sampling PDFs



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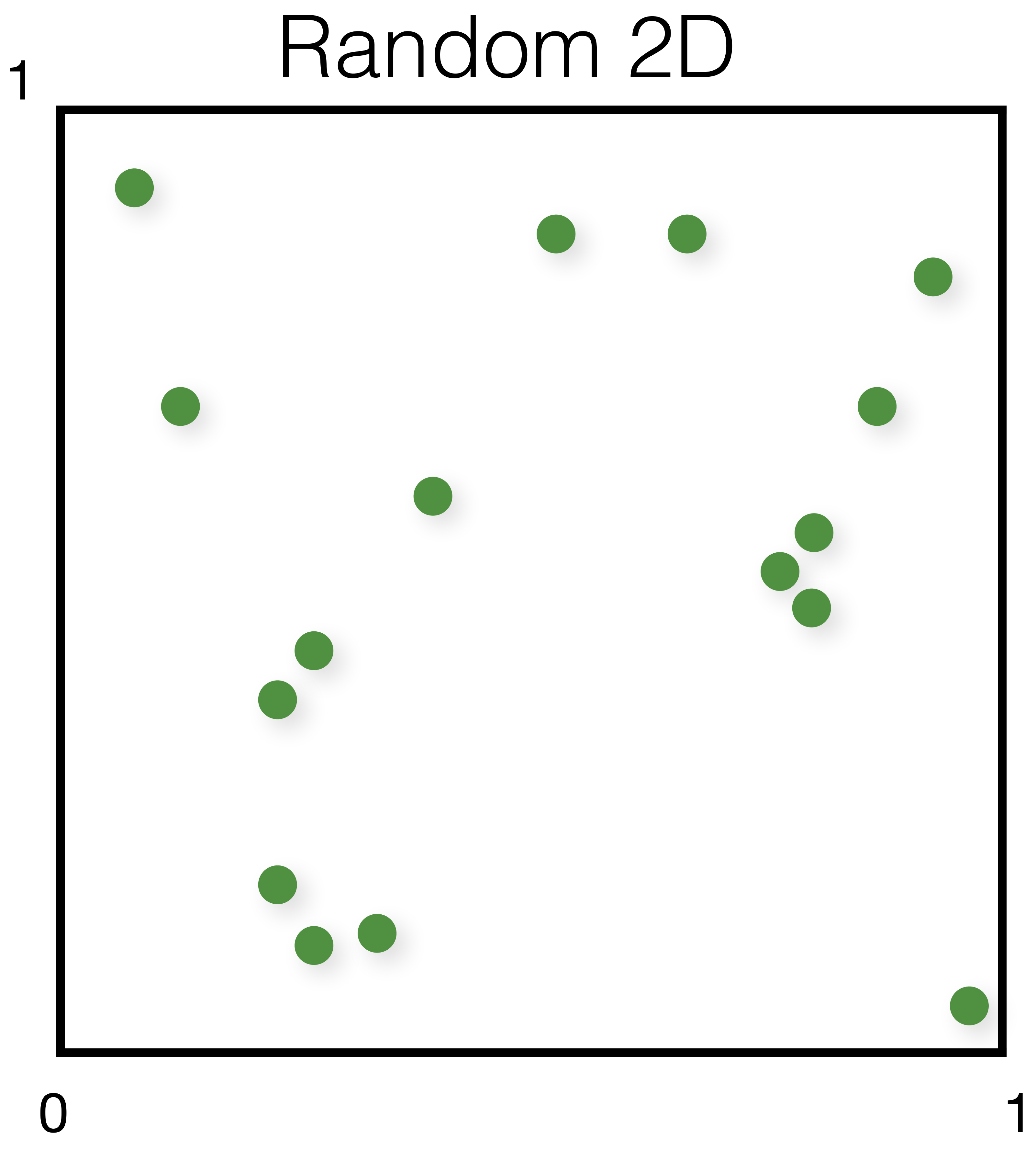

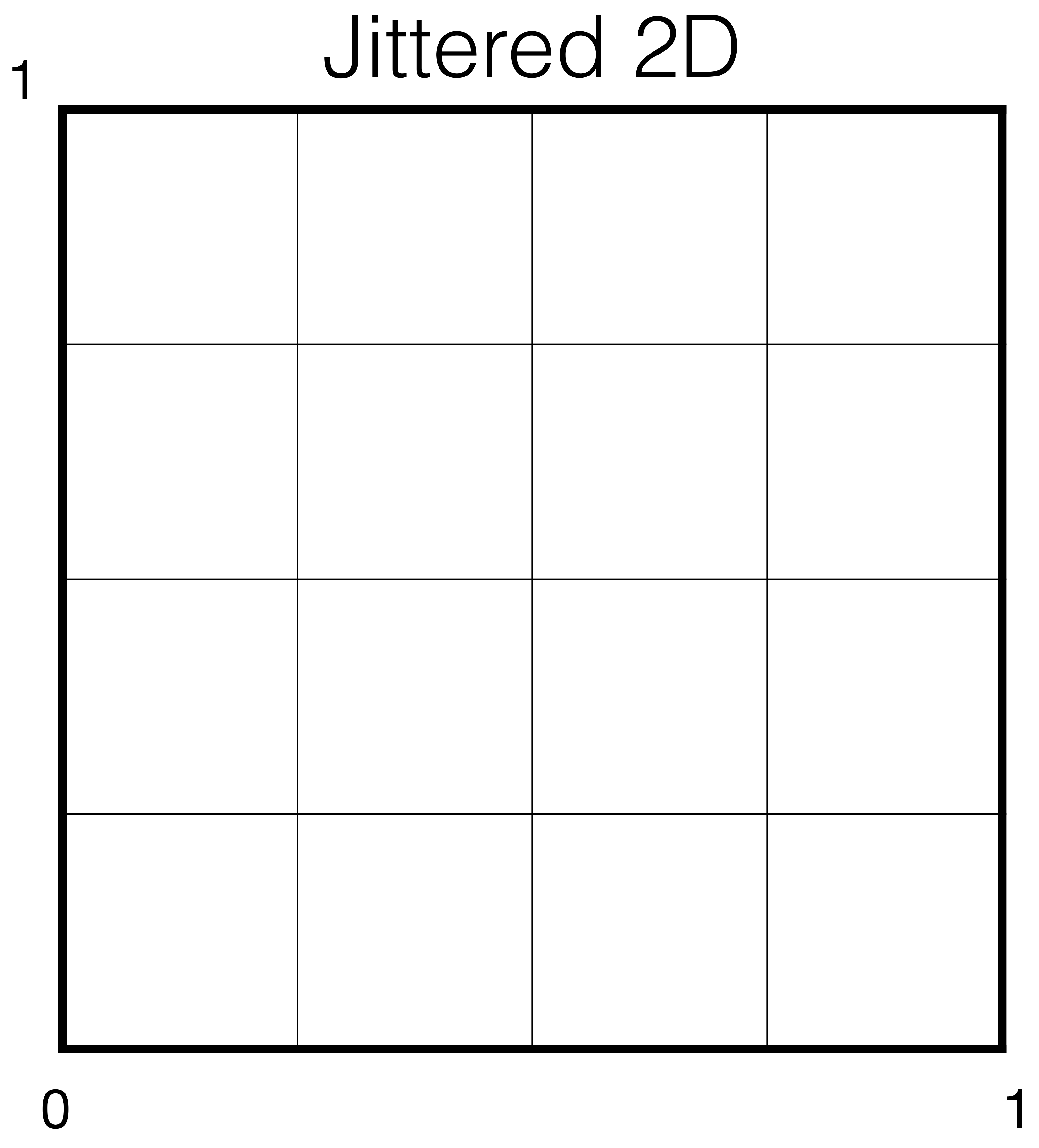

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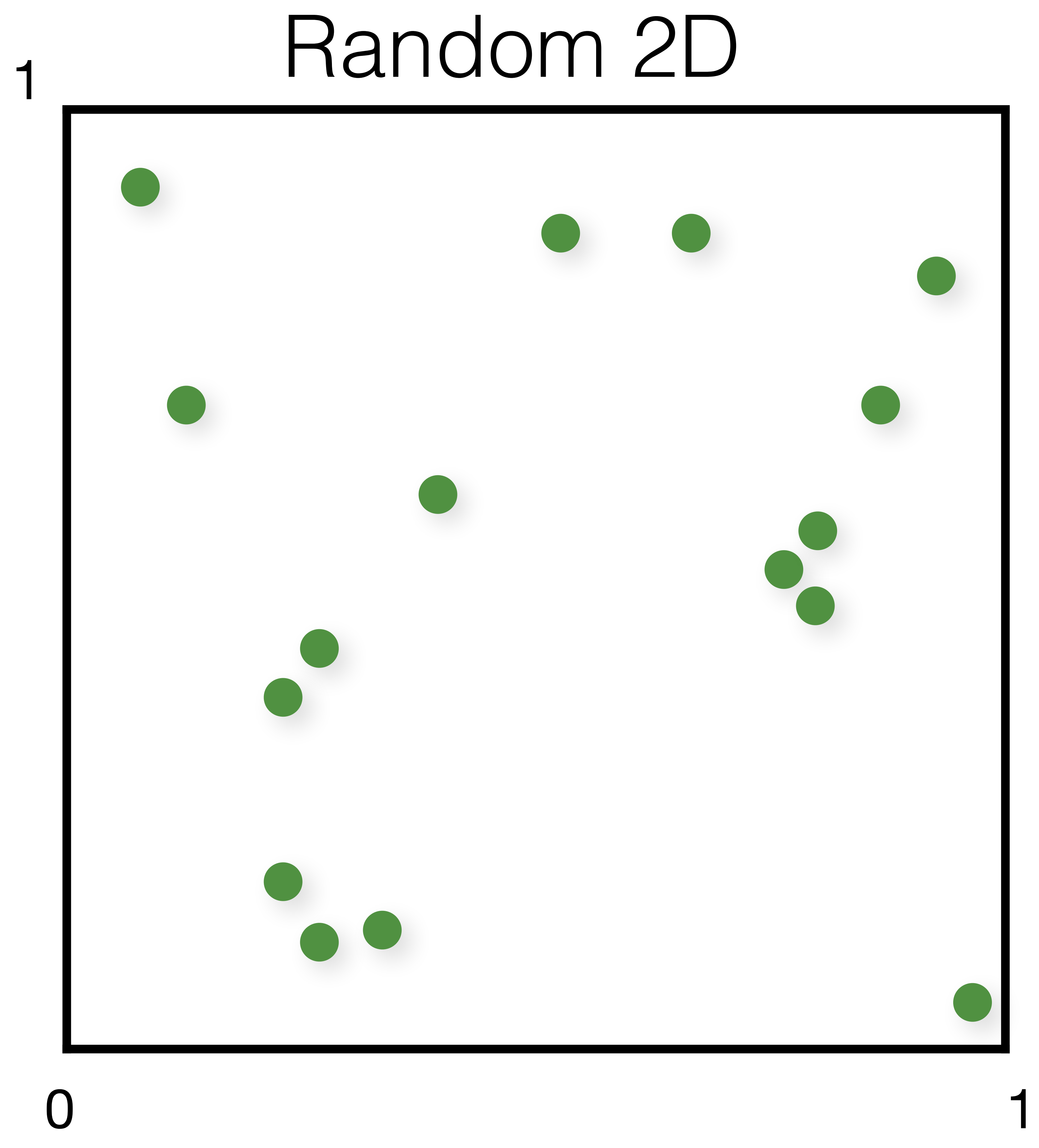
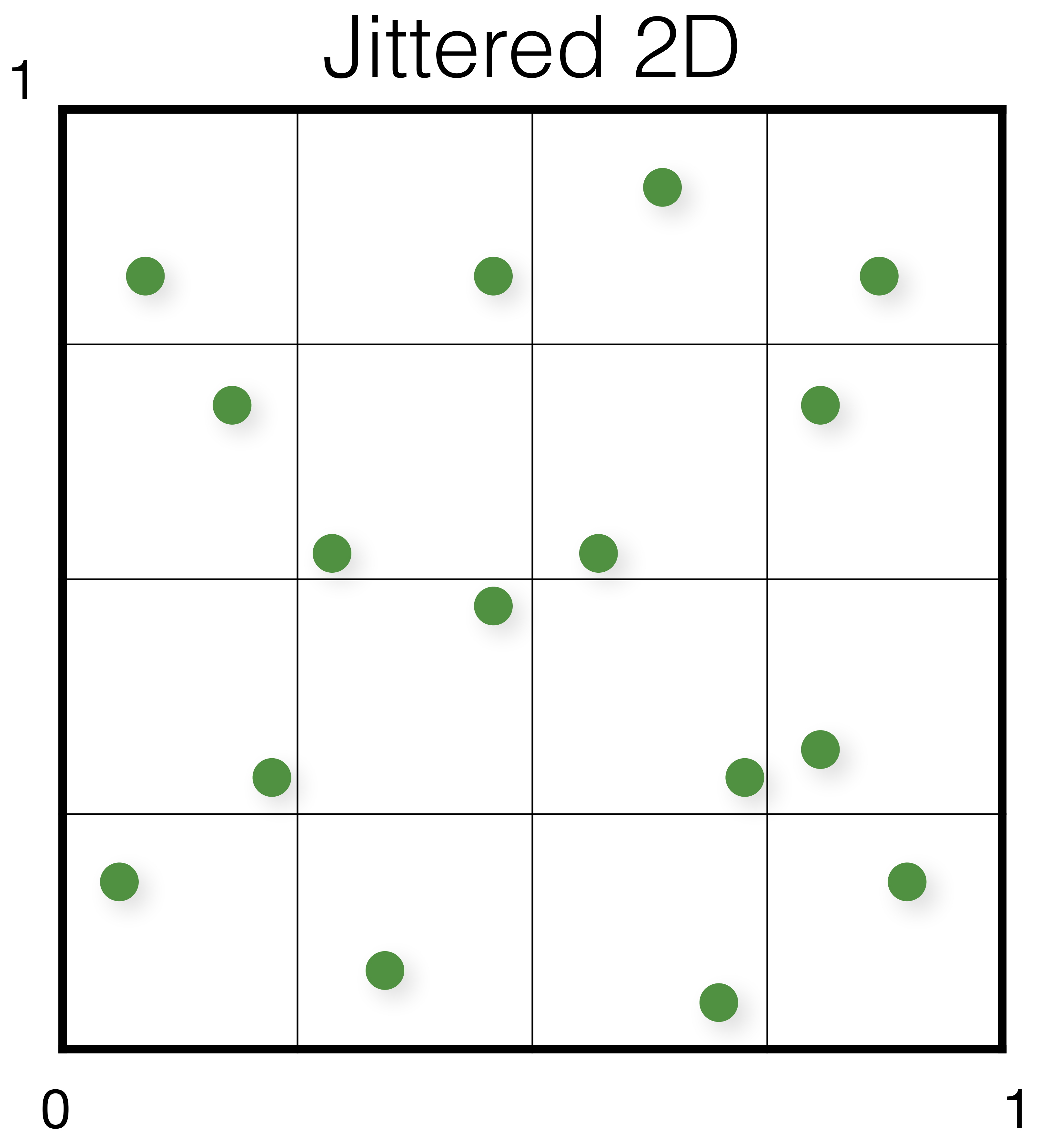
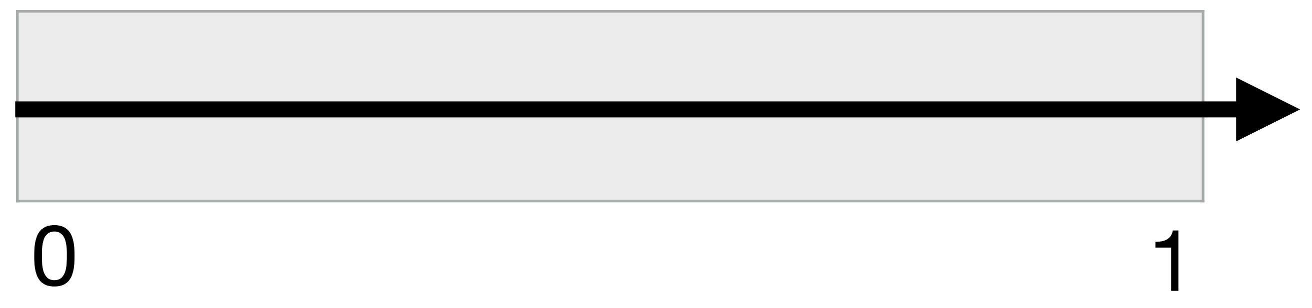




Constant Sampling PDFs

Random 1D

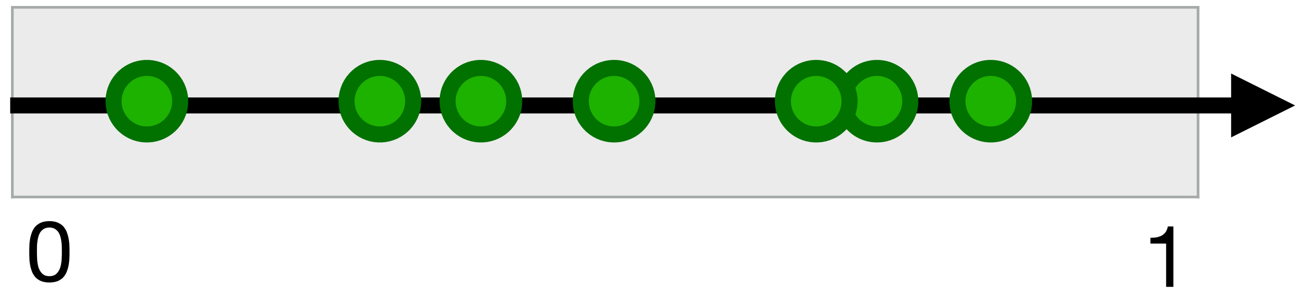


$$\xi \in [0, 1)$$

Sampling a unit domain with uniform random samples

Constant Sampling PDFs

Random 1D



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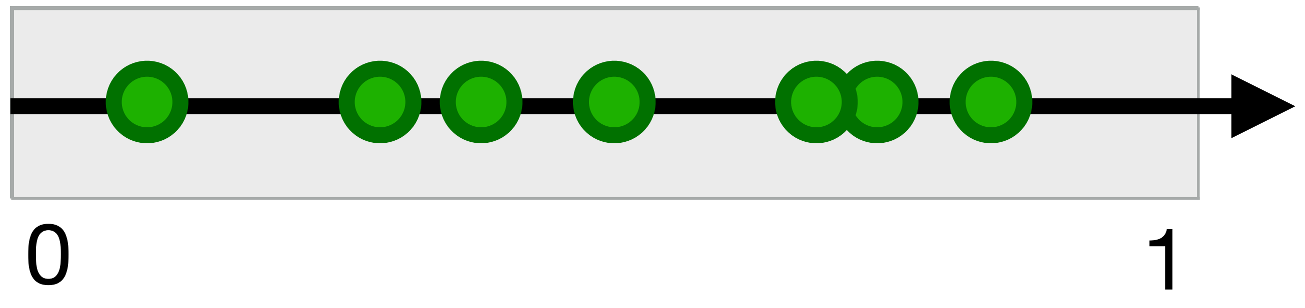

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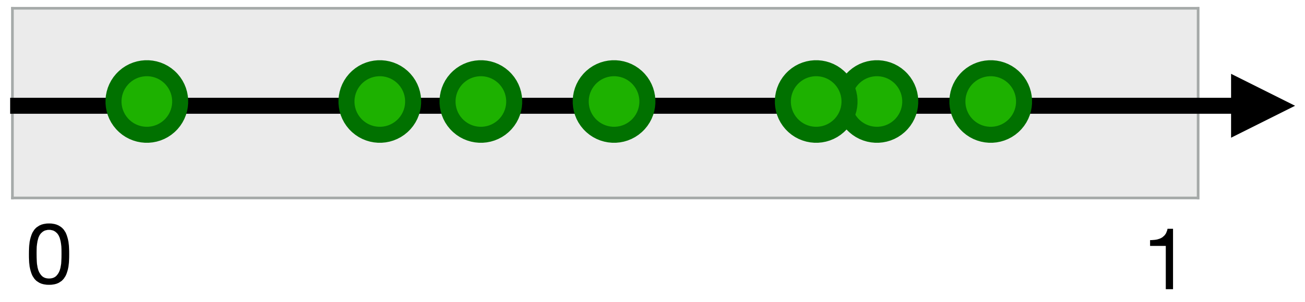

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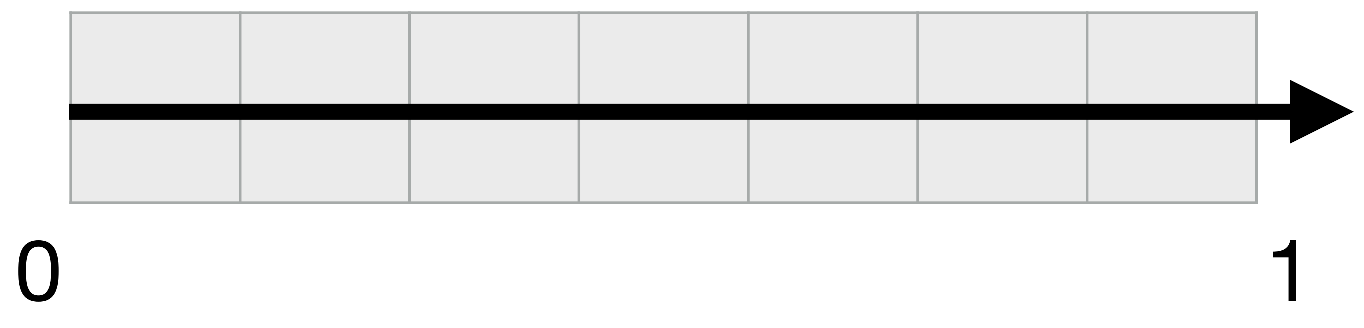
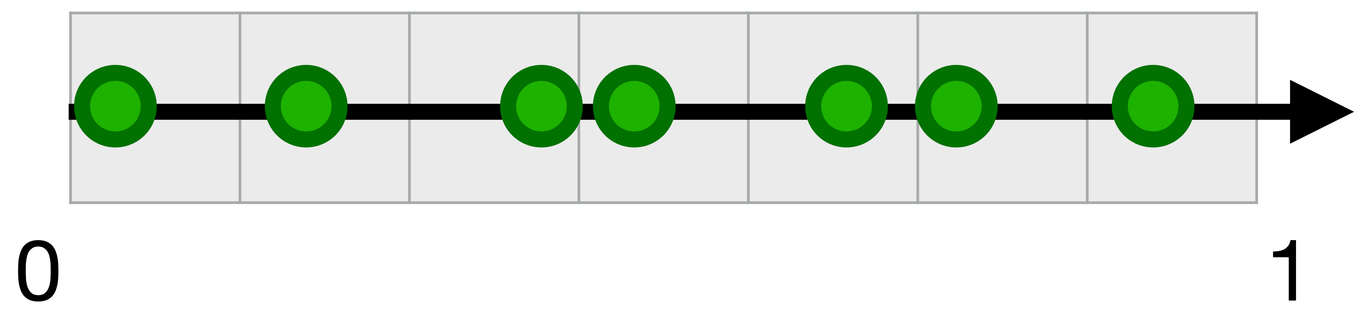
Jittered 1D



Sampling each stratum with uniform random samples

Constant Sampling PDFs

Jittered 1D



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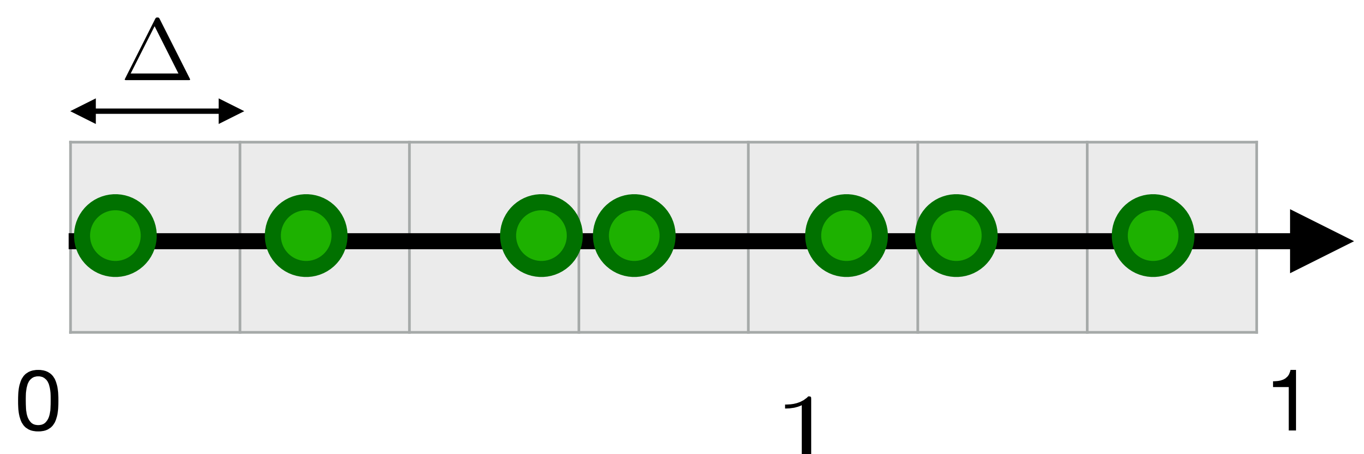
Jittered 1D

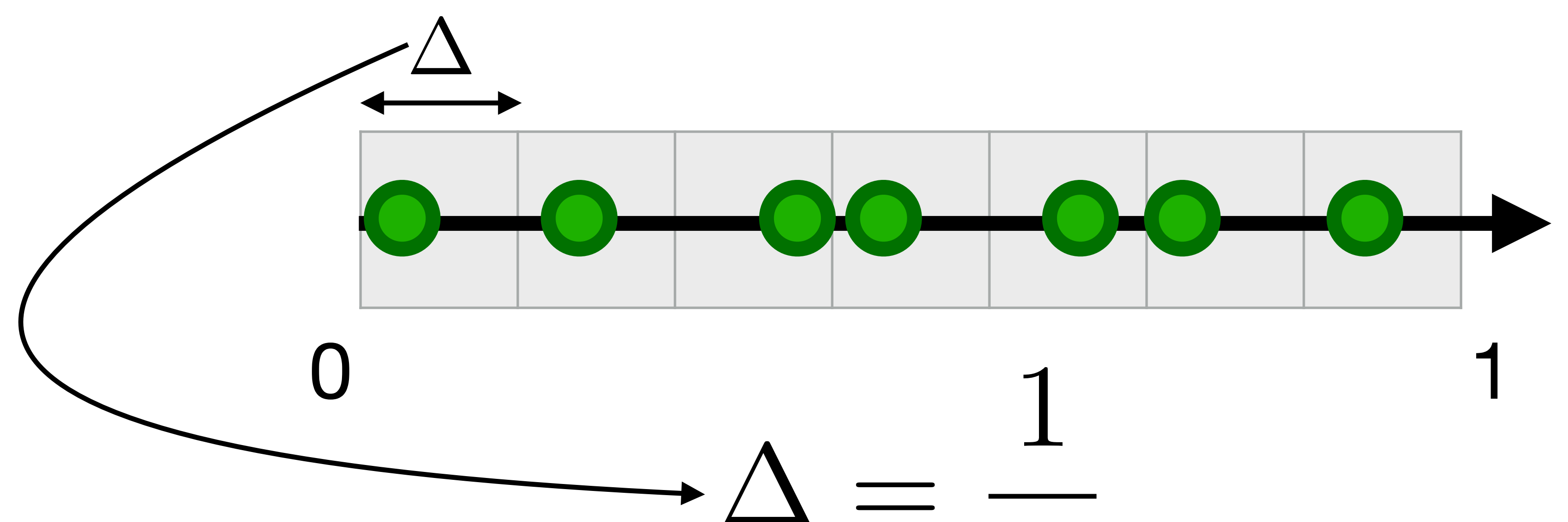


$$\Delta = \frac{1}{N}$$

Sampling each stratum with uniform random samples

Constant Sampling PDFs

Jittered 1D



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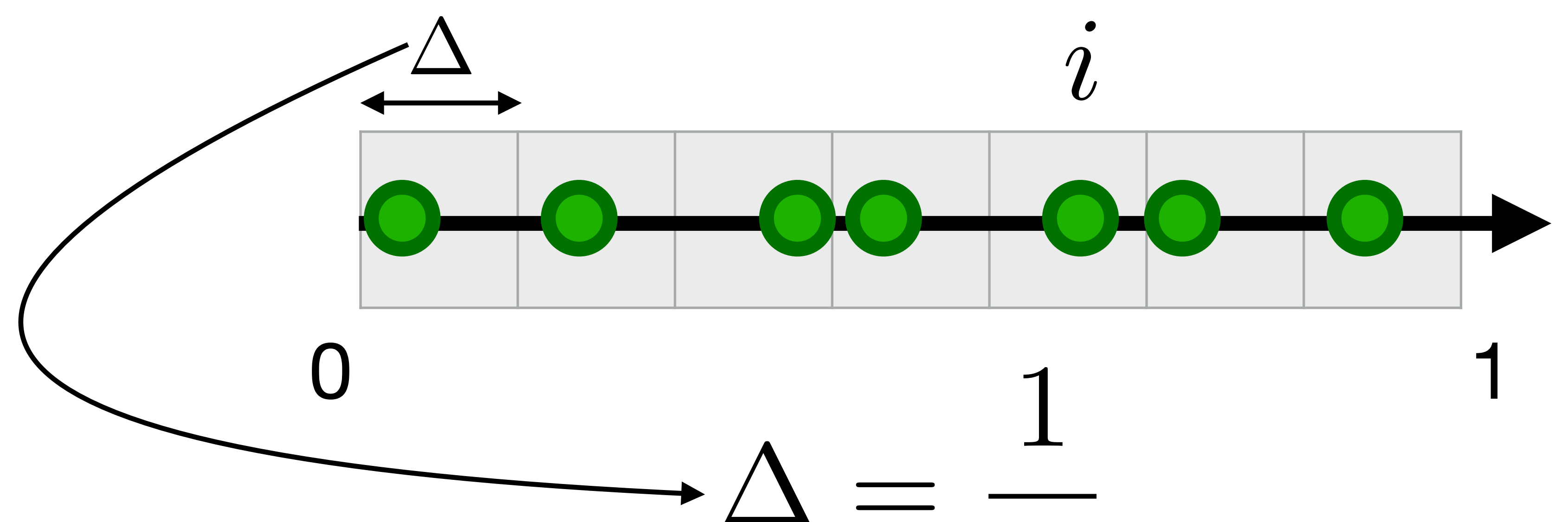
Constant Sampling PDFs

Jittered 1D



$$\Delta = \frac{1}{N}$$

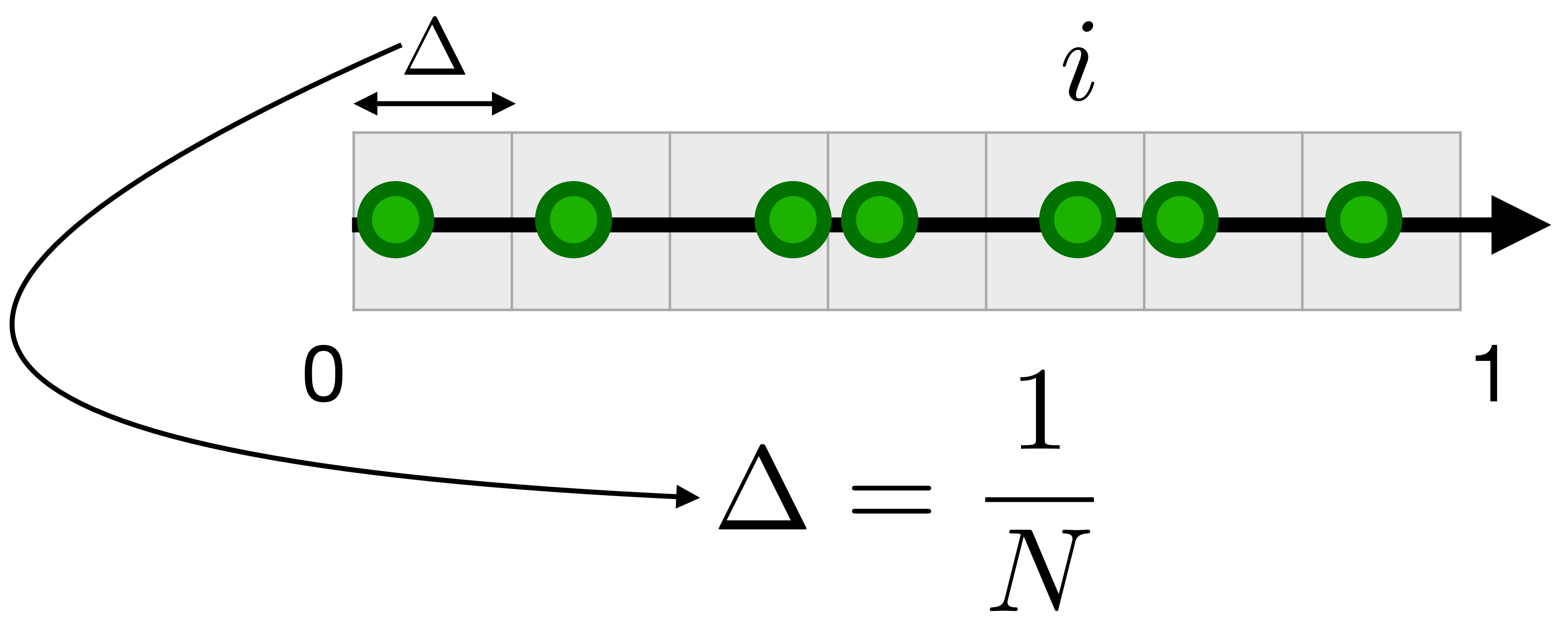
Probability density of generating a sample in an i-th stratum is given by:

$$p(x_i) = ???$$

Sampling each stratum with uniform random samples

Constant Sampling PDFs

Jittered 1D



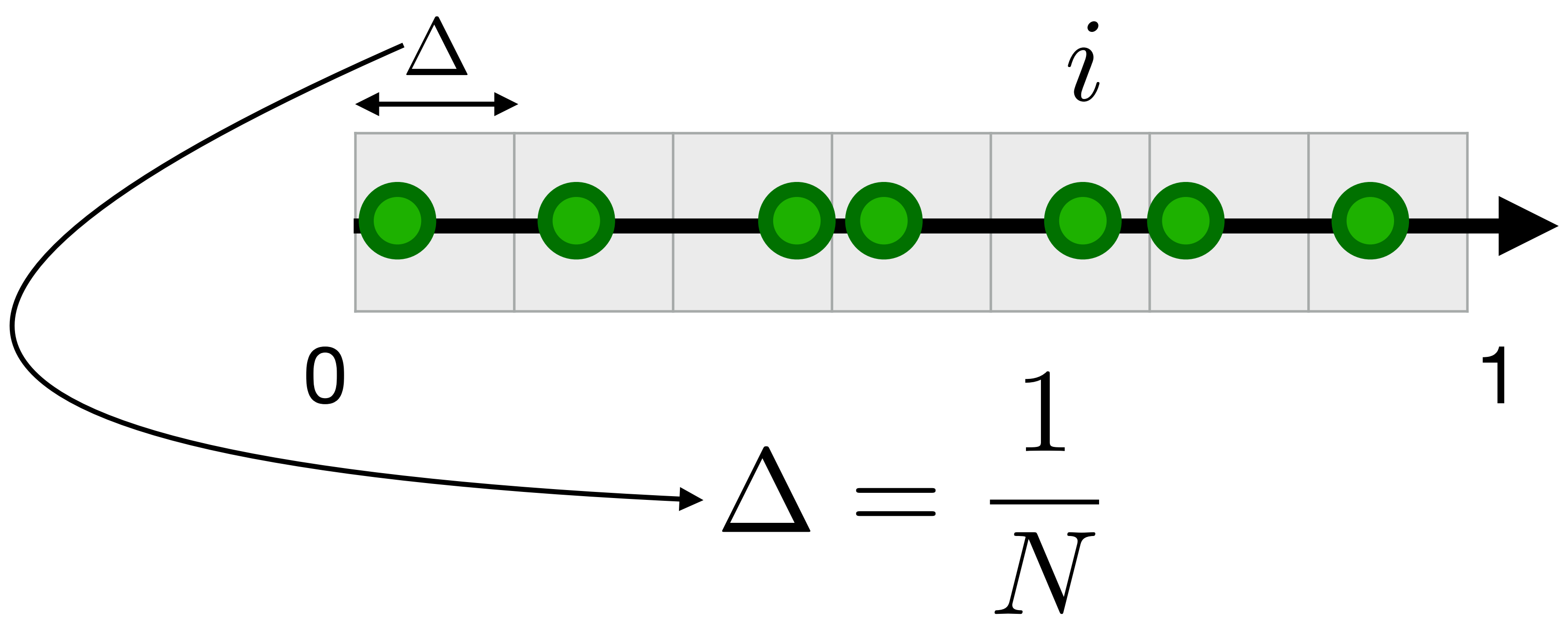
Probability density of generating a sample in an i-th stratum is given by:

$$p(x_i) = \begin{cases} N & x \in [\frac{i}{N}, \frac{i+1}{N}) \\ 0 & \text{otherwise} \end{cases}$$

Sampling each stratum with uniform random samples

Joint PDFs

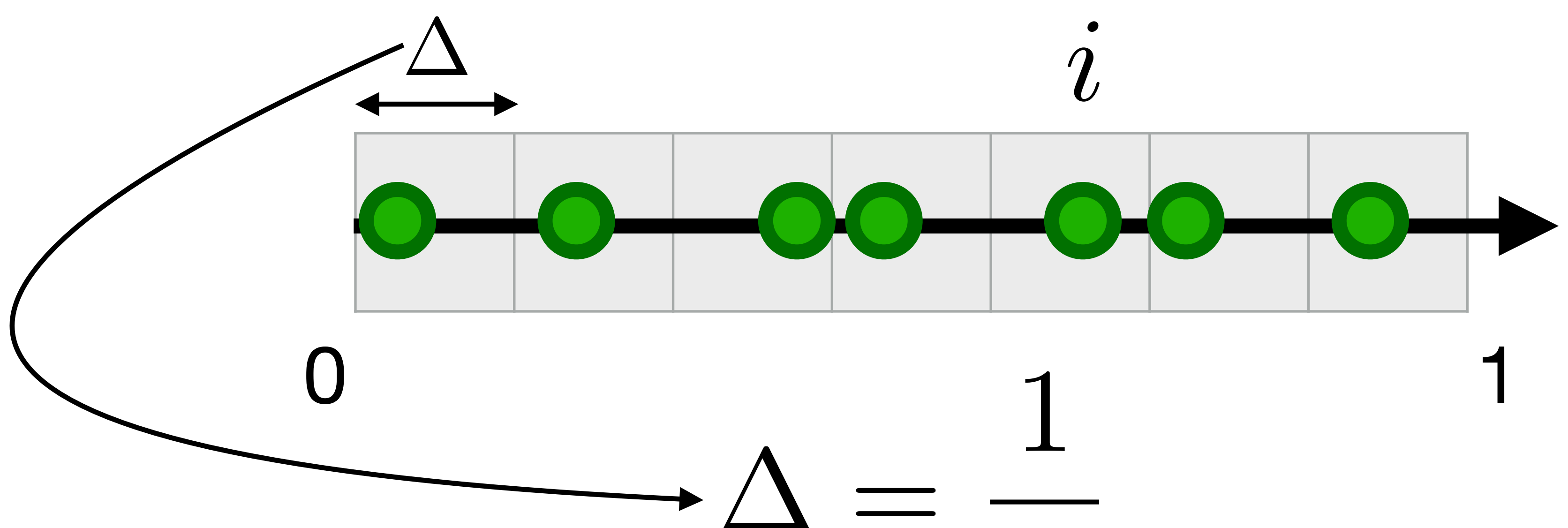
Jittered 1D



First, we divide the domain into equal strata.

Joint PDFs

Jittered 1D



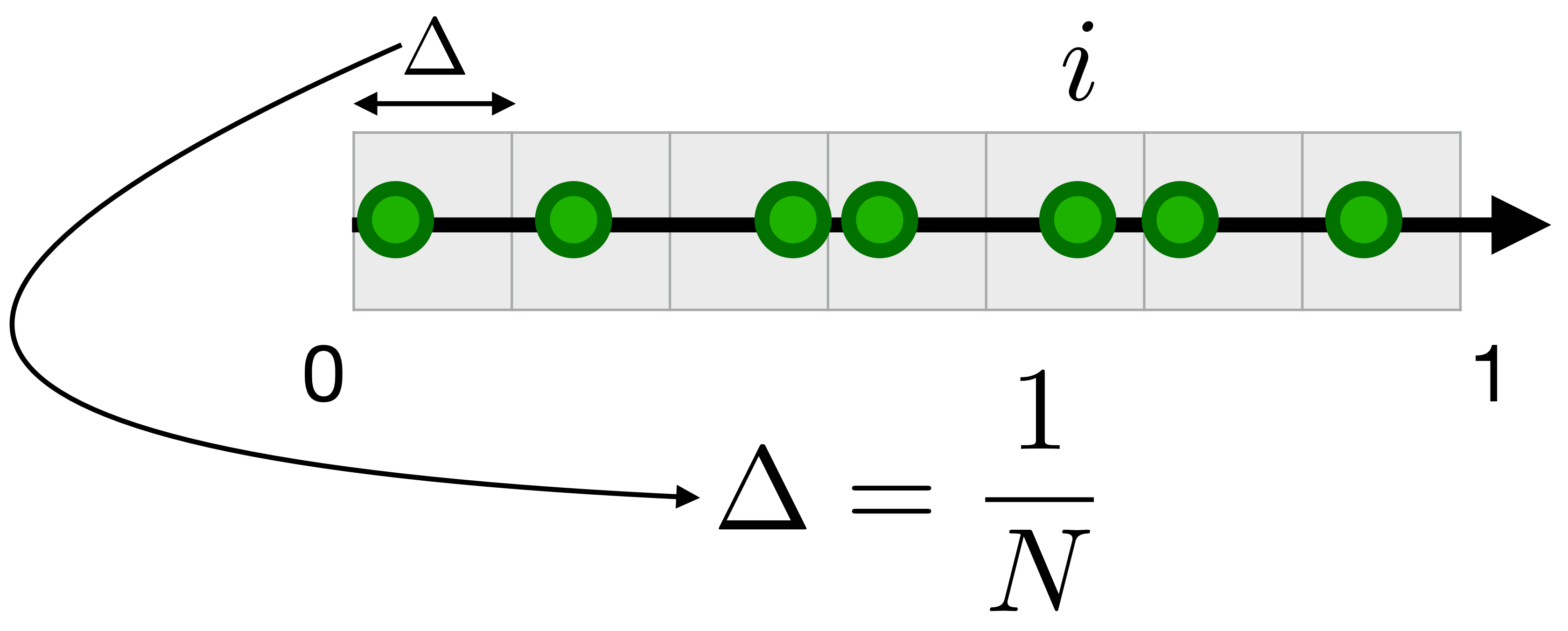
$$\Delta = \frac{1}{N}$$

First, we divide the domain into equal strata.

Second, we sample the domain.

Joint PDFs

Jittered 1D



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This implies that two samples are correlated to each other.

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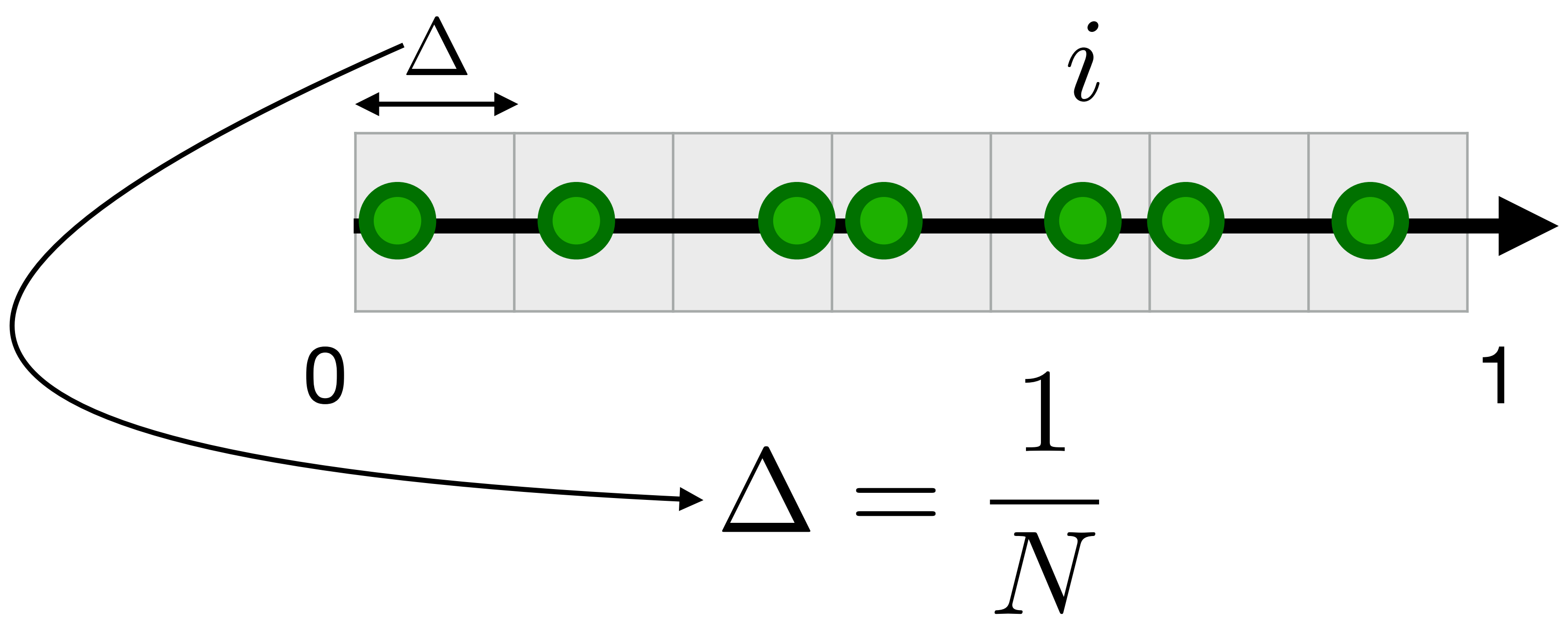

First, we divide the domain into equal strata.

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This implies that two samples are correlated to each other.

For two different strata i and j, what is the joint PDF for jittered sampling ?

$$p(x_i, x_j) = ???$$

Conditional and Marginal PDFs

Joint PDF

For two random variables X_1 and X_2, the joint PDF $p(x_1, x_2)$ is given by:

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$X_1 = x_1$ $\quad$ $p(x_2|x_1)$: conditional density function

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Marginal PDF

$$p(x_1) = \int_{\mathbb{R}} p(x_1, x_2) dx_2$$

$$p(x_2) = \int_{\mathbb{R}} p(x_1, x_2) dx_1$$

We integrate out one of the variable.

Conditional PDF

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)}$$

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}$$

The conditional density function is the density function for x_i given that some particular x_j has been chosen.

Conditional PDF

If both x_1 and x_2 are independent then:

$$p(x_1|x_2) = p(x_1)$$

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Conditional PDF

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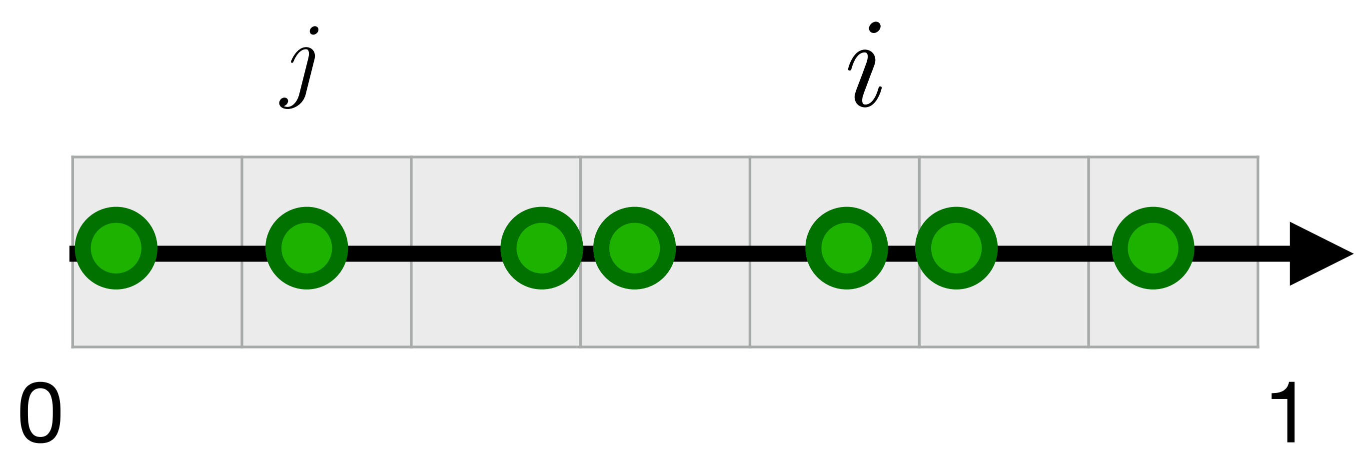
That gives:

$$p(x_1, x_2) = p(x_1)p(x_2)$$

Joint PDF of Jittered 1D Sampling



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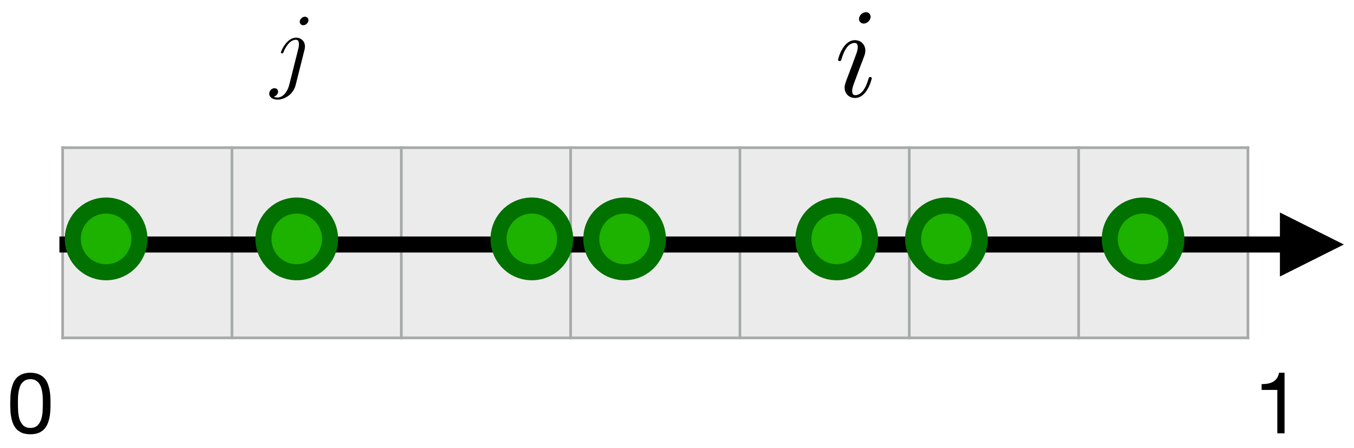

$$p(x_1, x_2) = p(x_1 | x_2) p(x_2)$$

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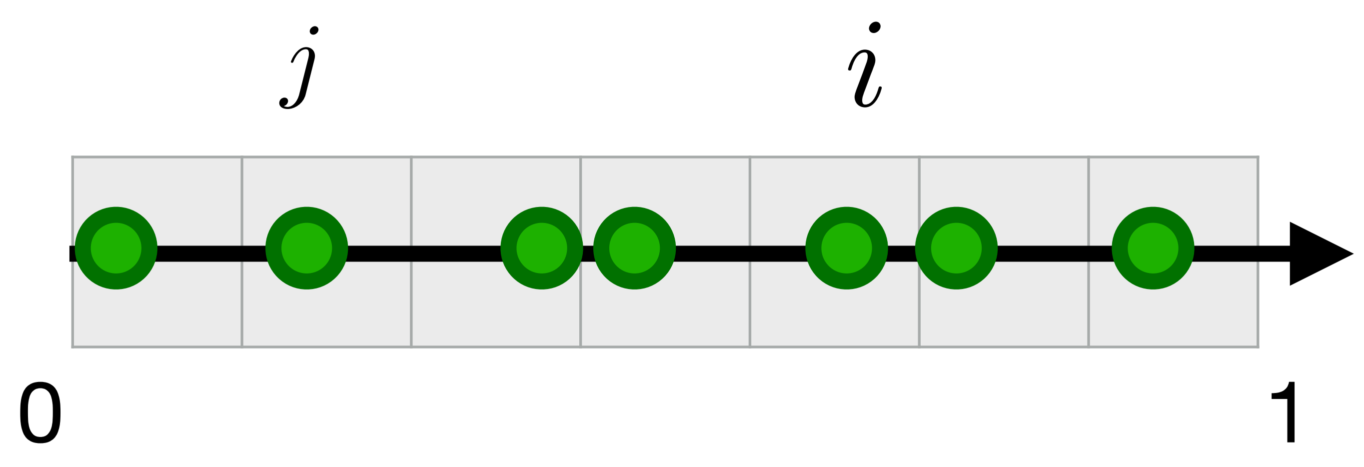

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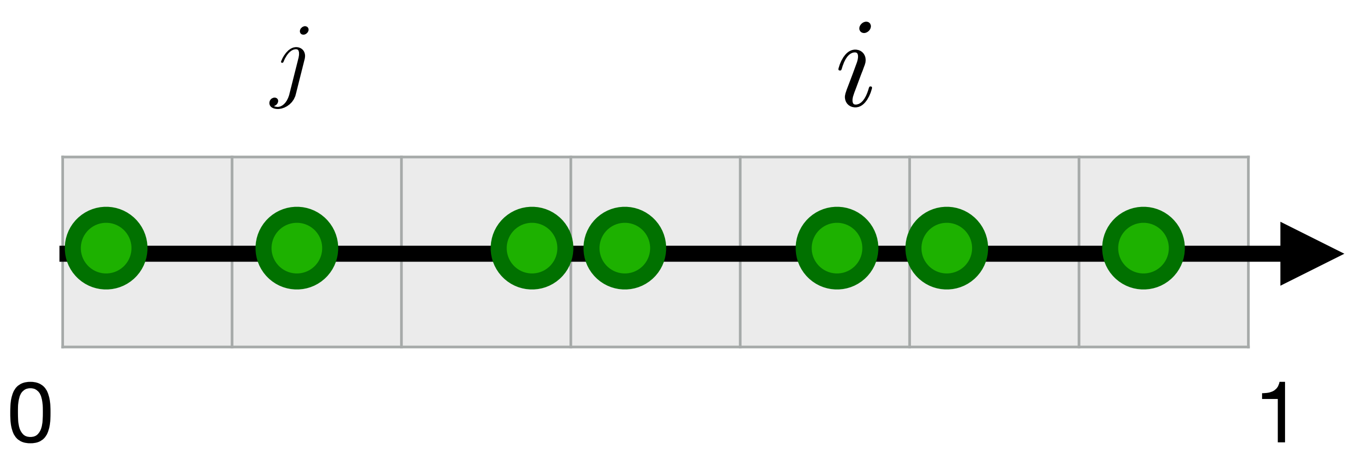
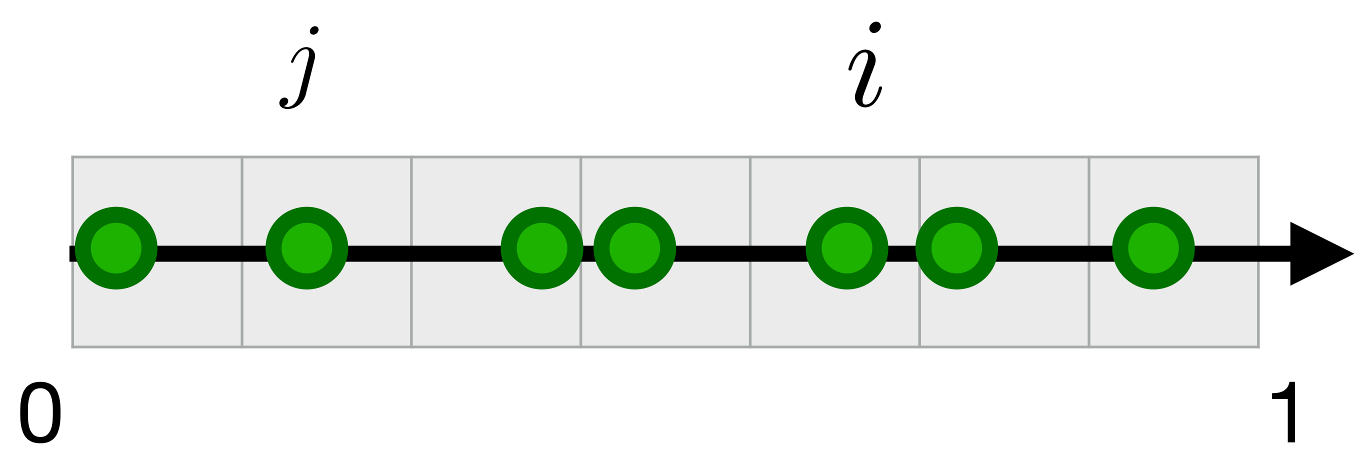


$$p(x_i, x_j) = \begin{cases} p(x_i)p(x_j) & i \neq j \\ 0 & otherwise \end{cases}$$

Joint PDF of Jittered 1D Sampling



$$p(x_i, x_j) = \begin{cases} p(x_i)p(x_j) & i \neq j \\ 0 & otherwise \end{cases}$$

$$p(x_i, x_j) = \begin{cases} N^2 & i \neq j \\ 0 & otherwise \end{cases}$$

Since, $p(x_i) = N$

Questions ?

Image rendered using PBRT

Questions ?

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Expected Value

Expected value

- Expected value: average value of the variable

$$E[X] = \sum_{i=1}^{N} x_i p_i$$

- example: rolling a die

$$E[X] =$$

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- example: rolling a die

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

Expected value

- Properties:

$$E[X+Y] = E[X] + E[Y]$$

Expected value

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$$E[X + Y] = E[X] + E[Y]$$
$$E[X + c] = E[X] + c$$

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Law of large numbers

- By taking *infinitely* many samples, the error between the estimate and the expected value is *statistically* zero
  - the estimate will converge to the right value

$$\text{probability}\left[E[x] = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x_i\right] = 1$$

Variance

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- Variance: how much different from the average

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Variance

- Variance: how much different from the average

$$\begin{aligned}\sigma^2[X] &= E[(X - E[X])^2] \\ &= E[X^2 + E[X]^2 - 2XE[X]] \\ &= E[X^2] + E[E[X]^2] - 2E[X]E[E[X]]] \\ &= E[X^2] + E[X]^2 - 2E[X]^2 \\ &= E[X^2] - E[X]^2\end{aligned}$$

$$\boxed{\sigma^2[X] = E[X^2] - E[X]^2}$$

Variance

- example: Rolling a die
  - variance:

$$\sigma^2[X] = \ldots =$$

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Variance

- example: Rolling a die
  - variance:

$$\boxed{\sigma^2[X] = E[X^2] - E[X]^2}$$

$$\sigma^2[X] = \ldots =$$

$$E[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5$$

Variance

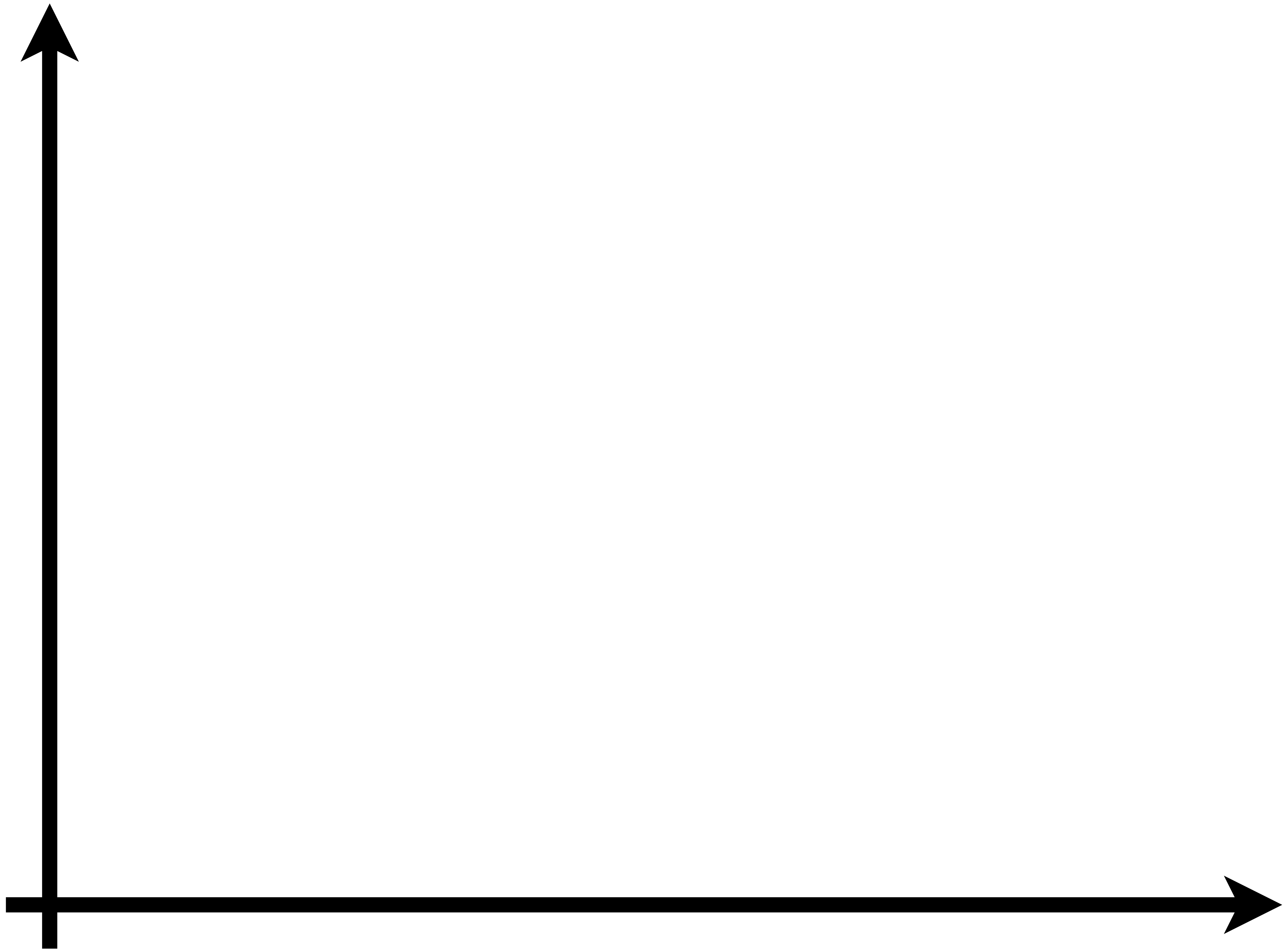
- example: Rolling a die
  - variance:

$$\boxed{\sigma^2[X] = E[X^2] - E[X]^2}$$

$$\sigma^2[X] = \ldots = 2.917$$

Monte Carlo Integration

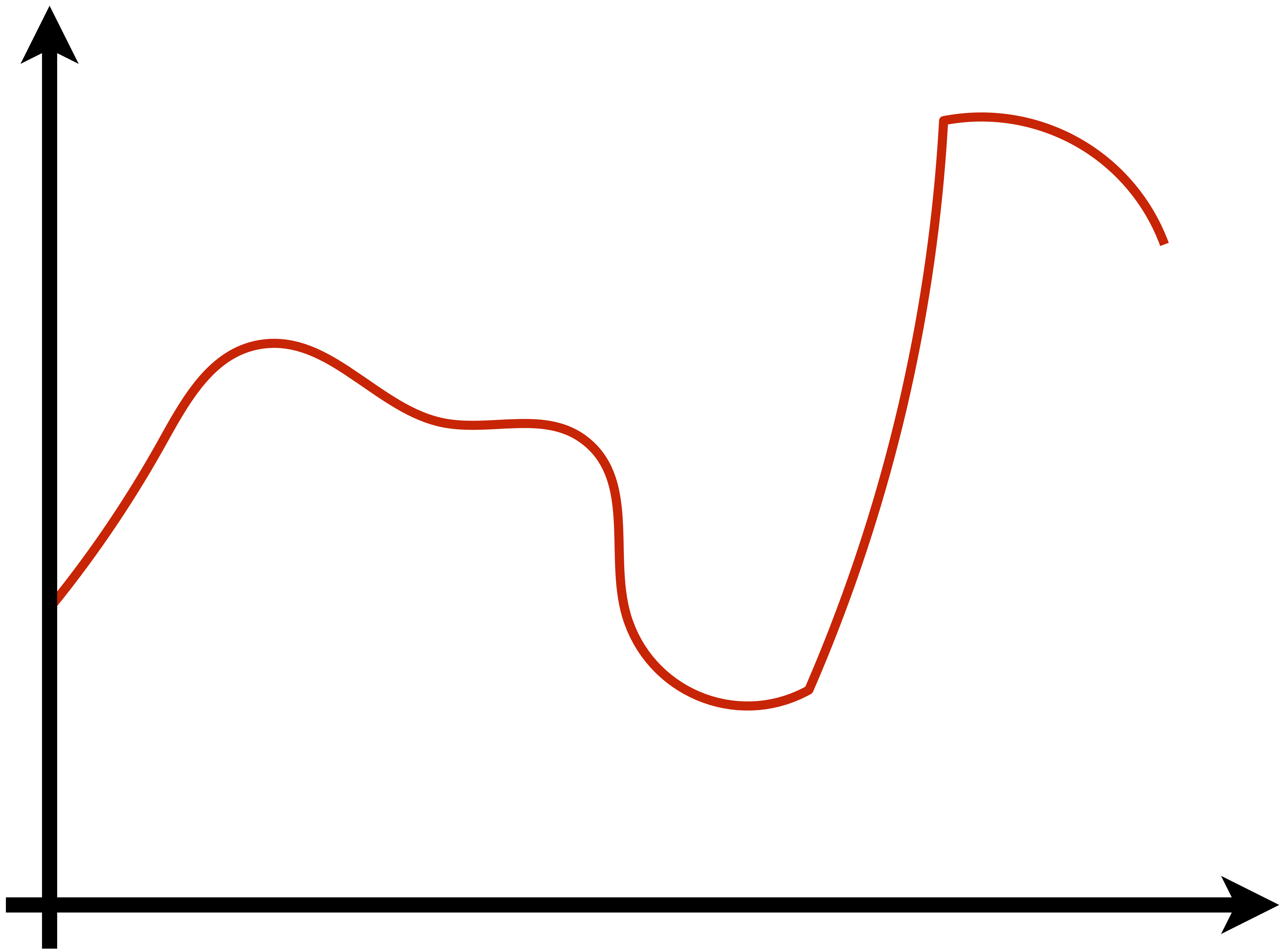
$$I = \int_D f(x)\,\mathrm{d}x$$



Slide after Wojciech Jarosz

Monte Carlo Integration

$$I = \int_D f(x)\, \mathrm{d}x$$



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Monte Carlo Integration

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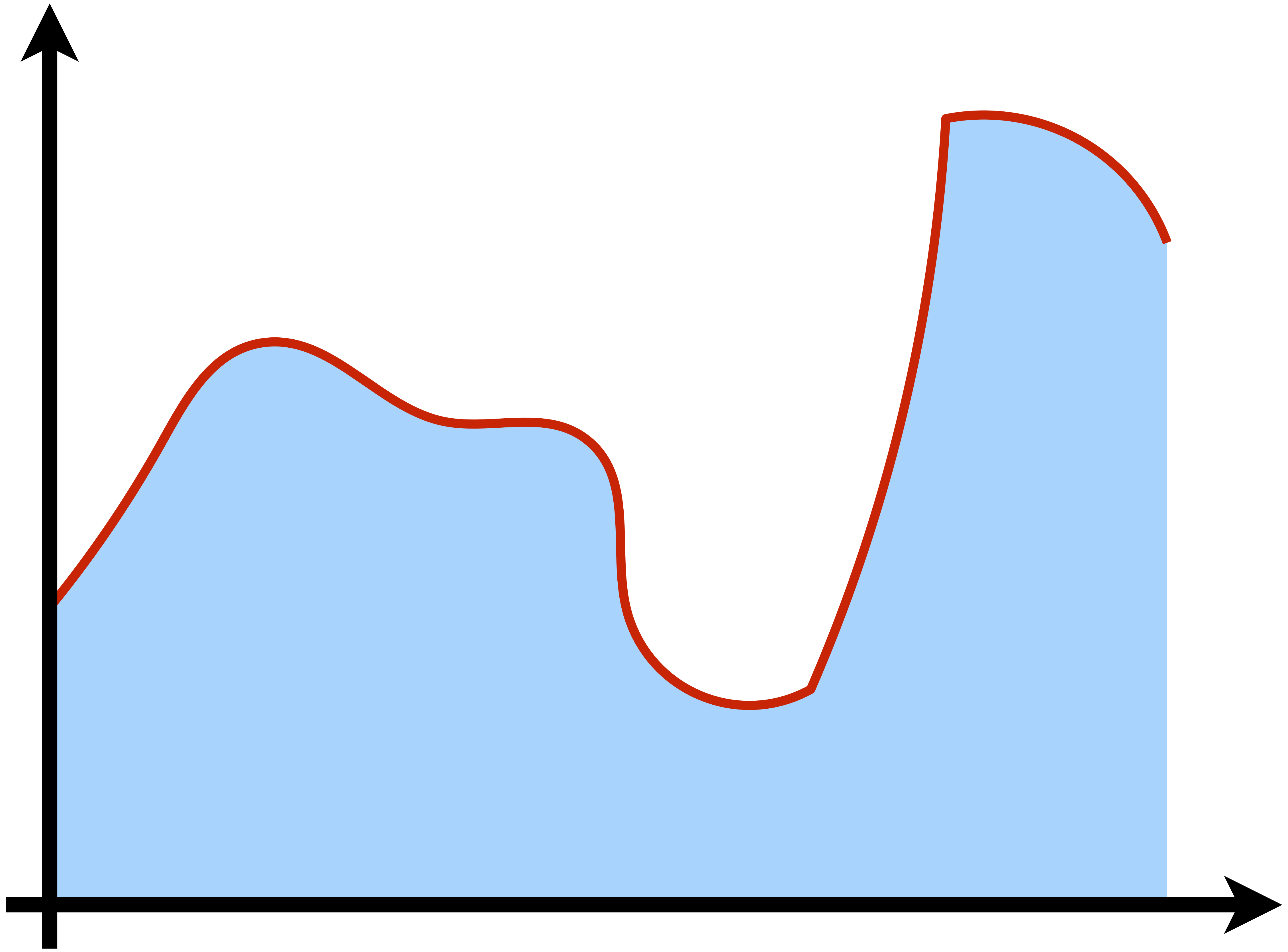

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Image rendered using PBRT