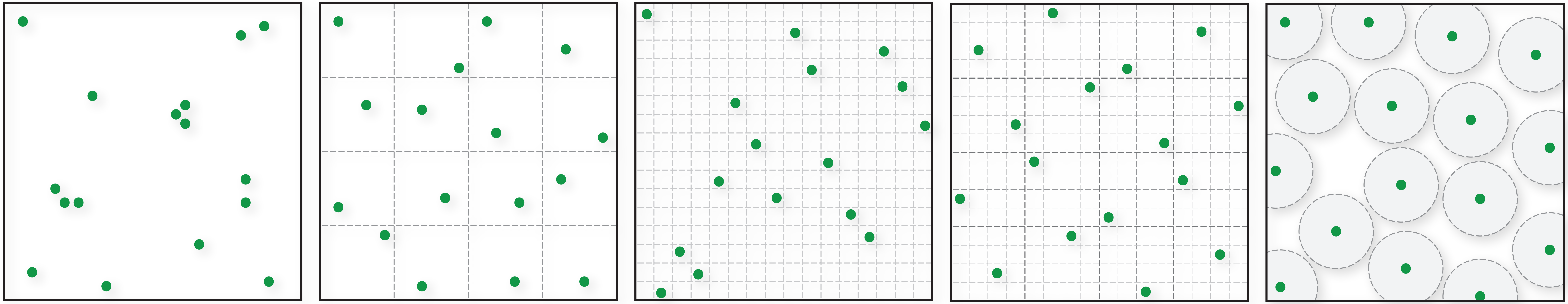
ADVANCED SAMPLING



Philipp Slusallek *Karol Myszkowski*
Gurprit Singh

UNIVERSITÄT DES SAARLANDES

Part of Siggraph 2016 Course

Fourier Analysis of Numerical Integration in Monte Carlo Rendering

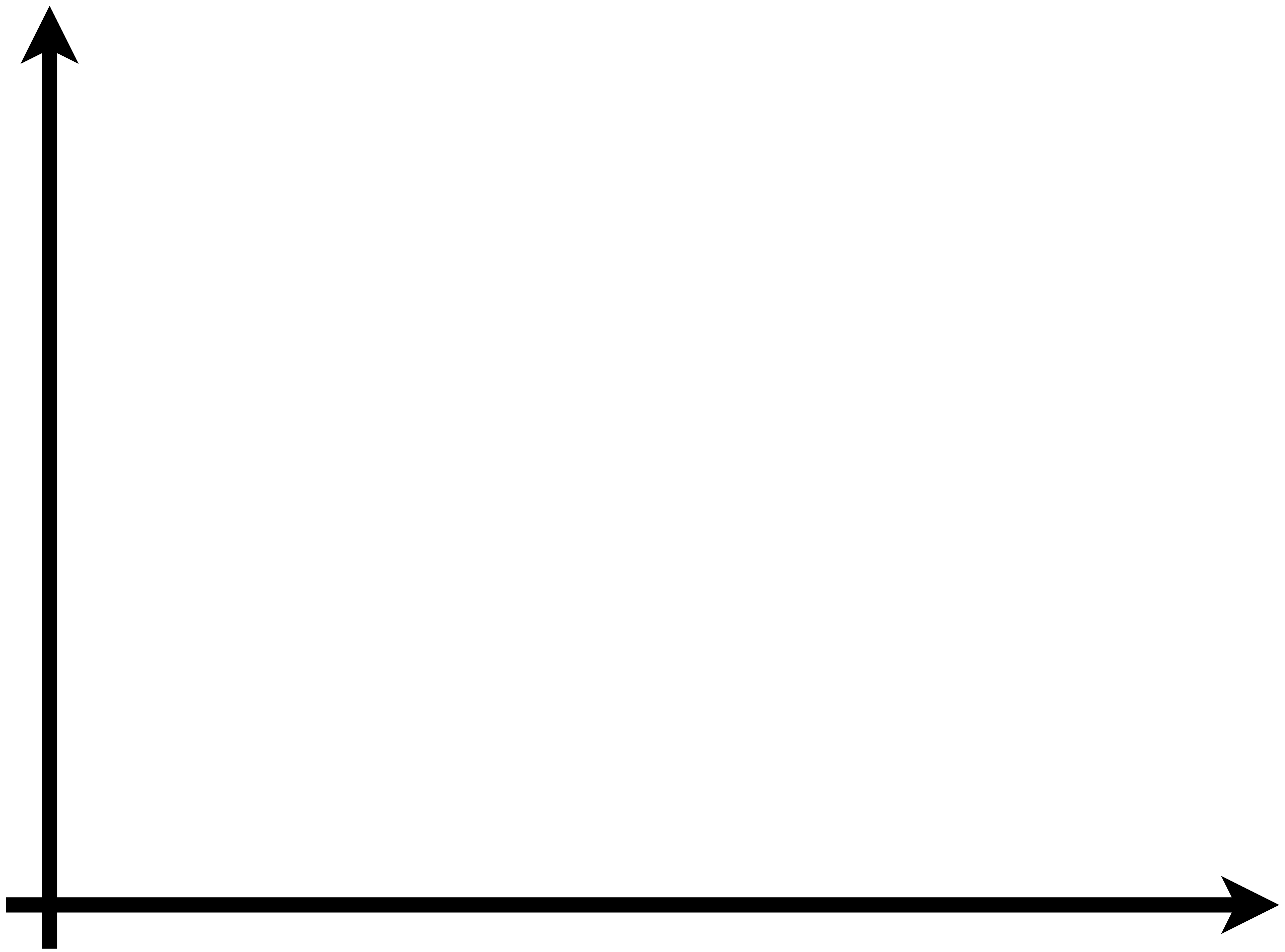
Kartic Subr Gurprit Singh *Wojciech Jarosz

Render the Possibilities SIGGRAPH2016

*First part slides are from Wojciech Jarosz

Recall: Monte Carlo Integration

$$I = \int_D f(x) \, \mathrm{d}x$$



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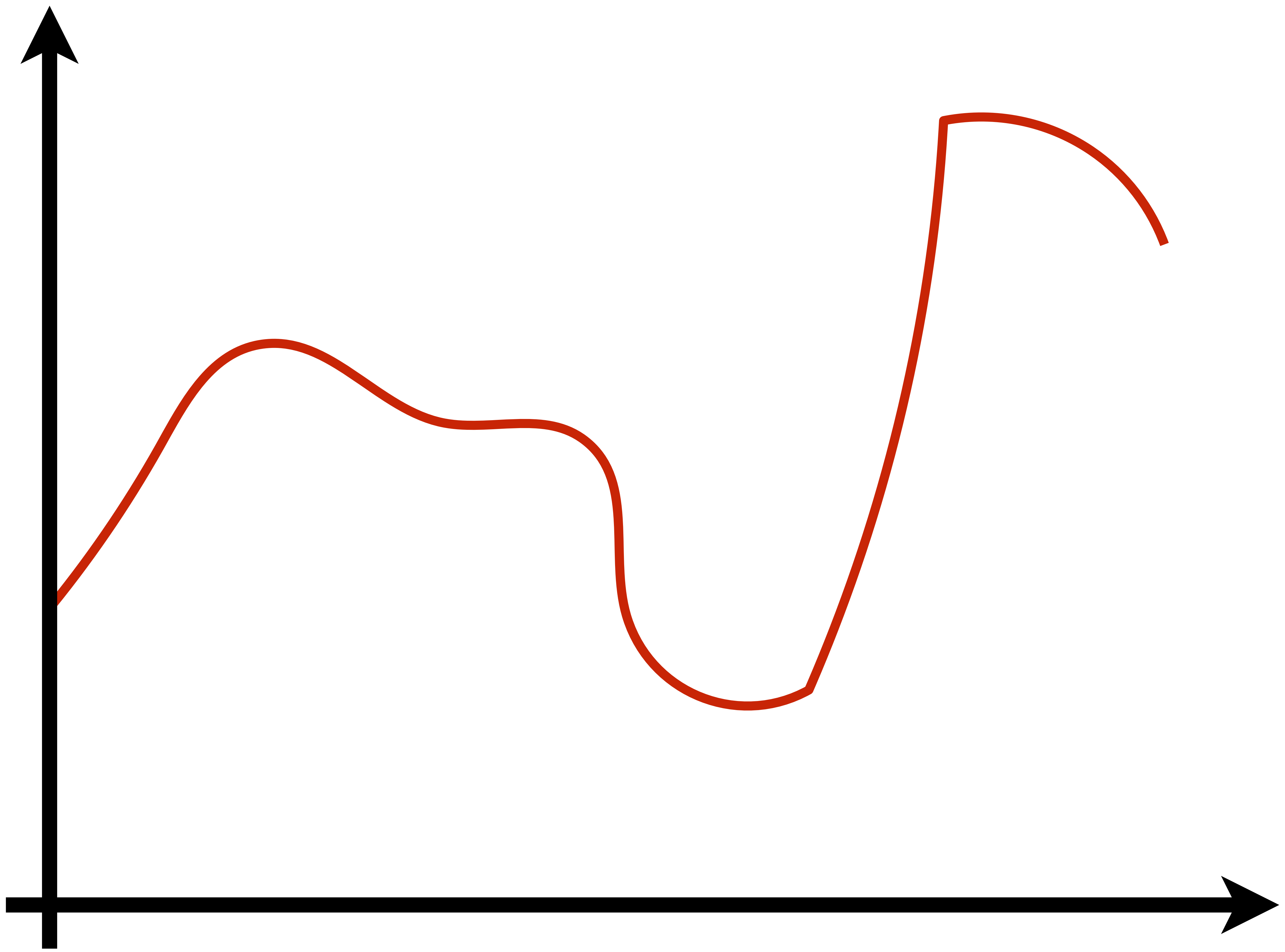

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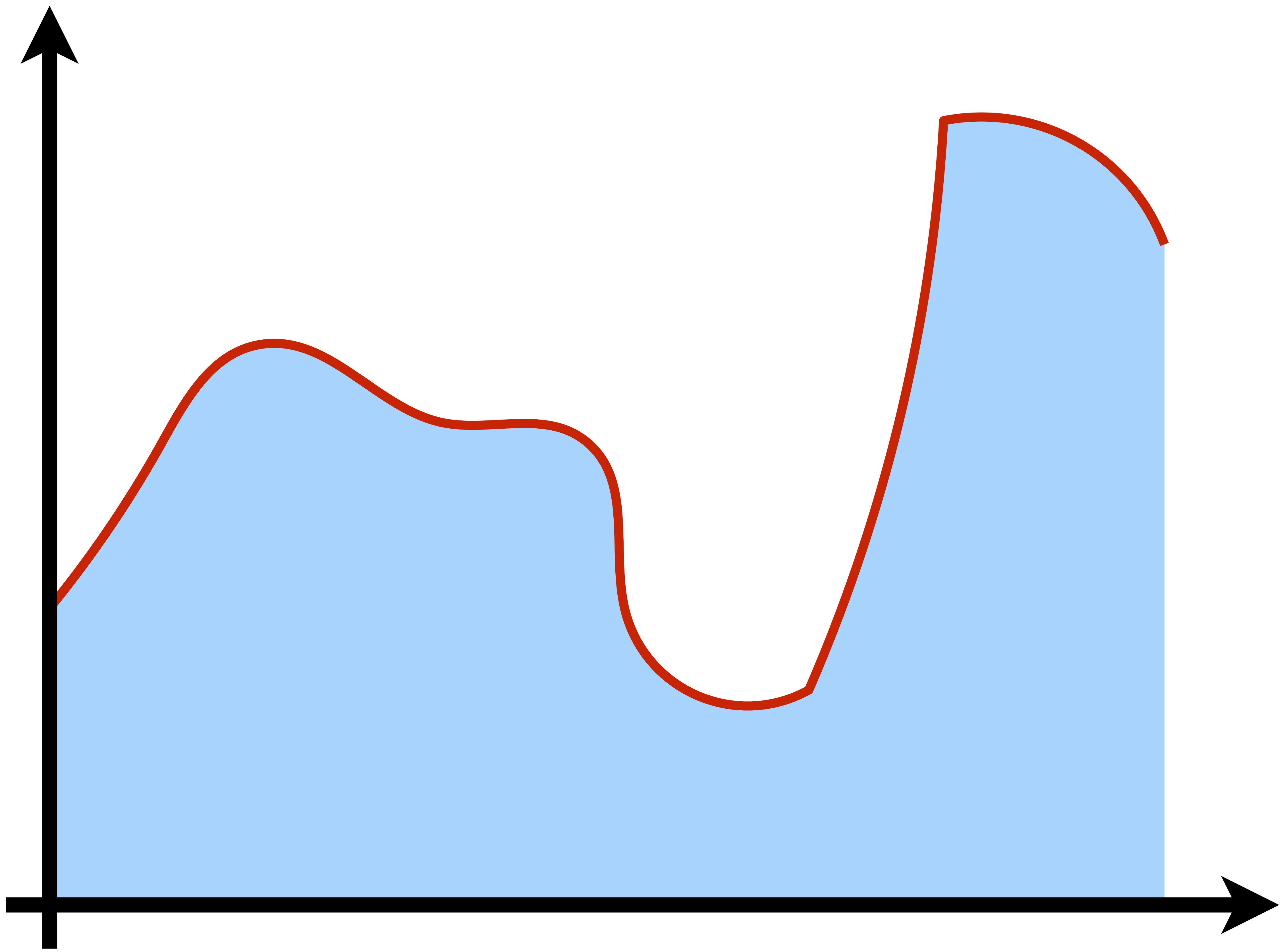
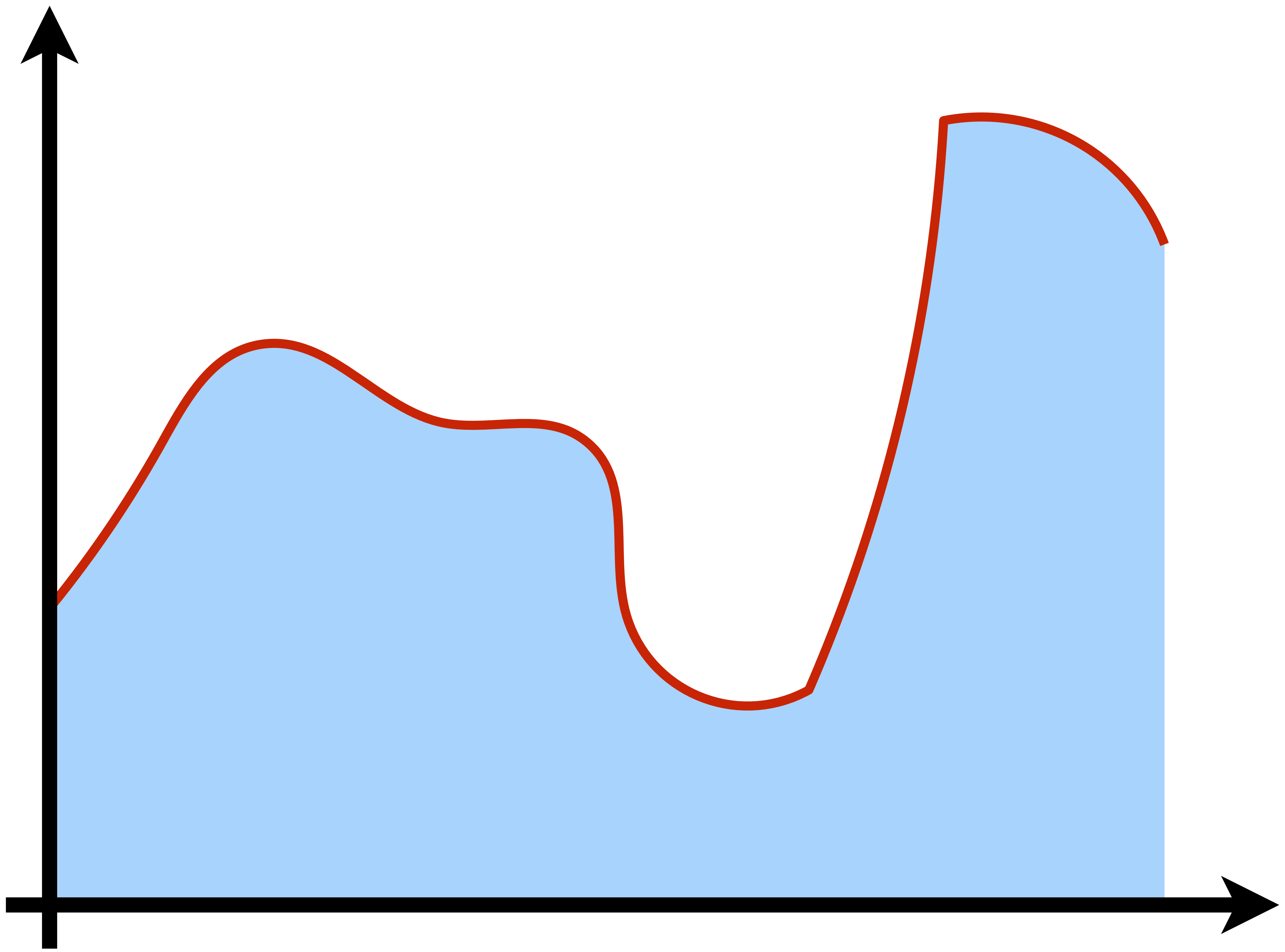


Recall: Monte Carlo Integration

$$I = \int_D f(x)\,\mathrm{d}x$$

$$\approx \int_D f(x)\,\mathbf{S}(x)\,\mathrm{d}x$$



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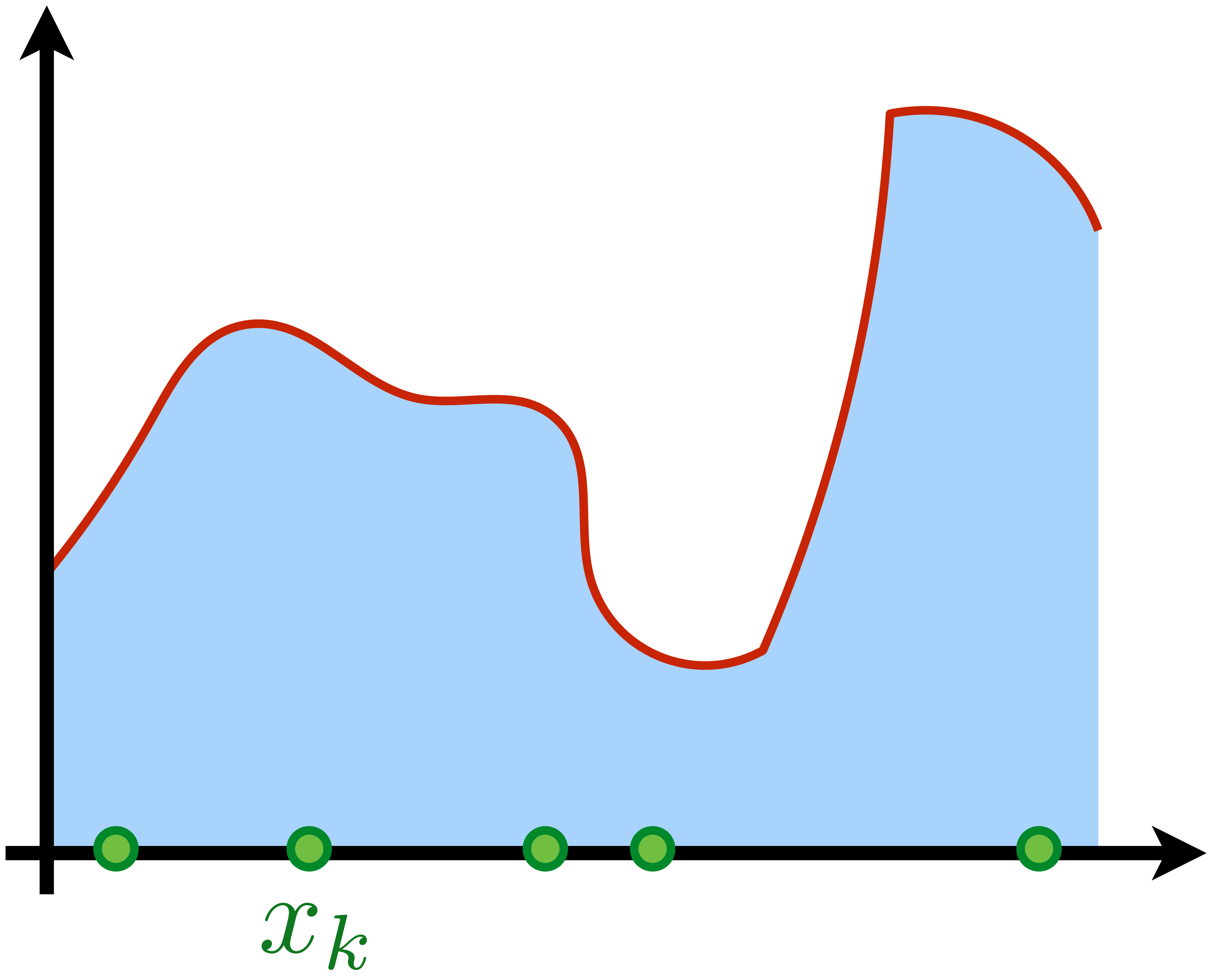

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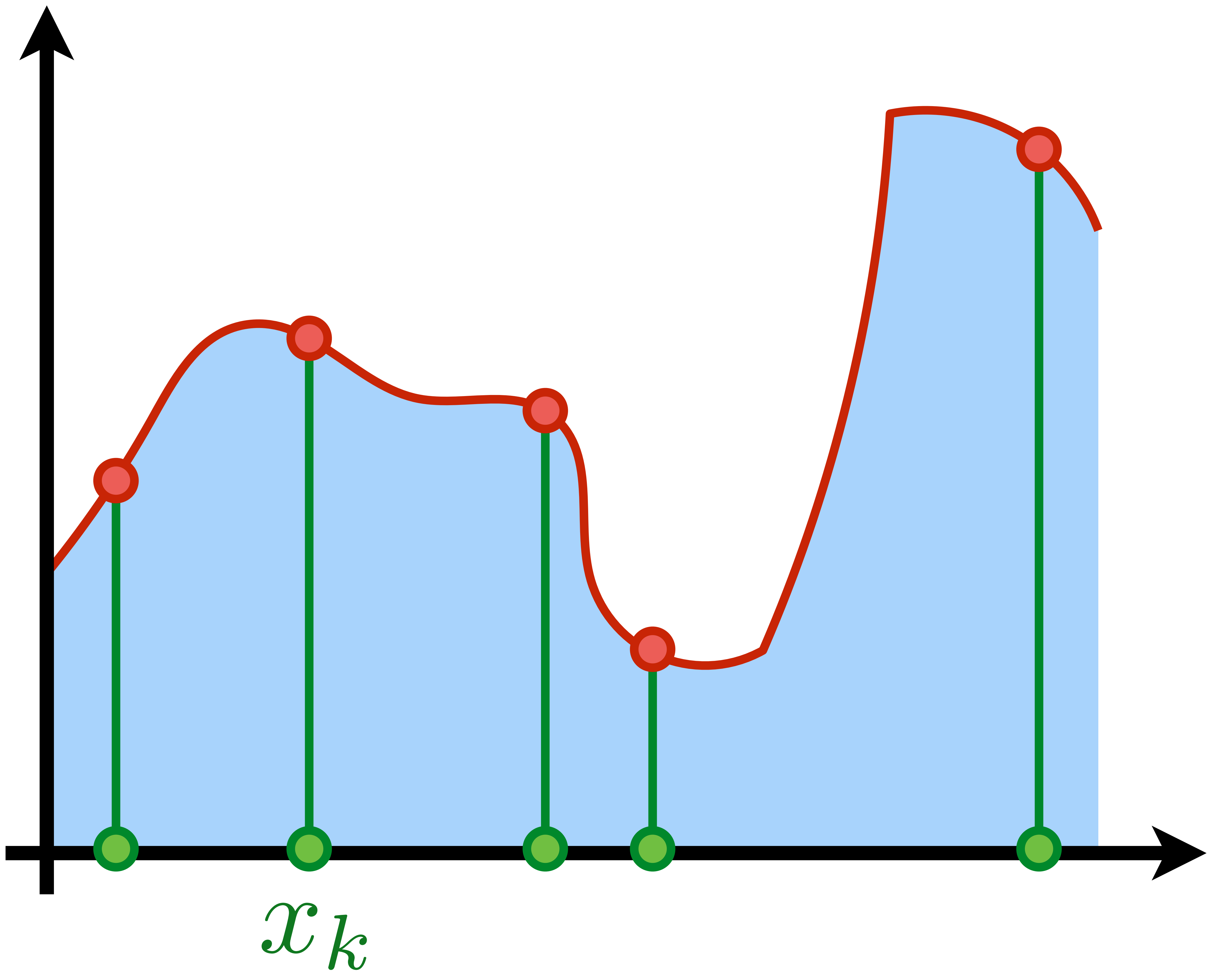

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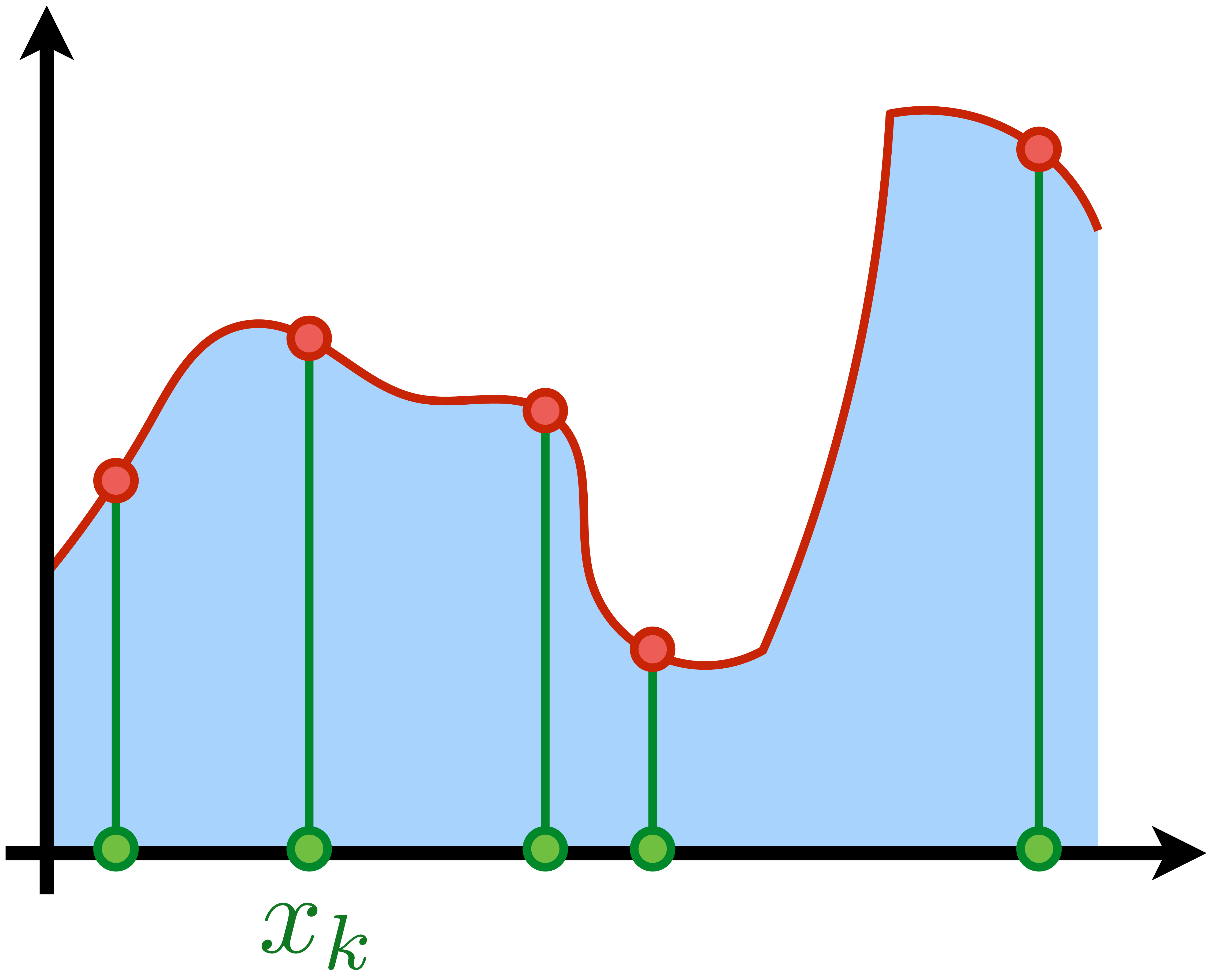
How to generate the locations x_k?



Independent Random Sampling

```
for (int k = 0; k < num; k++)
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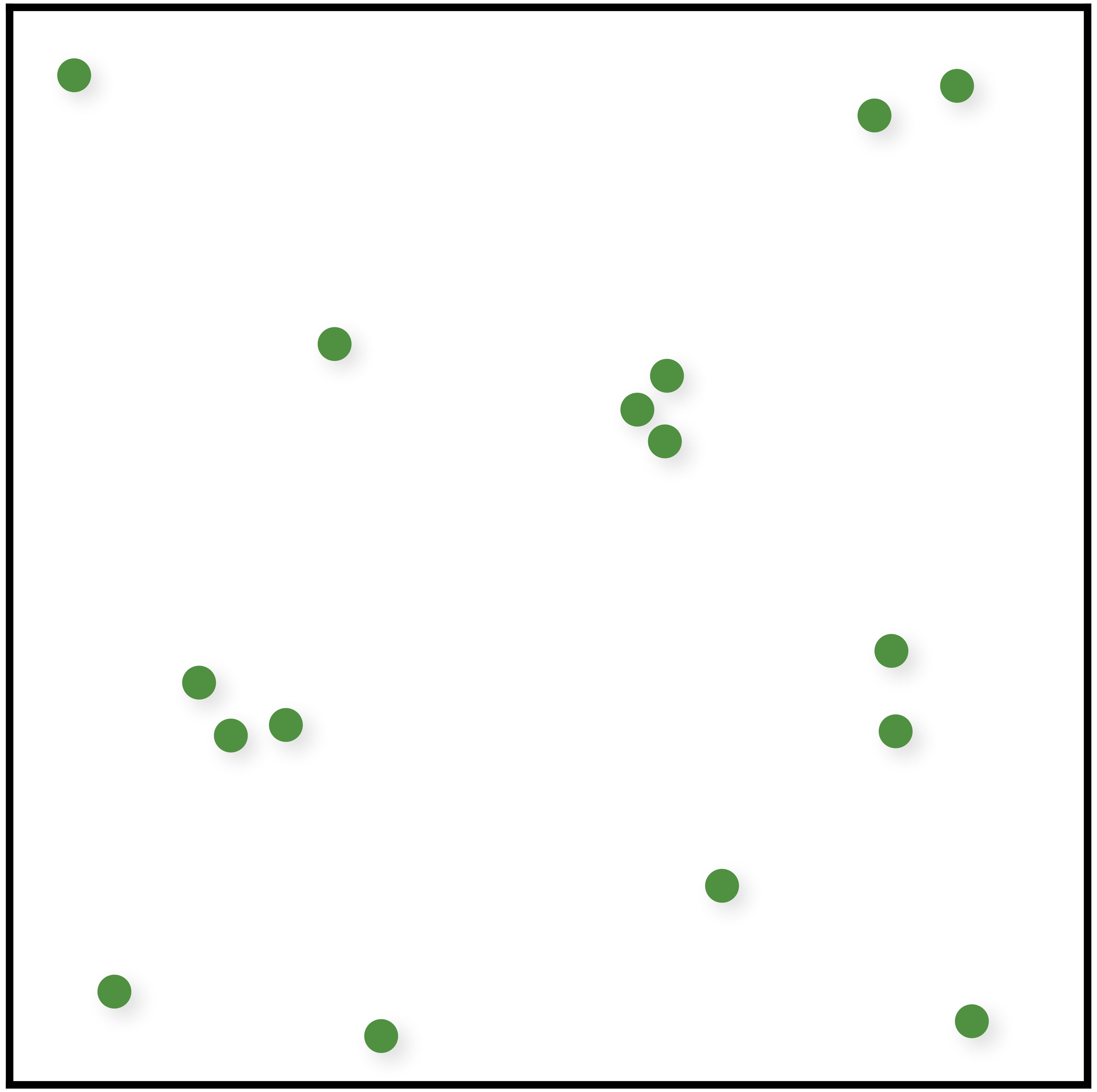

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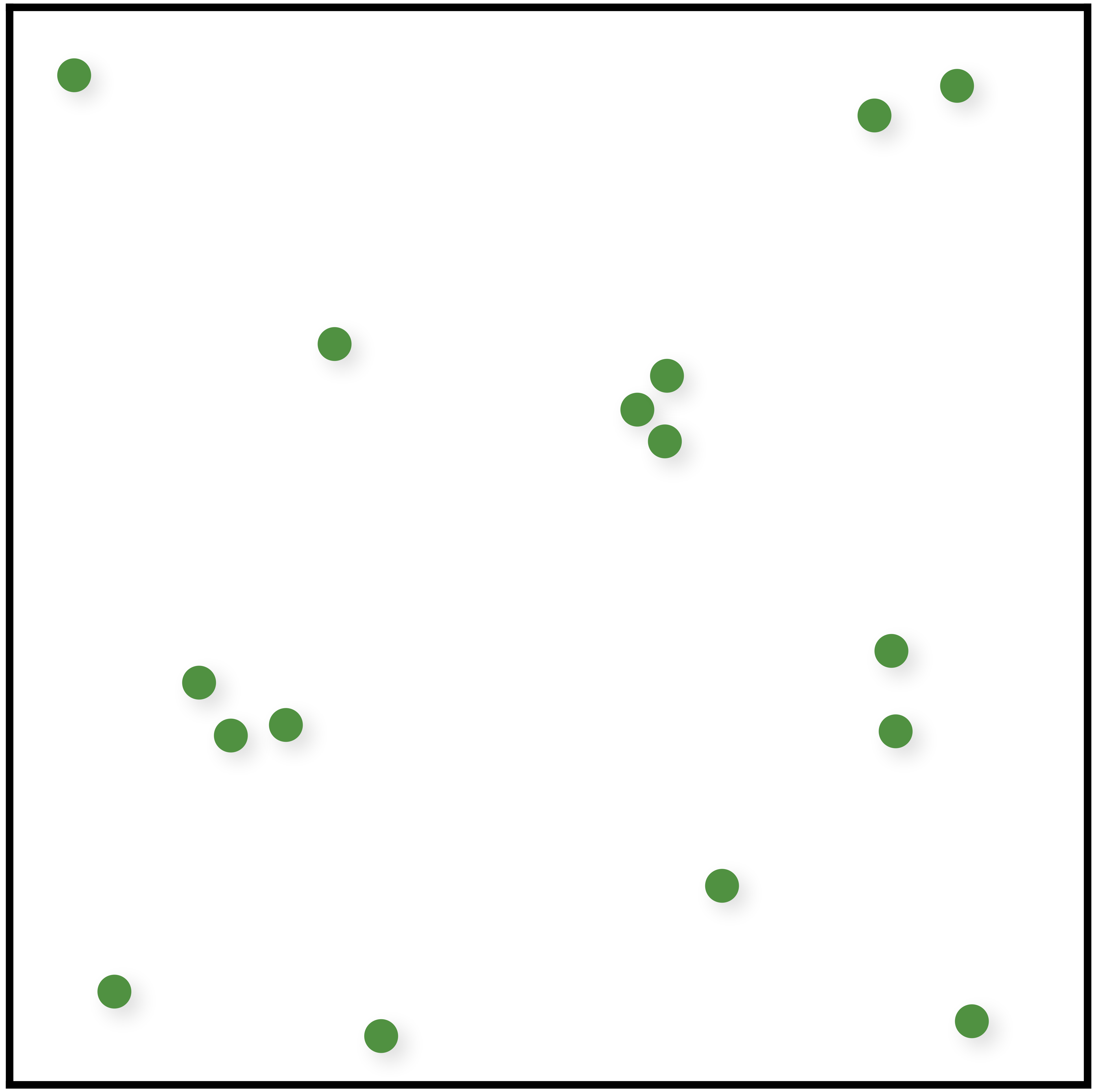
✔ Trivially extends to higher dimensions



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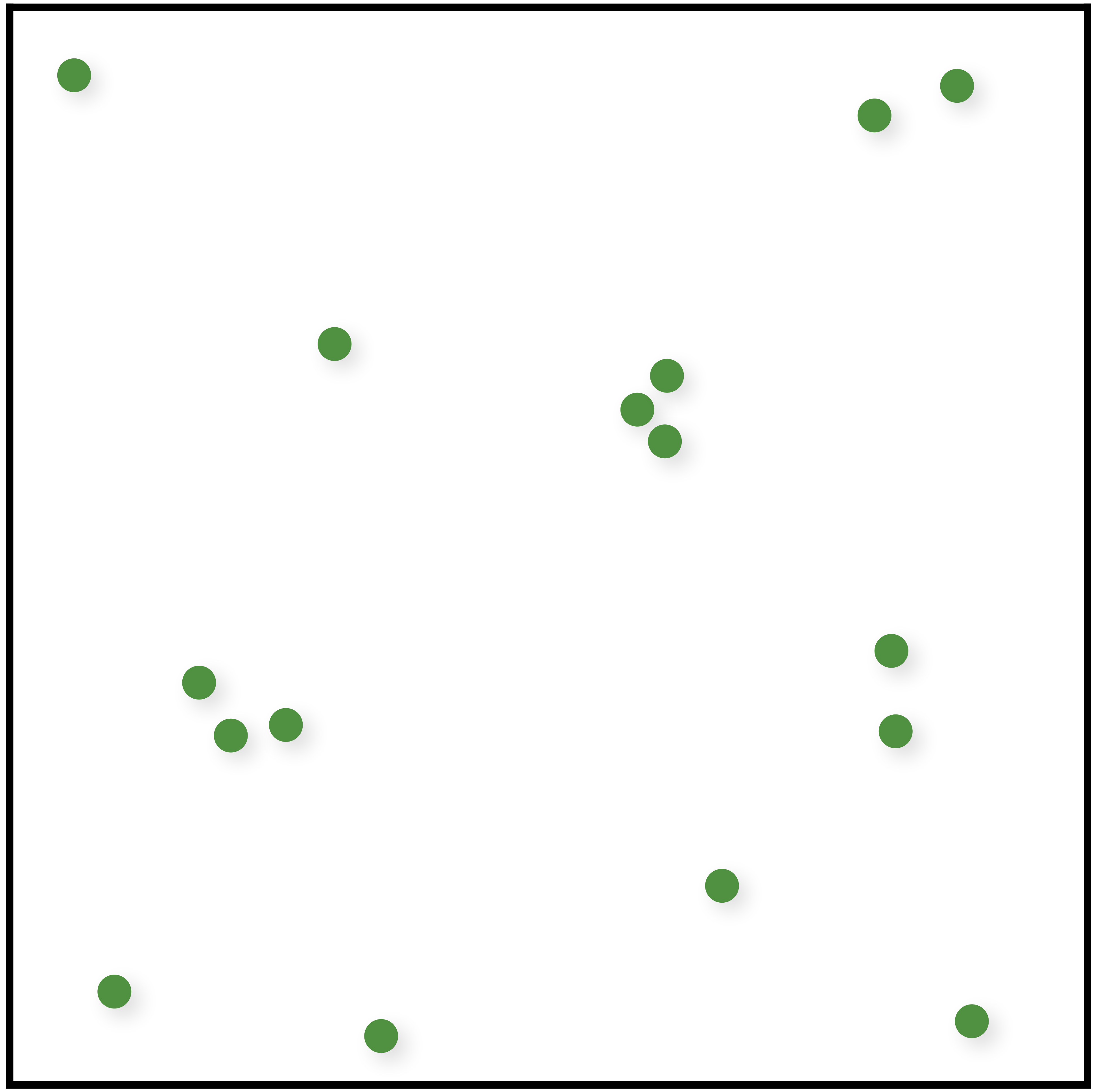

✔ Trivially extends to higher dimensions

✔ Trivially progressive and memory-less

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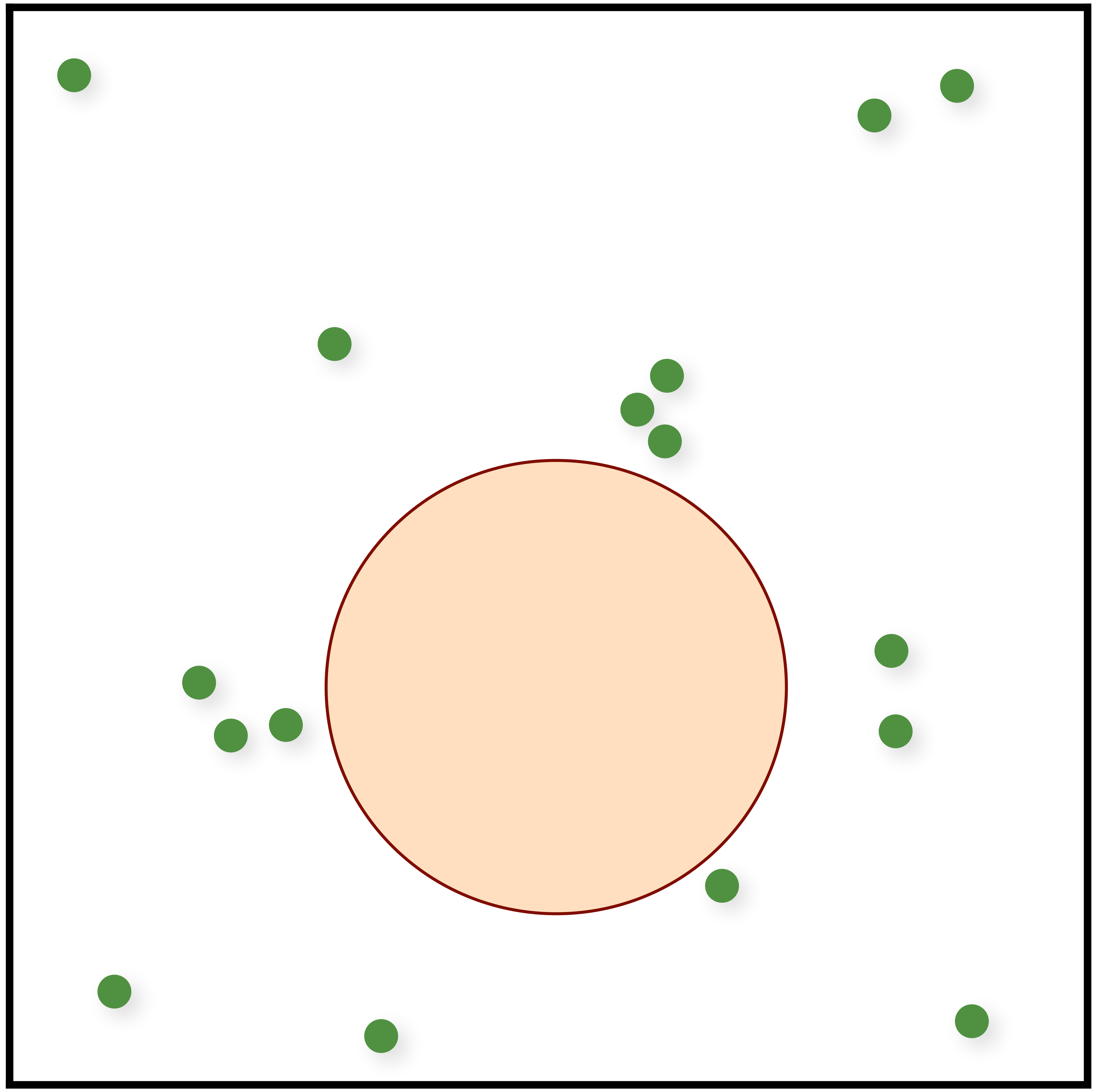

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✘ Big gaps

Independent Random Sampling

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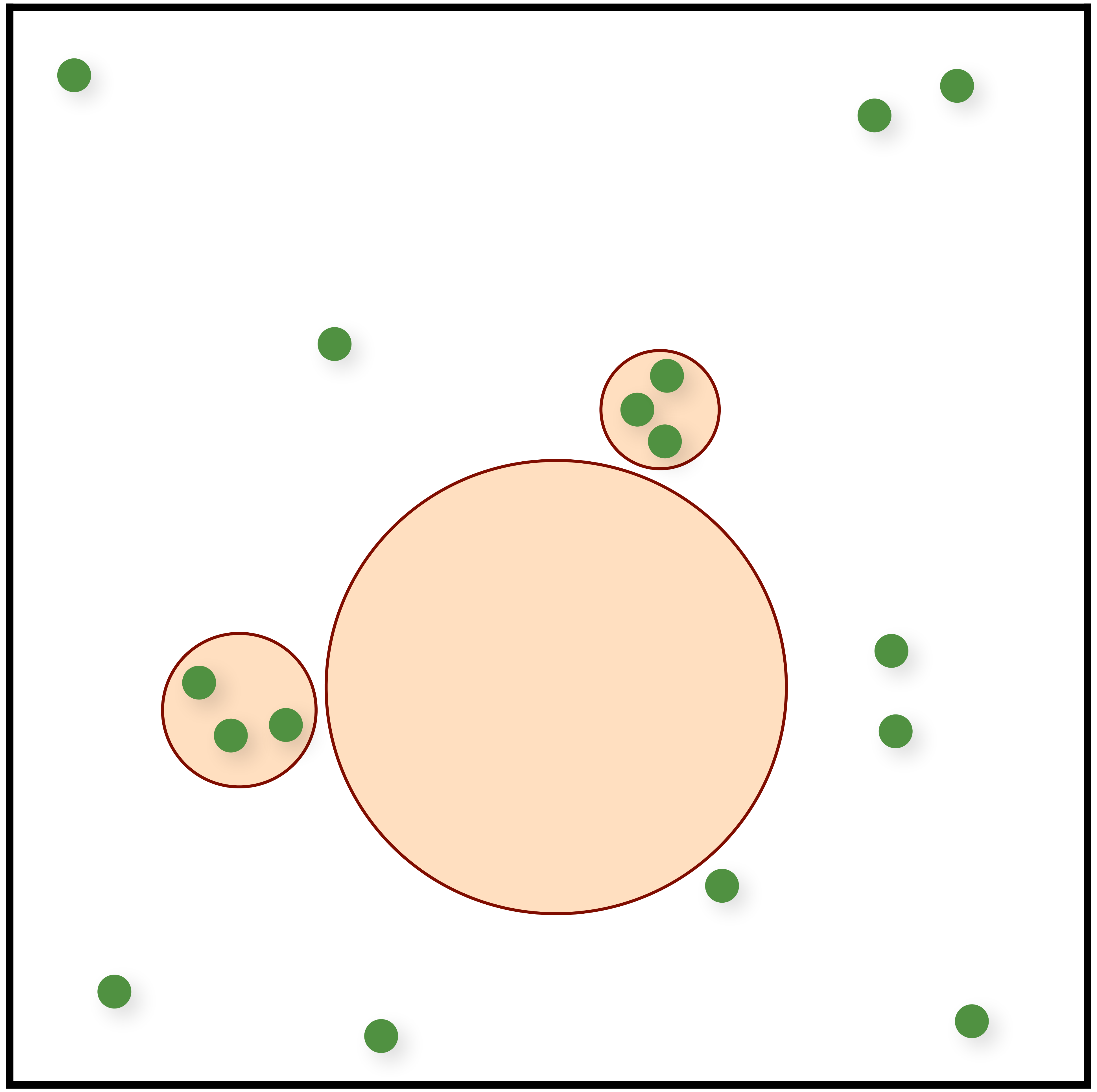
✘ Big gaps

✘ Clumping



Recall: Fourier theory

Fourier transform: $$\hat{f}(\omega) = \int_D f(x)\, \mathrm{e}^{-2\,\pi\,\imath\,\omega\,x}\, \mathrm{d}x$$

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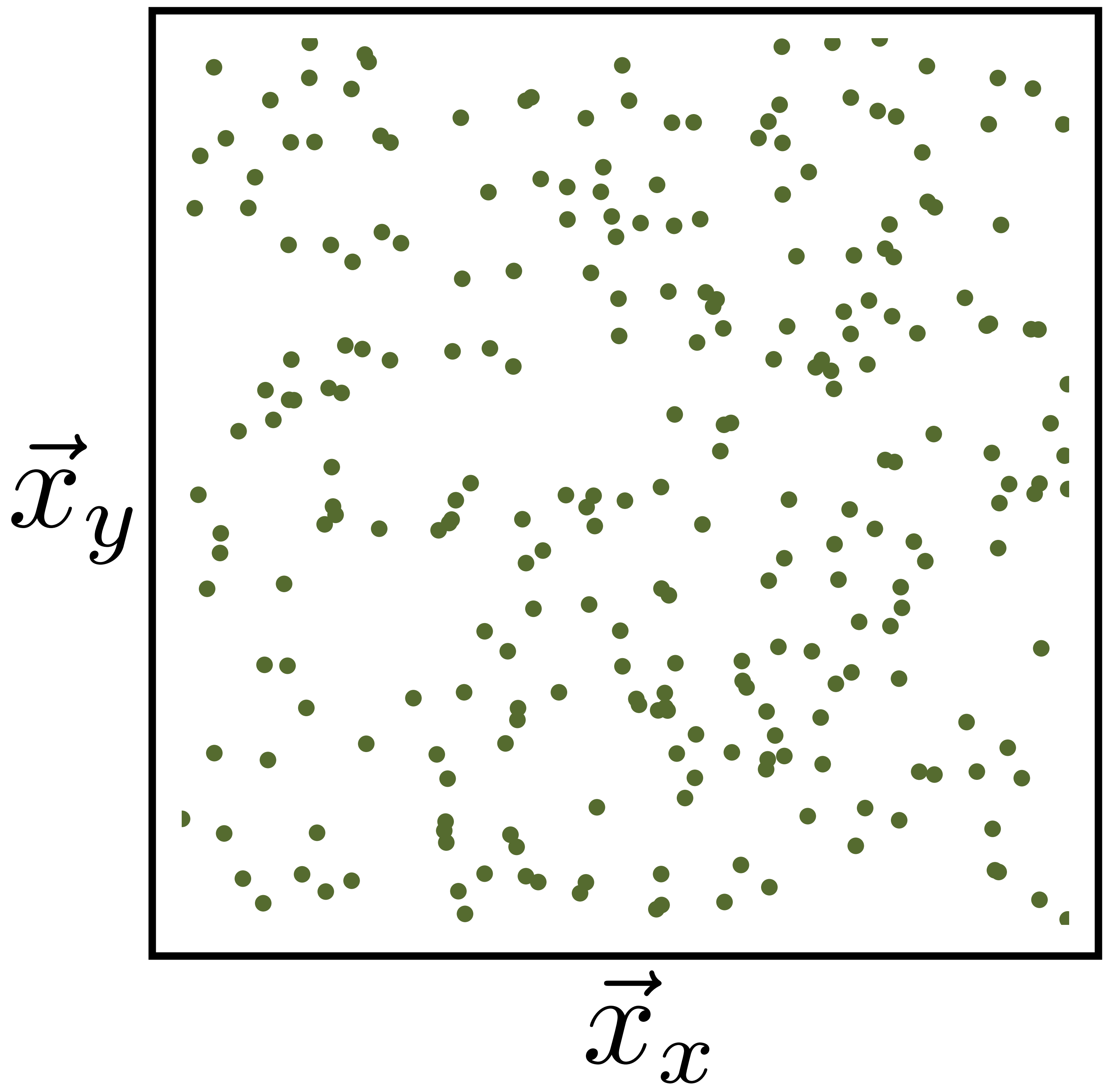
$$= \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x}_k)}$$

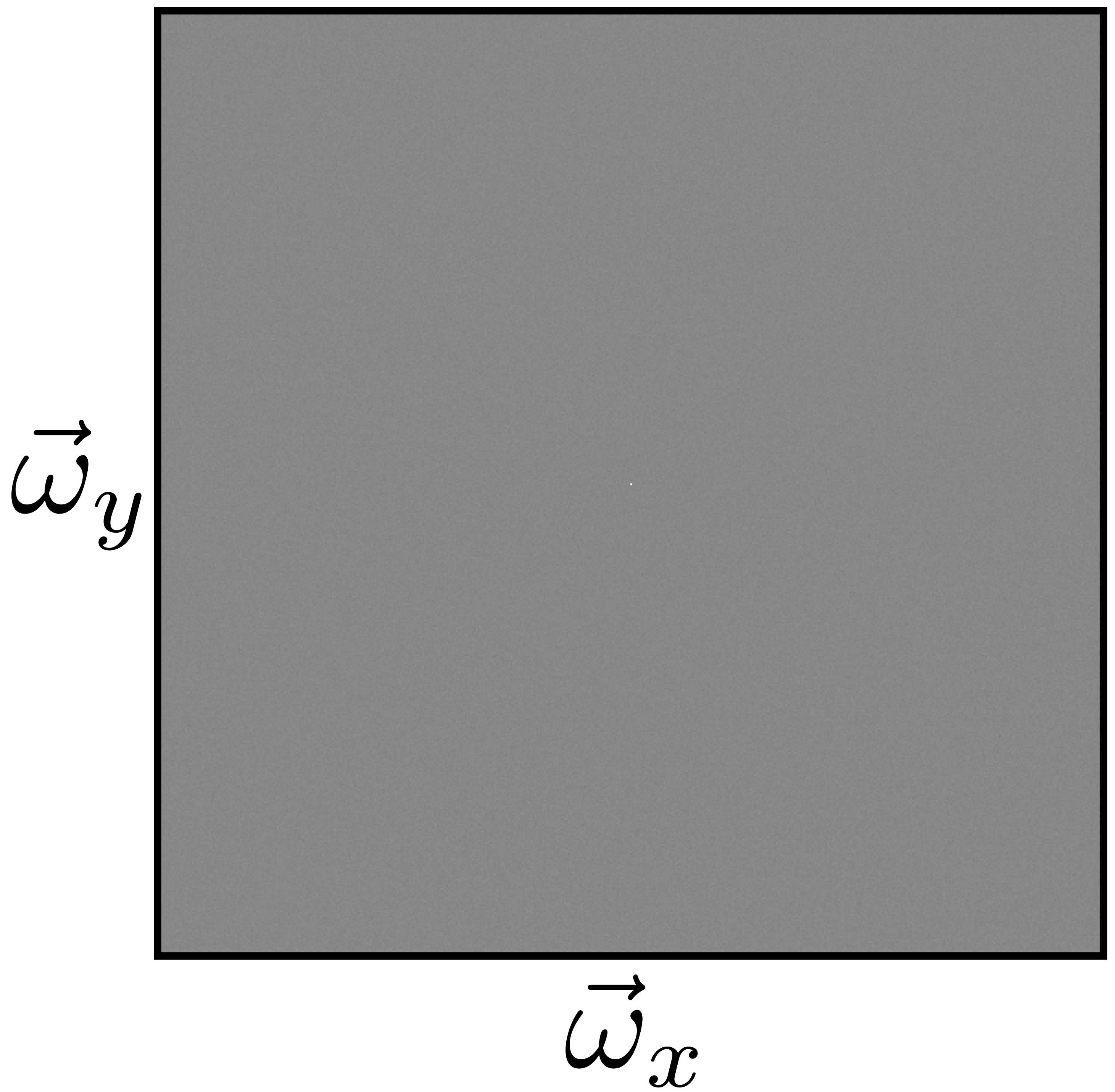
Independent Random Sampling

Samples



$$\frac{1}{N}\sum_{k=1}^{N}\delta(|\vec{x}-\vec{x}_k|)$$

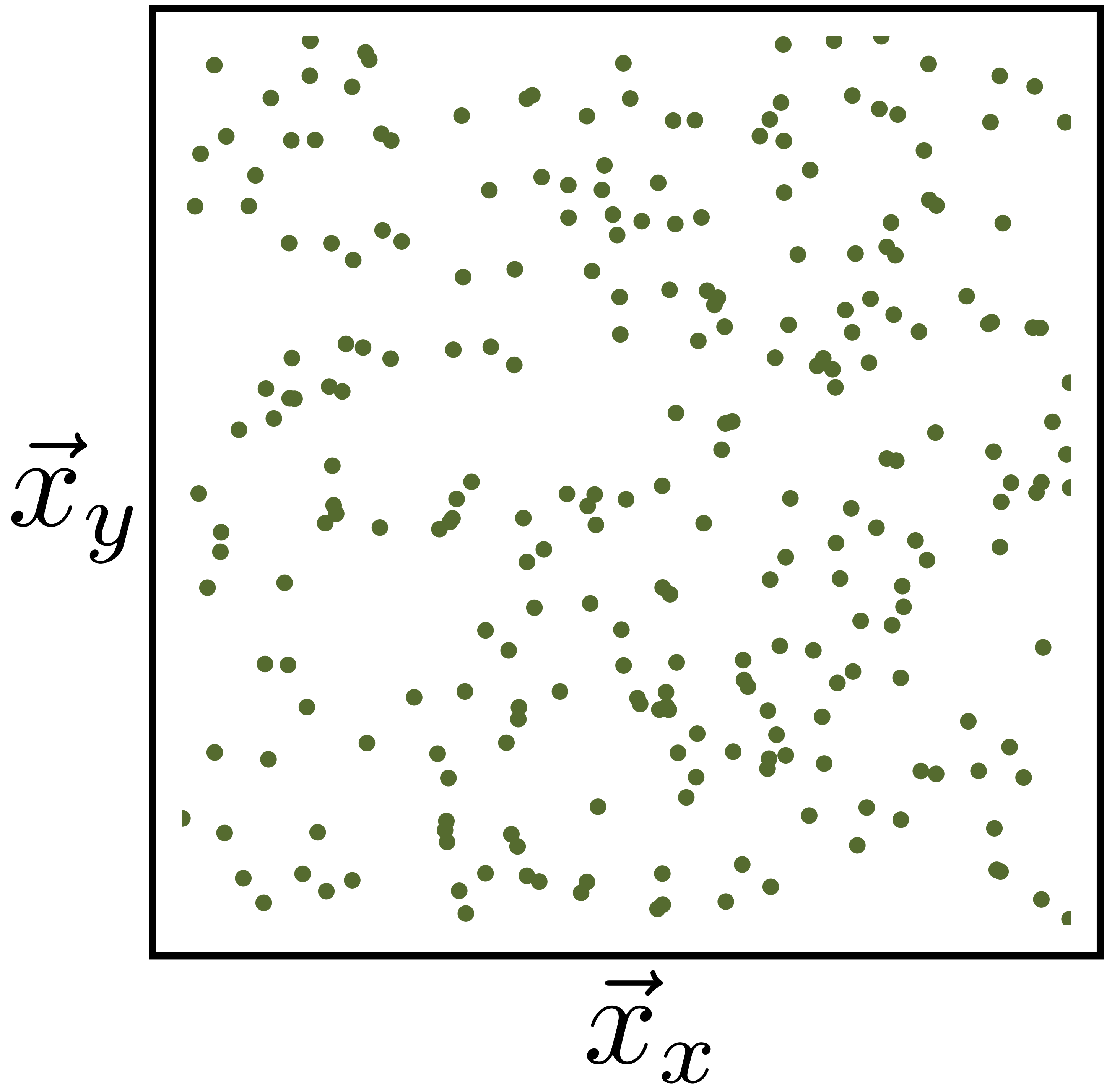
Power spectrum



$$\left|\frac{1}{N}\sum_{k=1}^{N}\mathrm{e}^{-2\pi\imath(\vec{\omega}\cdot\vec{x}_k)}\right|^2$$

Independent Random Sampling

Samples



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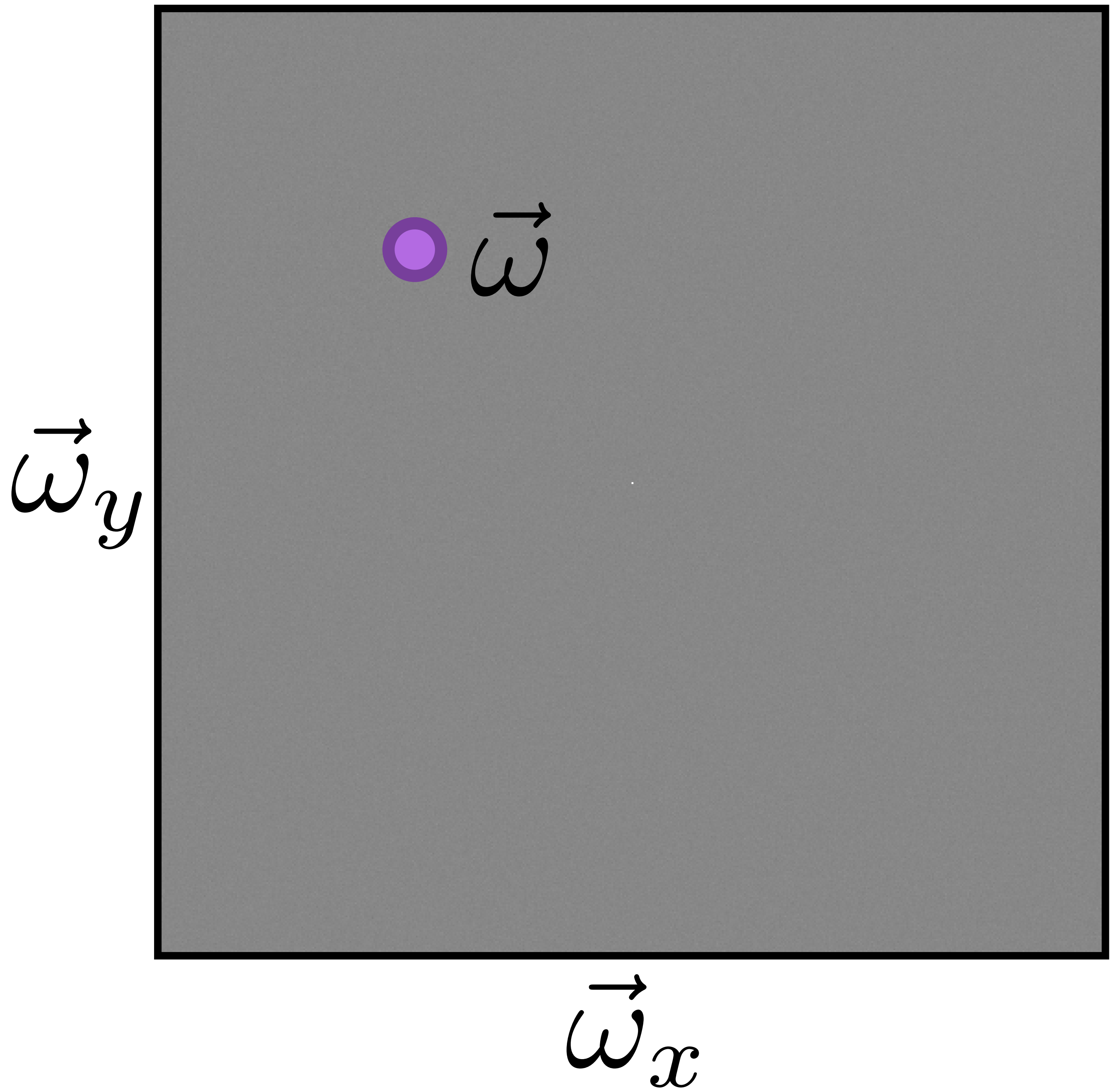

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Independent Random Sampling

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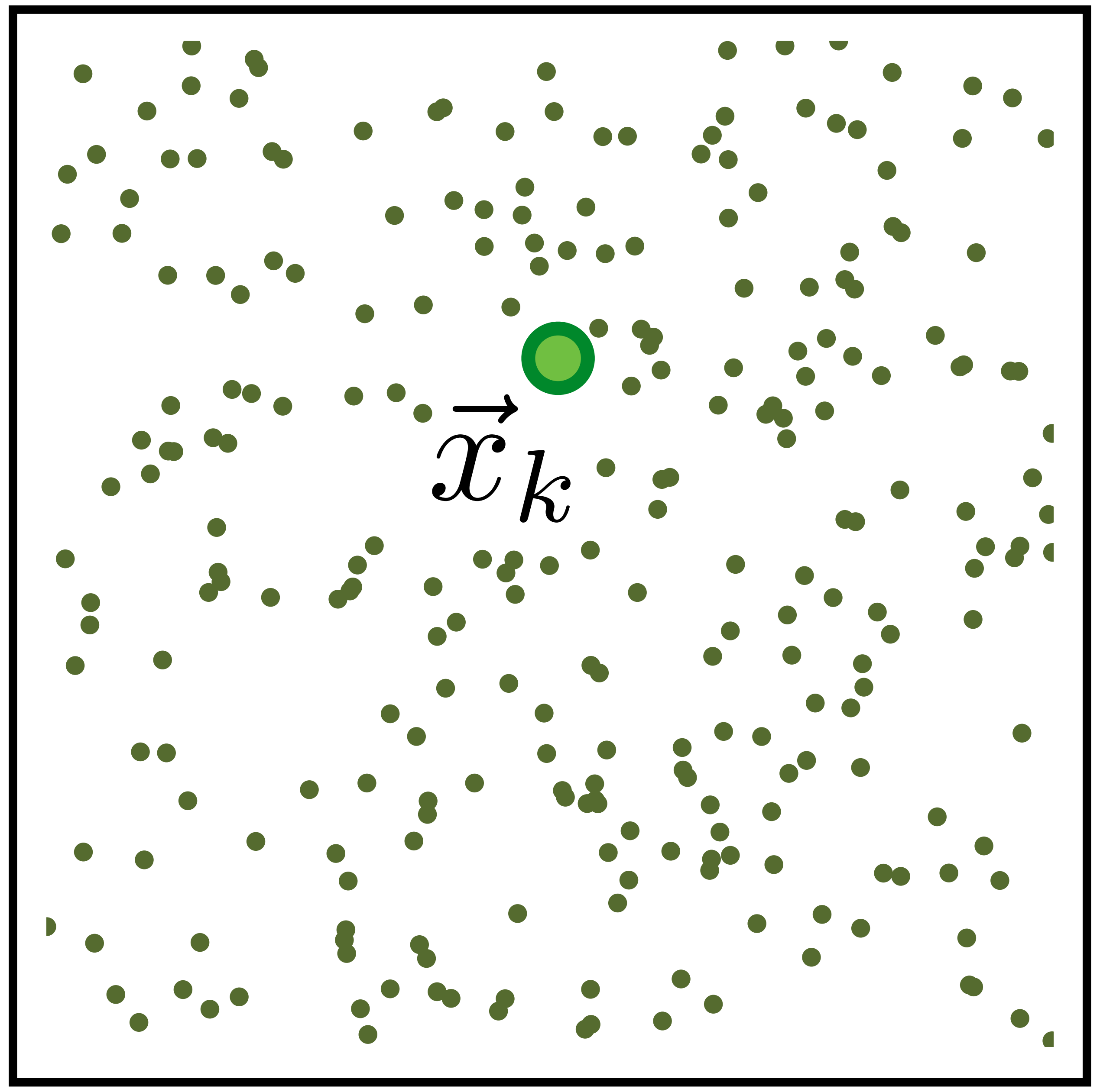

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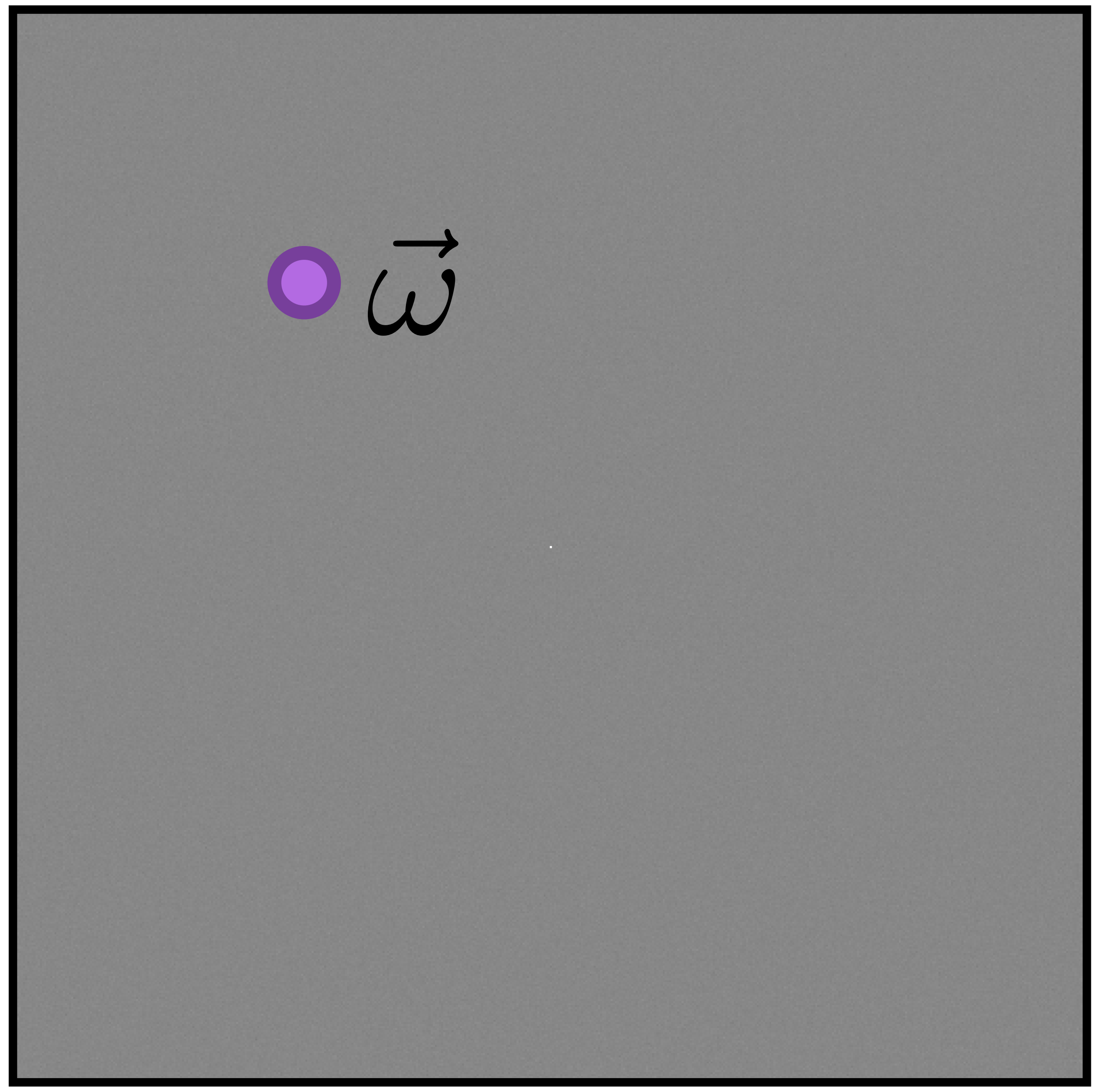

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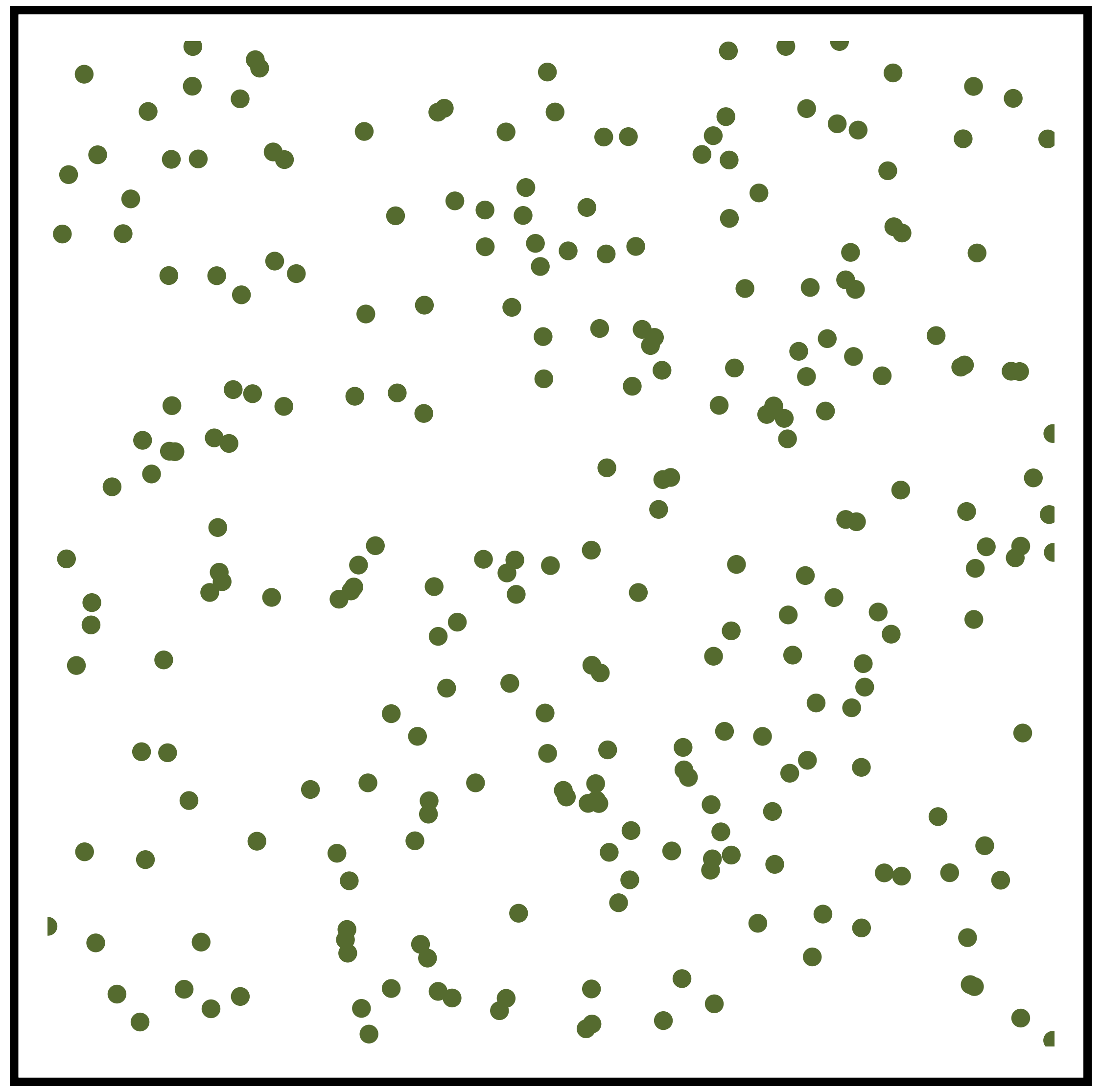
$$\left|\frac{1}{N}\sum_{k=1}^{N}e^{-2\pi\imath(\vec{\omega}\cdot\vec{x}_k)}\right|^2$$

Independent Random Sampling

Many sample set realizations



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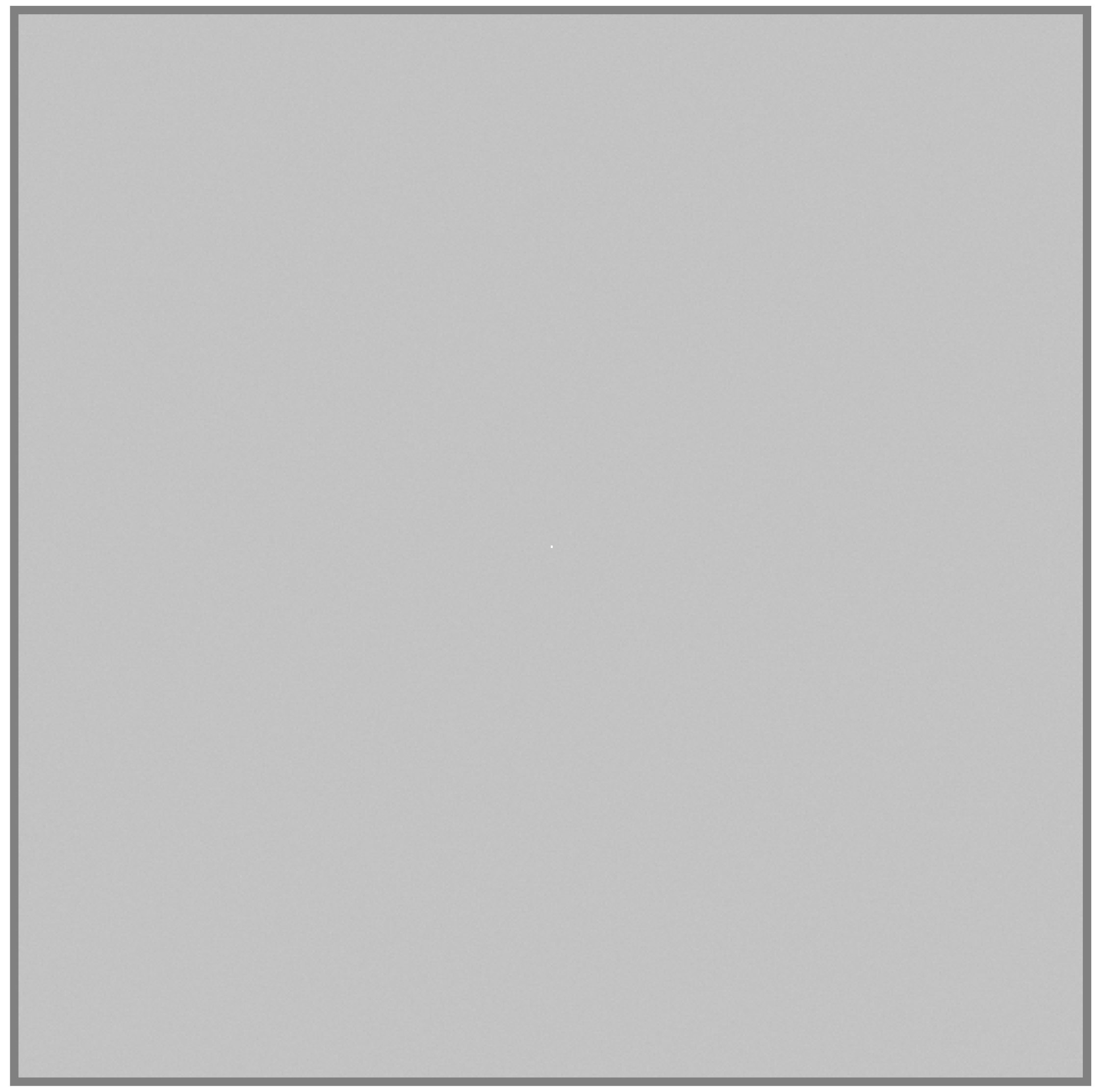
Expected power spectrum



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Independent Random Sampling

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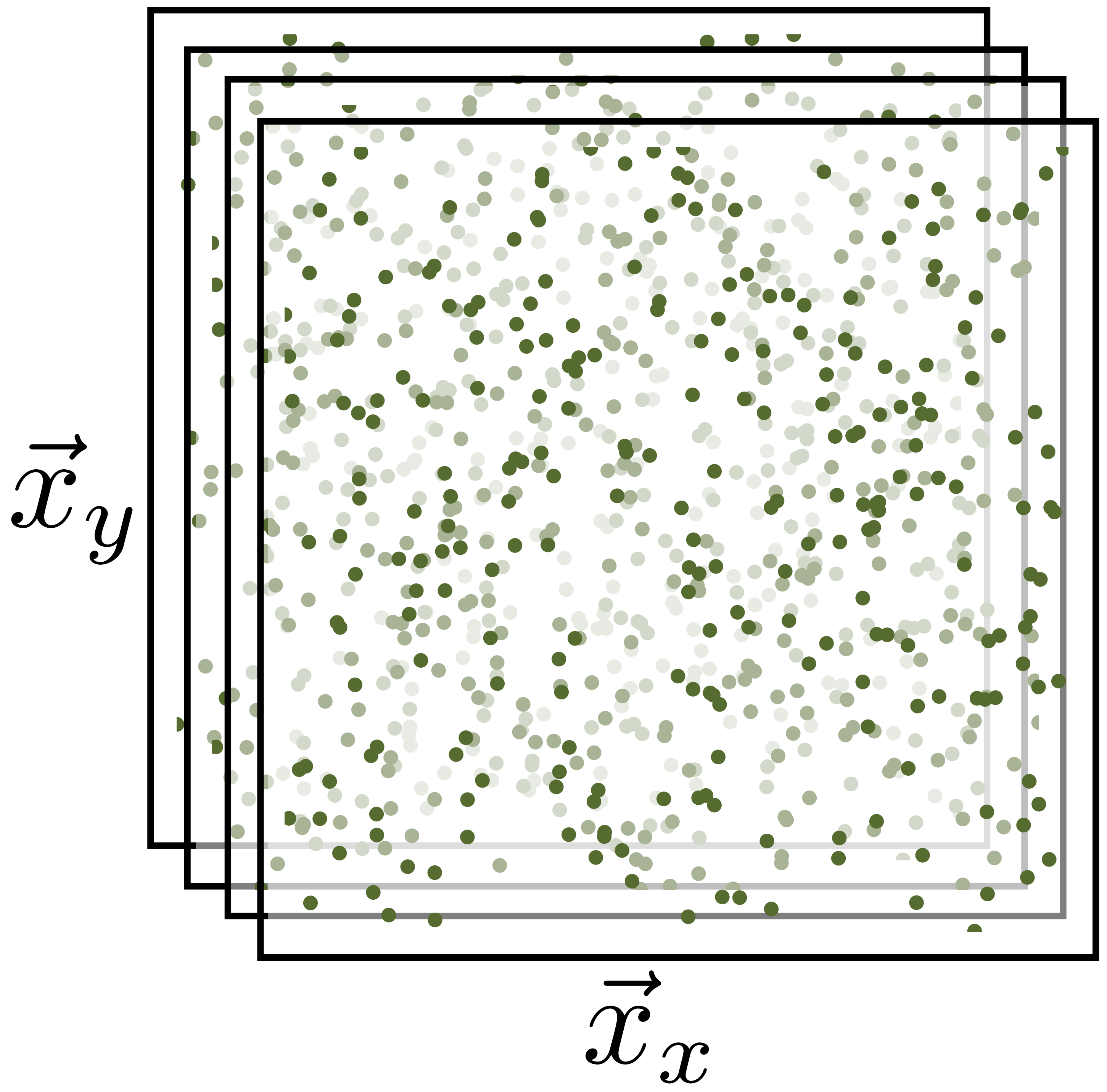

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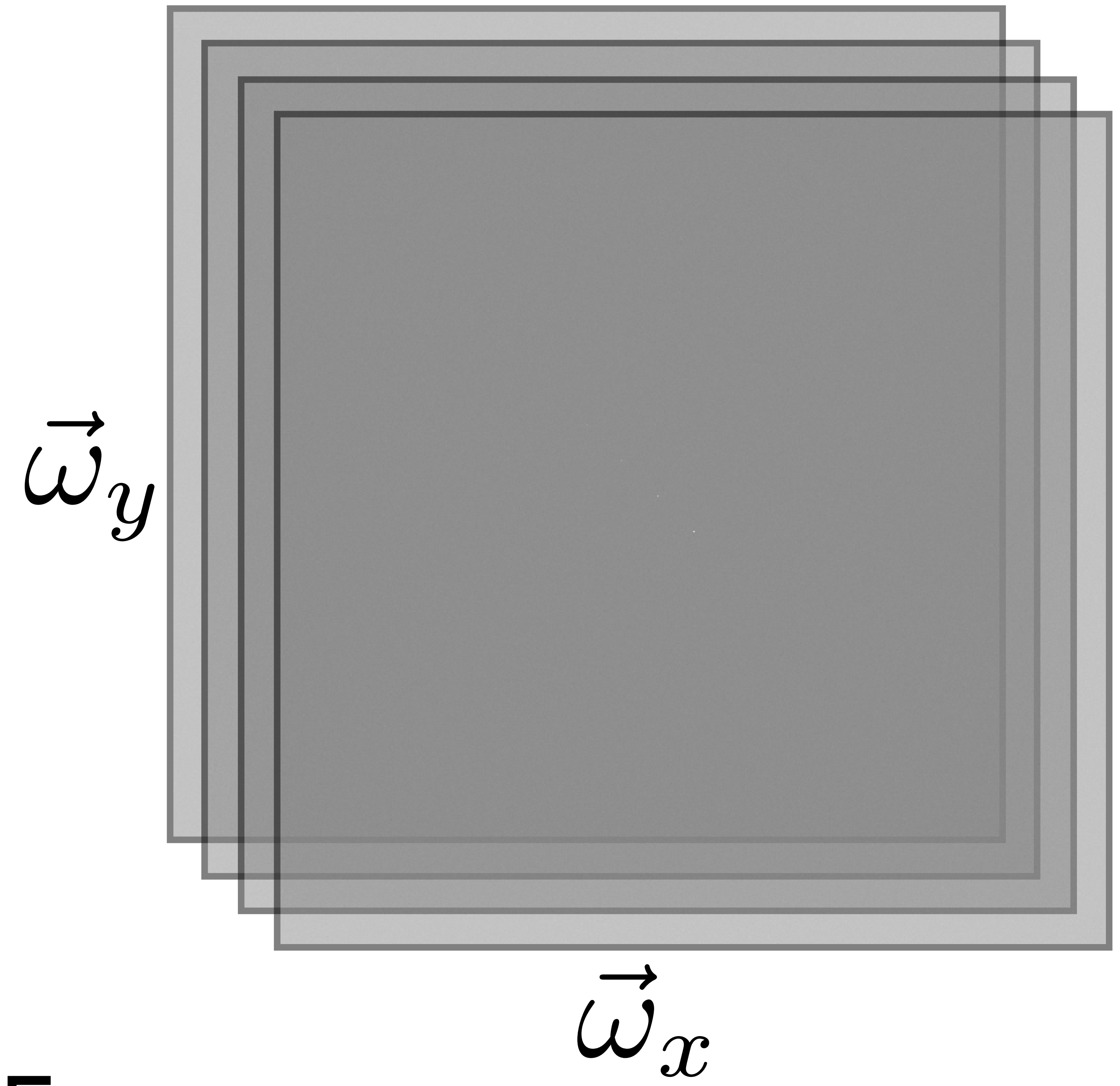
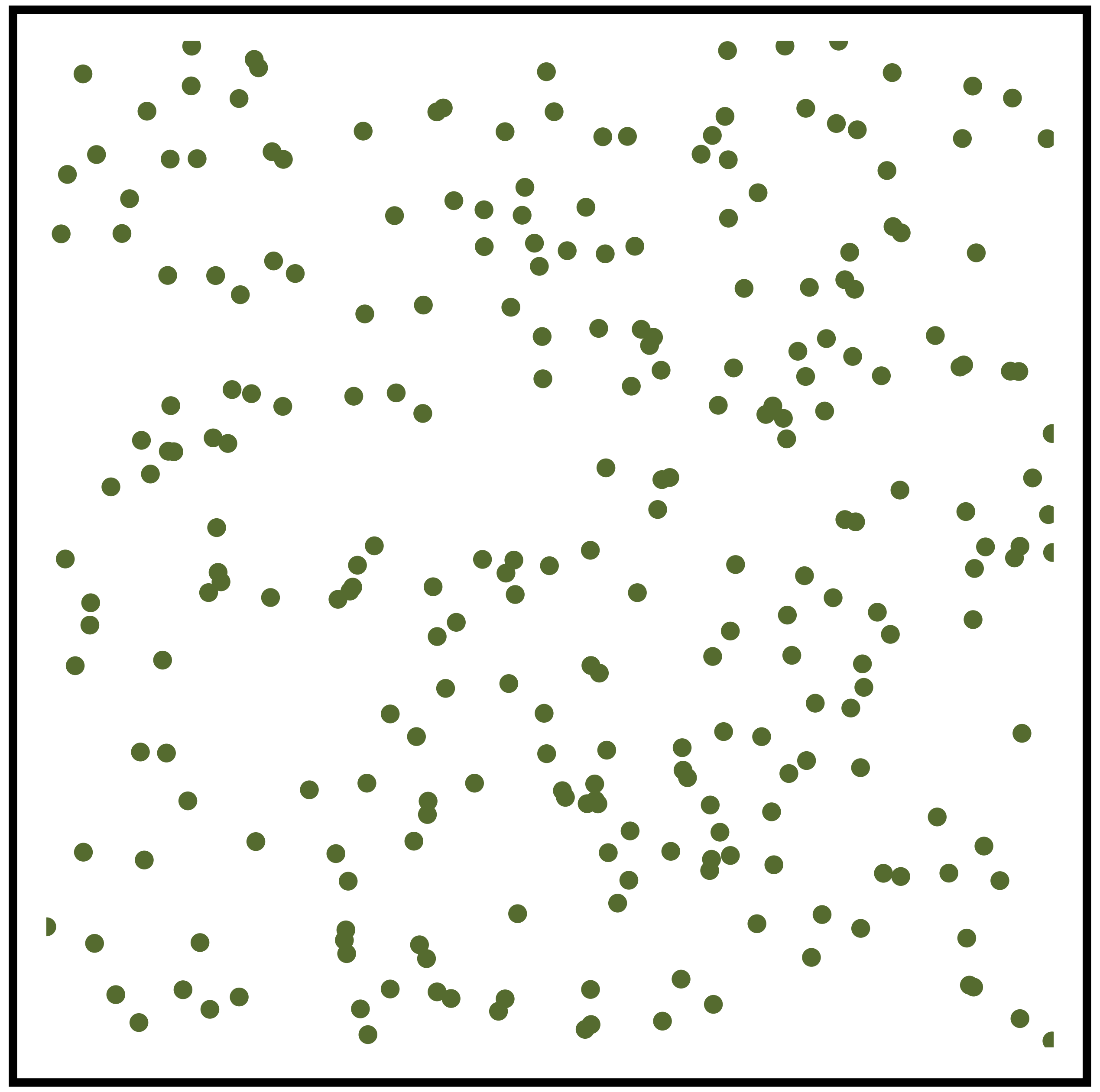


$$\mathrm{E}\left[\left|\frac{1}{N}\sum_{k=1}^{N}\mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x}_k)}\right|^2\right]$$

Independent Random Sampling

Samples



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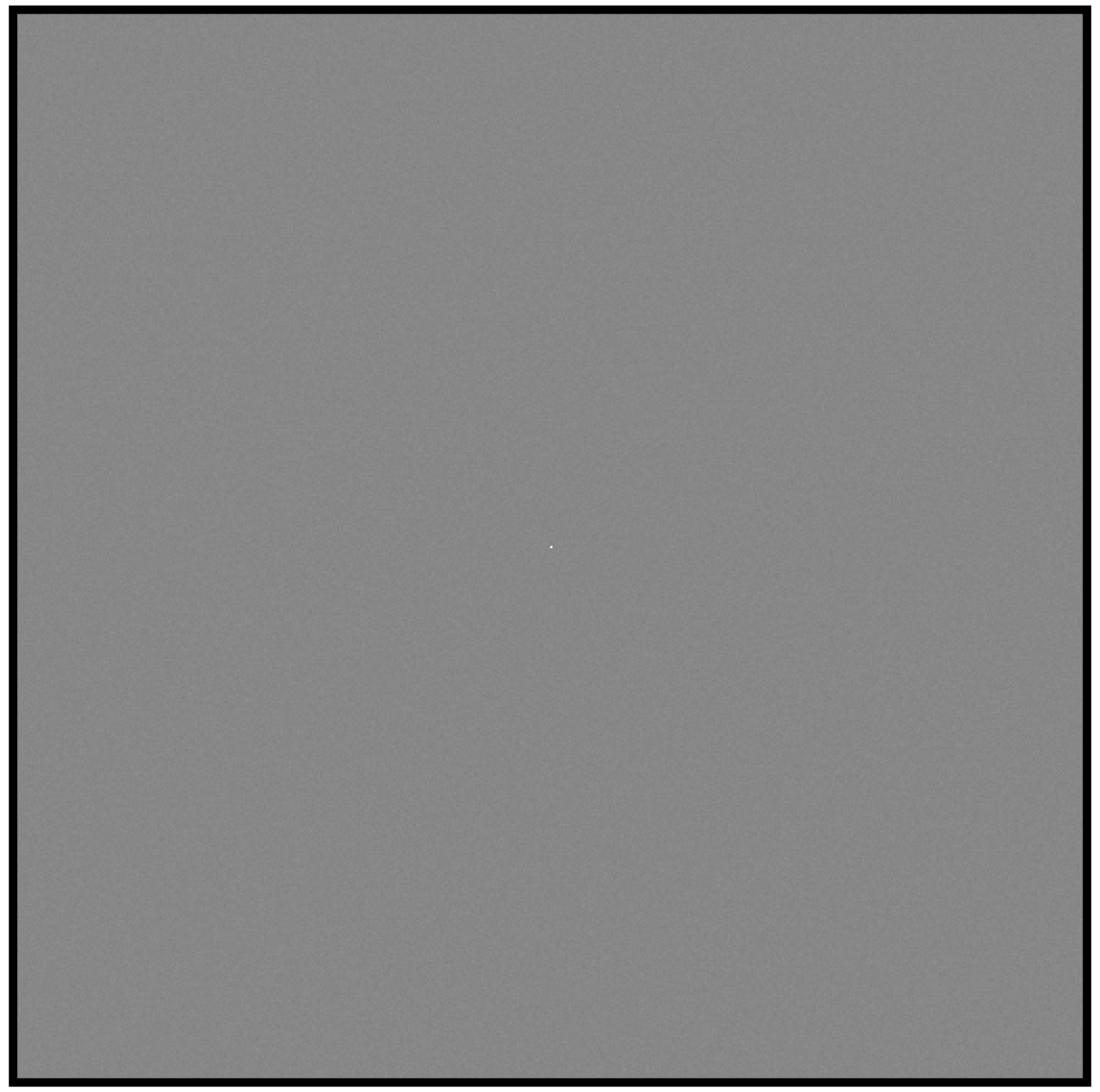

$$\frac{1}{N}\sum_{k=1}^{N}\delta(|\vec{x}-\vec{x}_k|) \quad \mathrm{E}\left[\left|\frac{1}{N}\sum_{k=1}^{N}\mathrm{e}^{-2\,\pi\,\imath\,(\vec{\omega}\cdot\vec{x}_k)}\right|^2\right]$$

Independent Random Sampling

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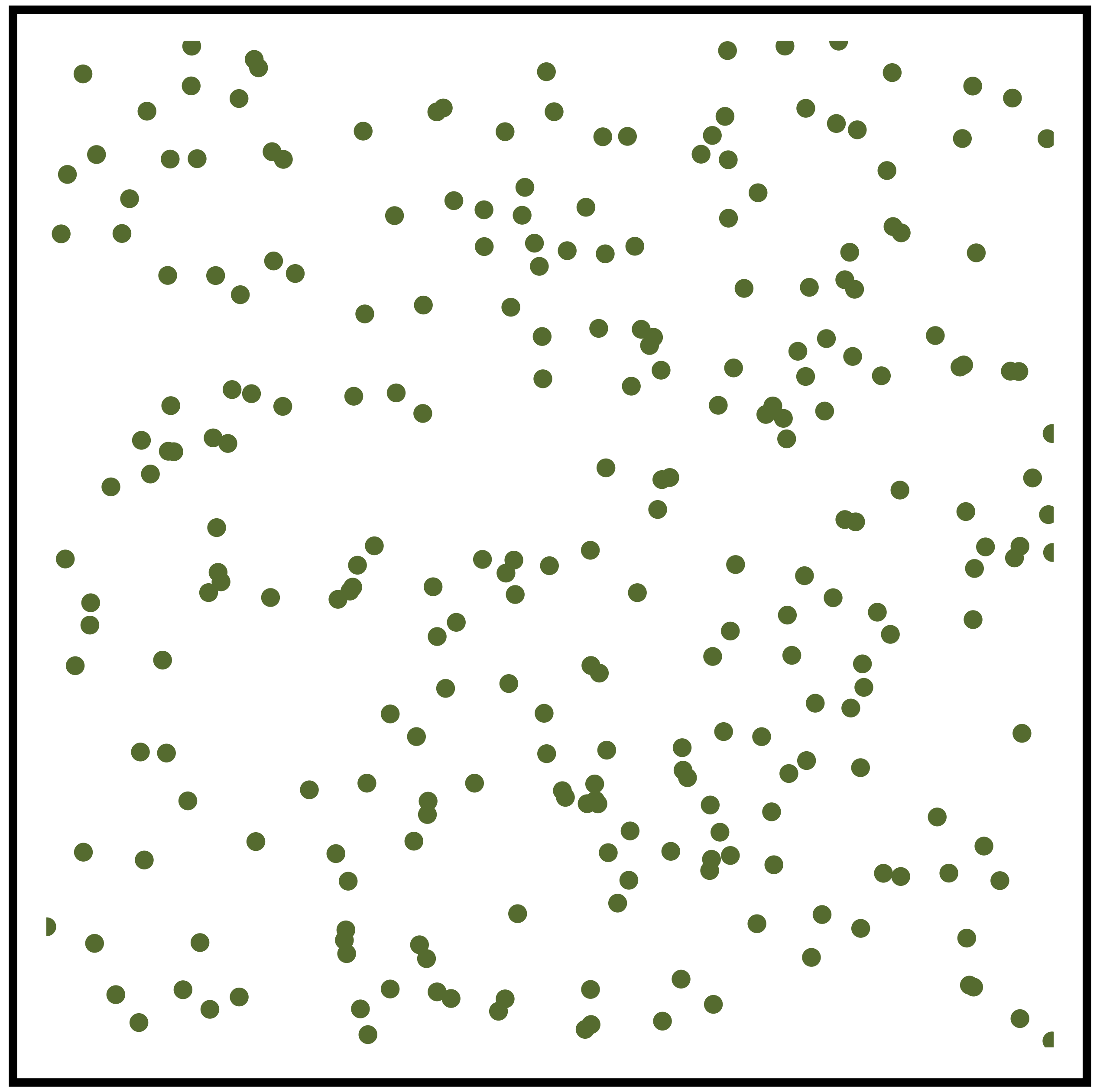

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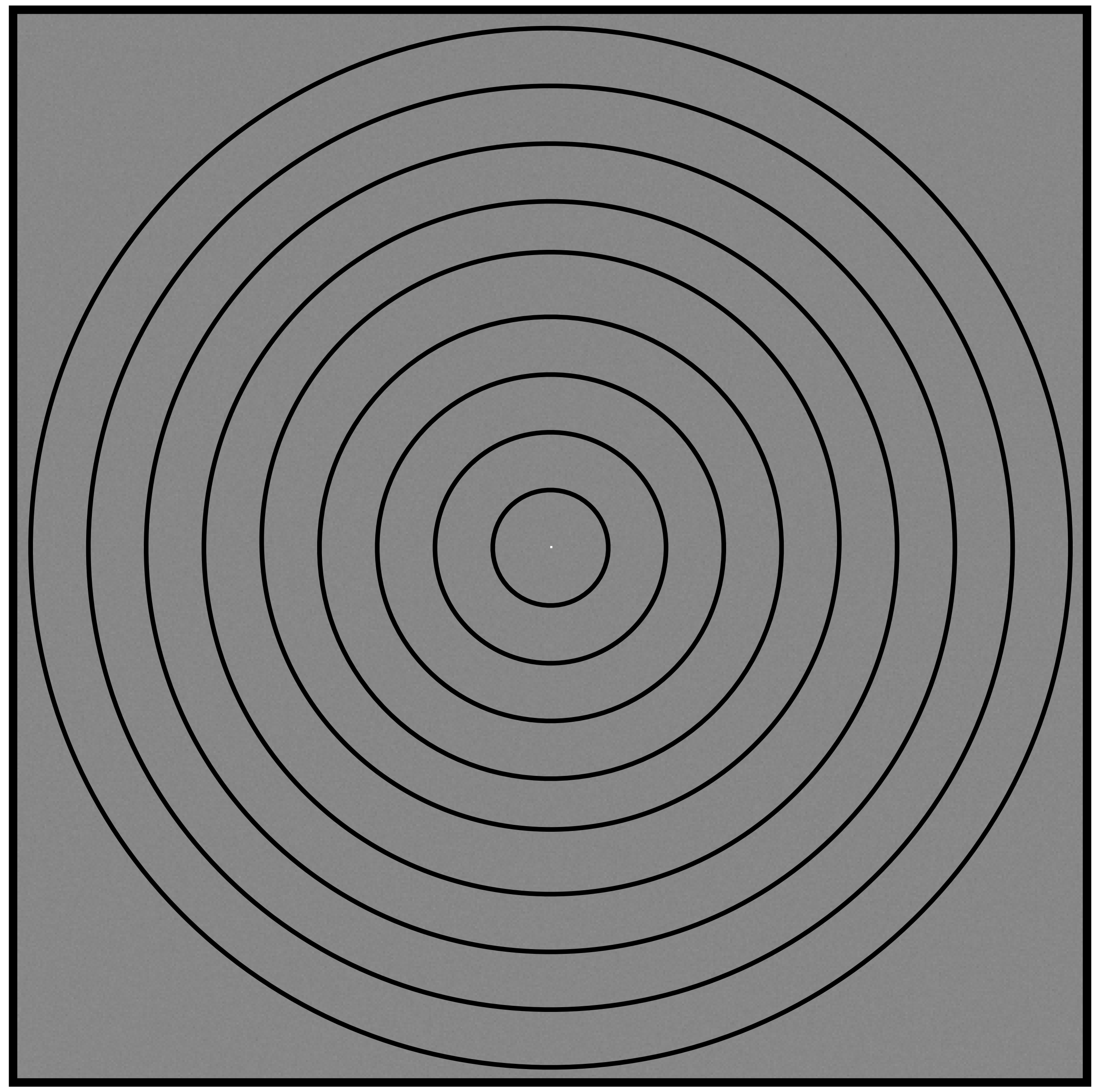
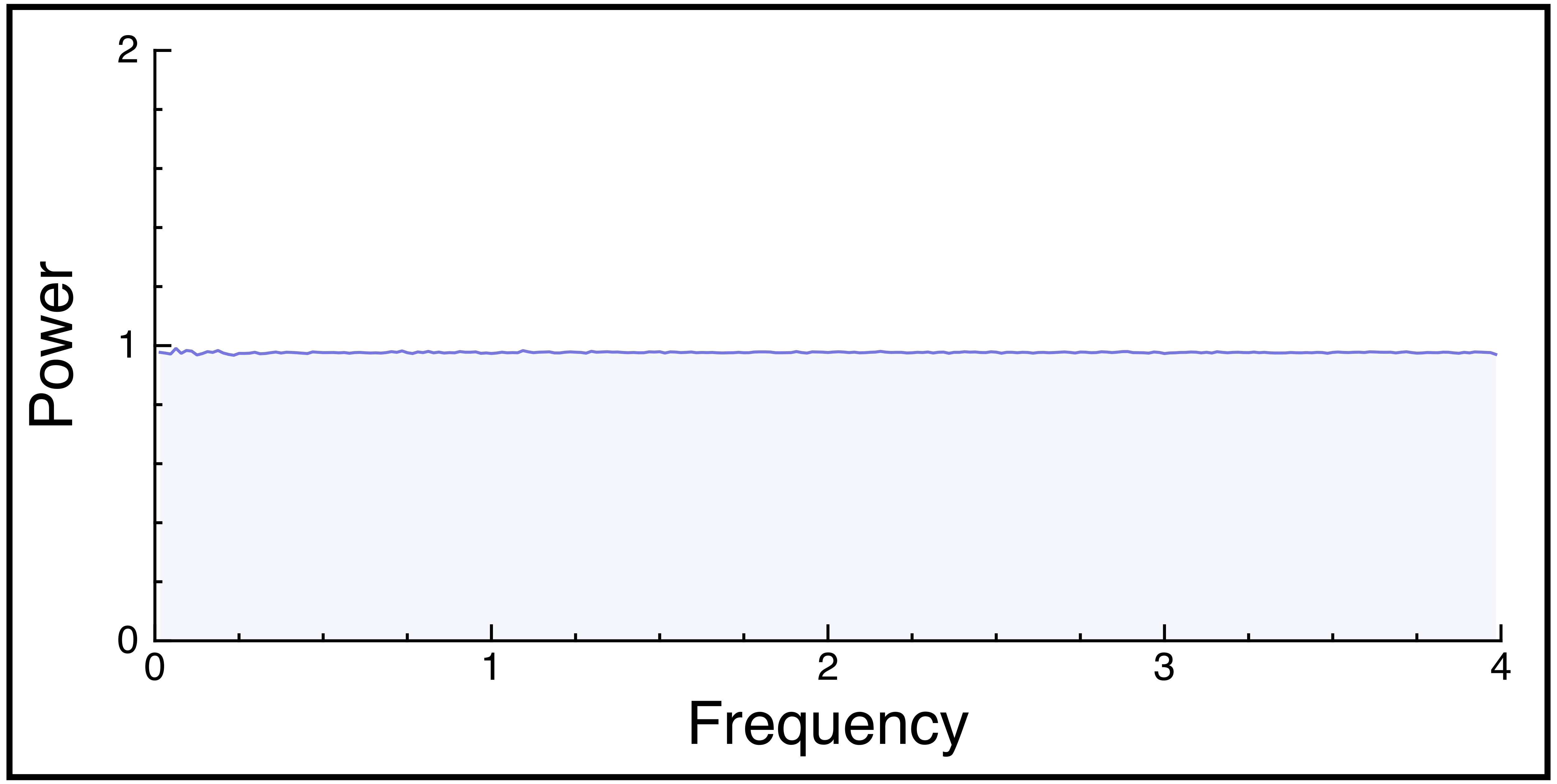


Radial mean



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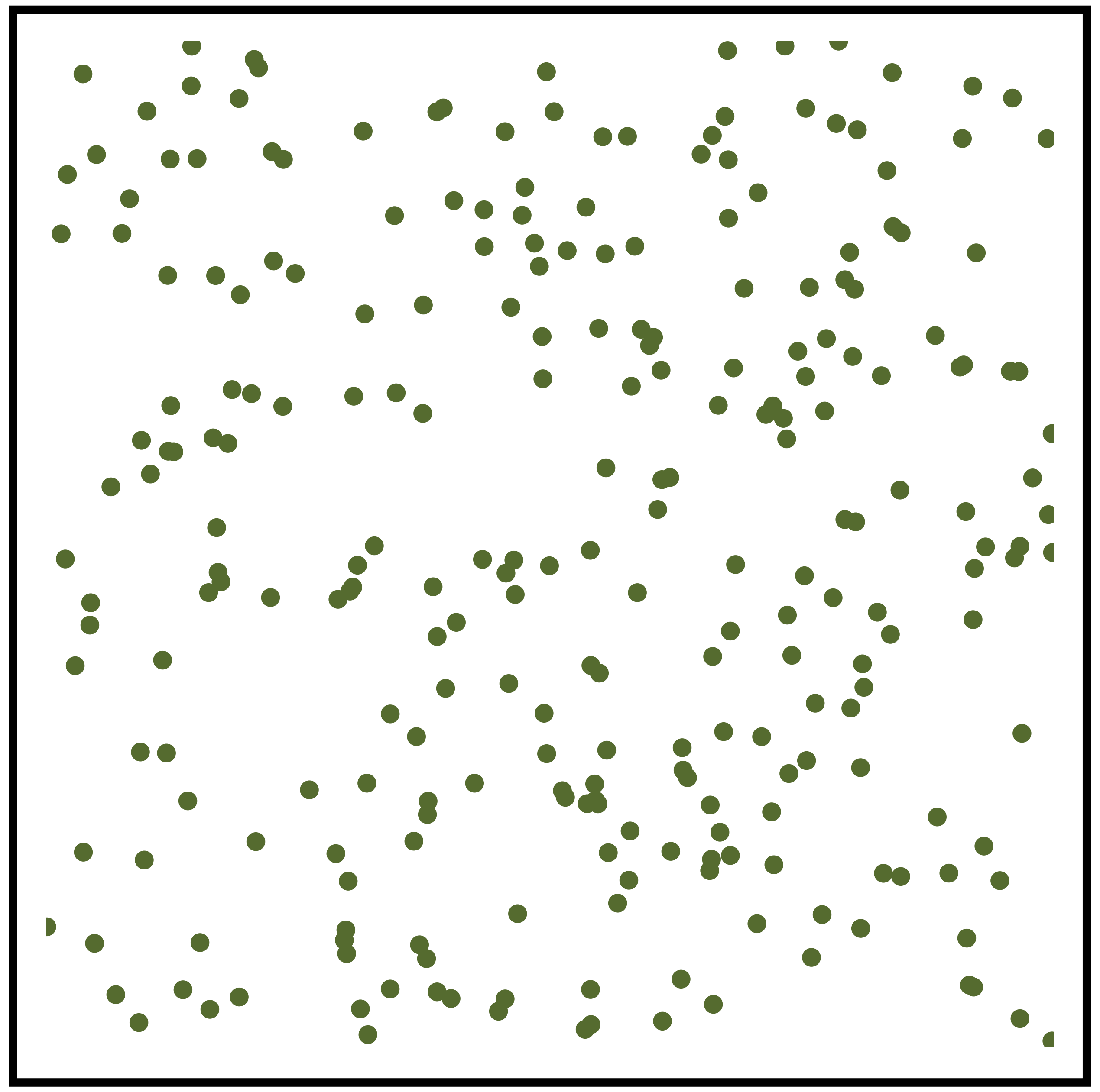
Independent Random Sampling

Samples



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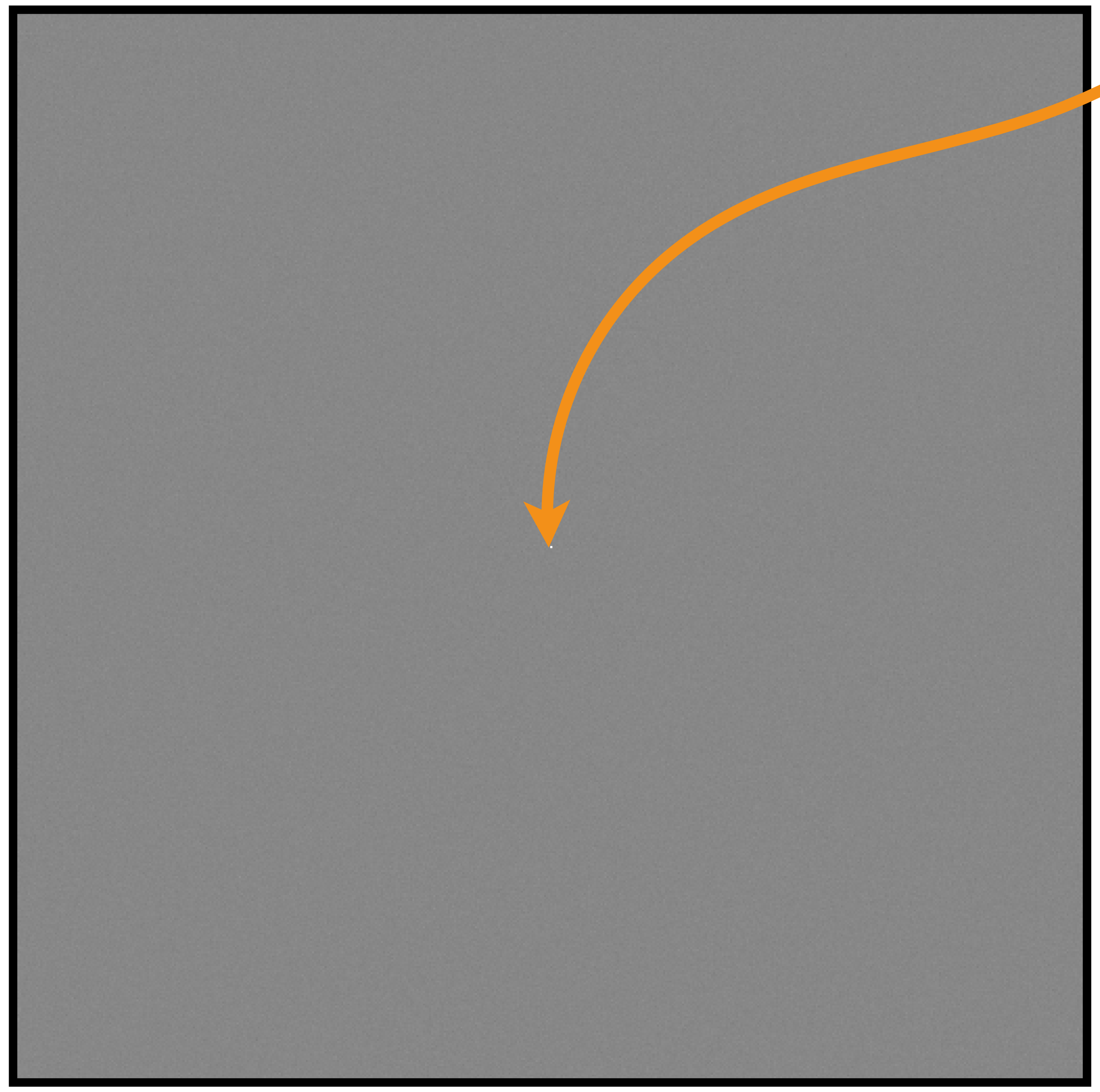

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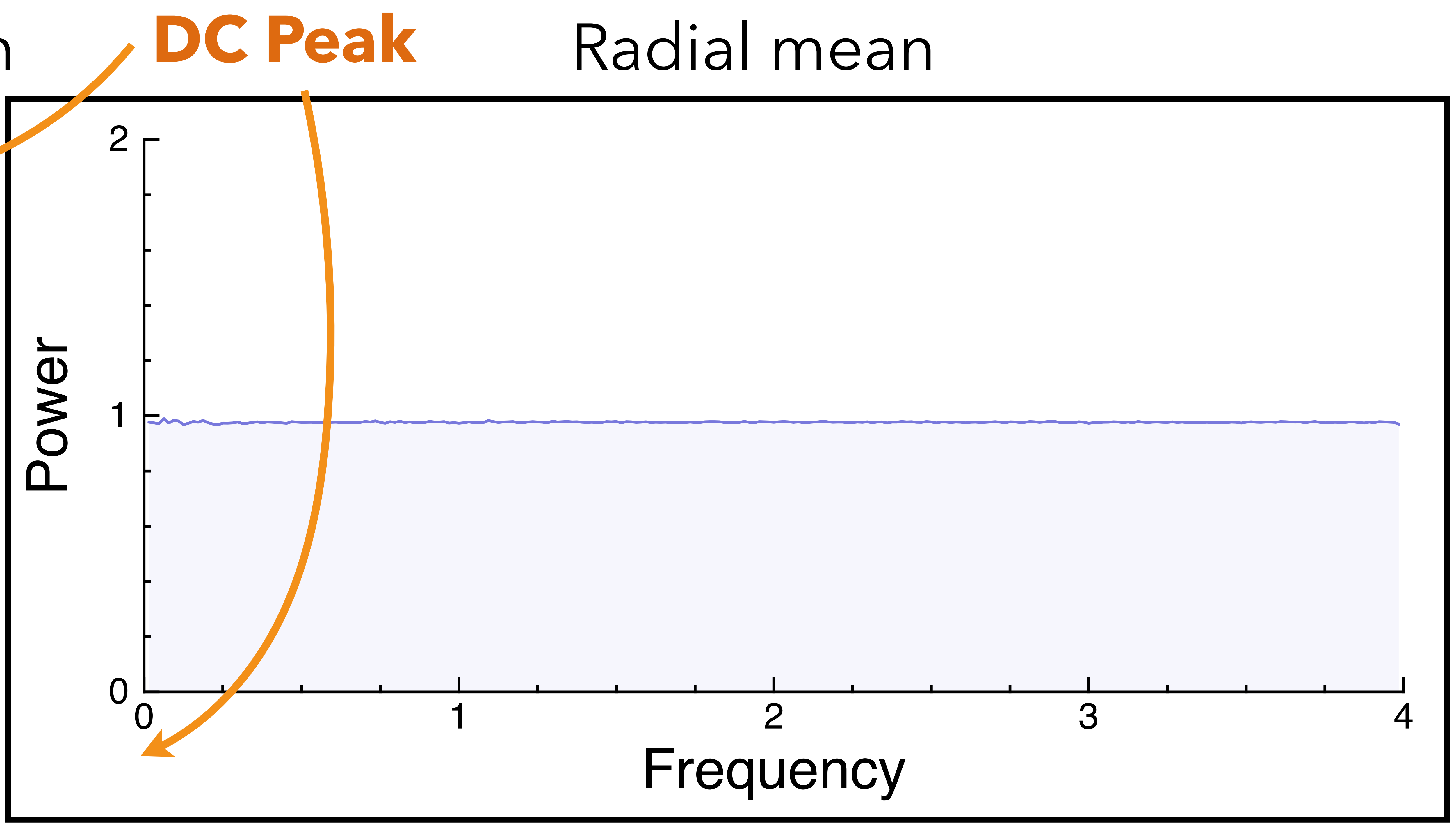

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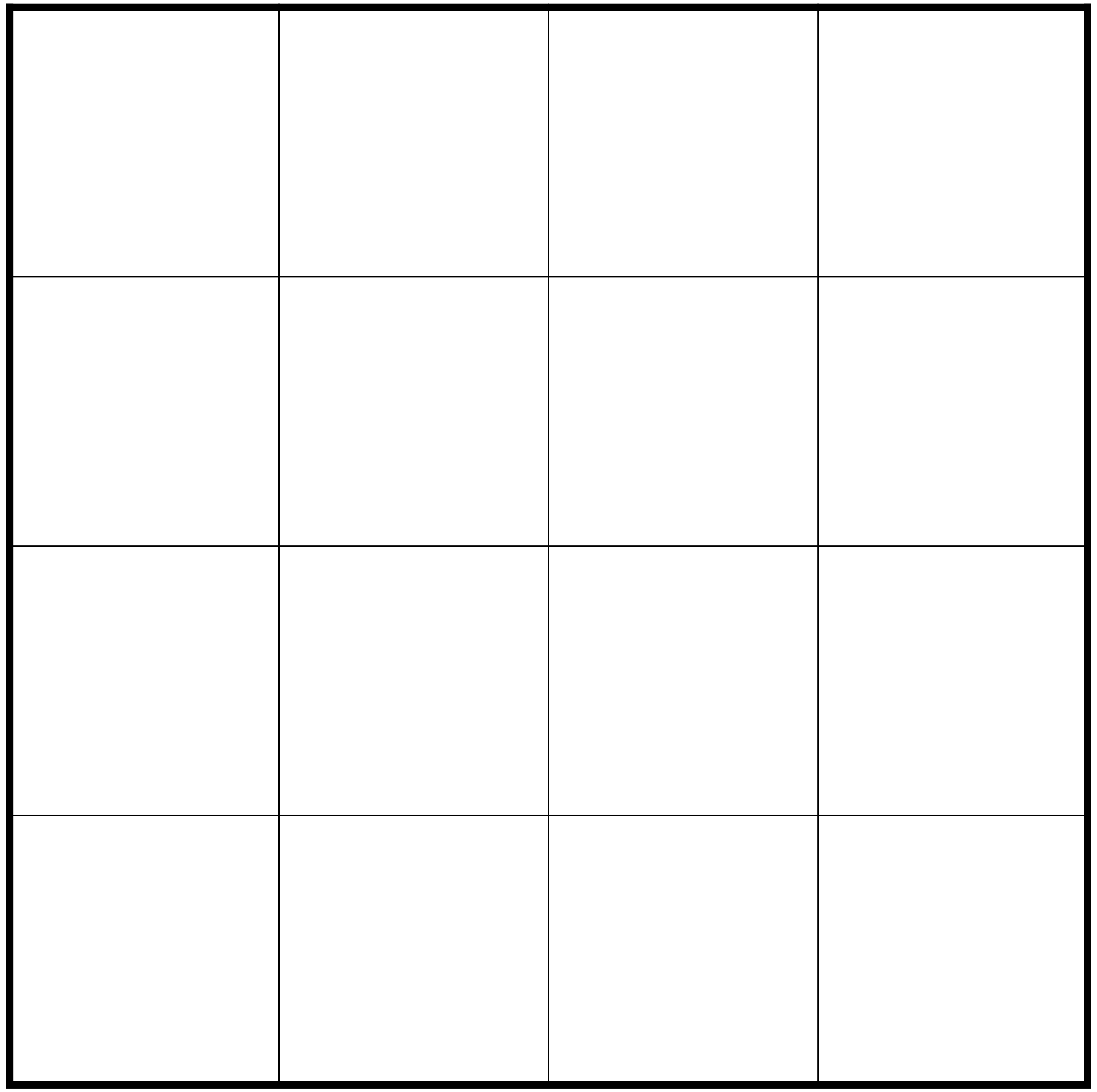
Regular Sampling

```
for (uint i = 0; i < numX; i++)
	for (uint j = 0; j < numY; j++)
	{
		samples(i,j).x = (i + 0.5)/numX;
		samples(i,j).y = (j + 0.5)/numY;
	}
```



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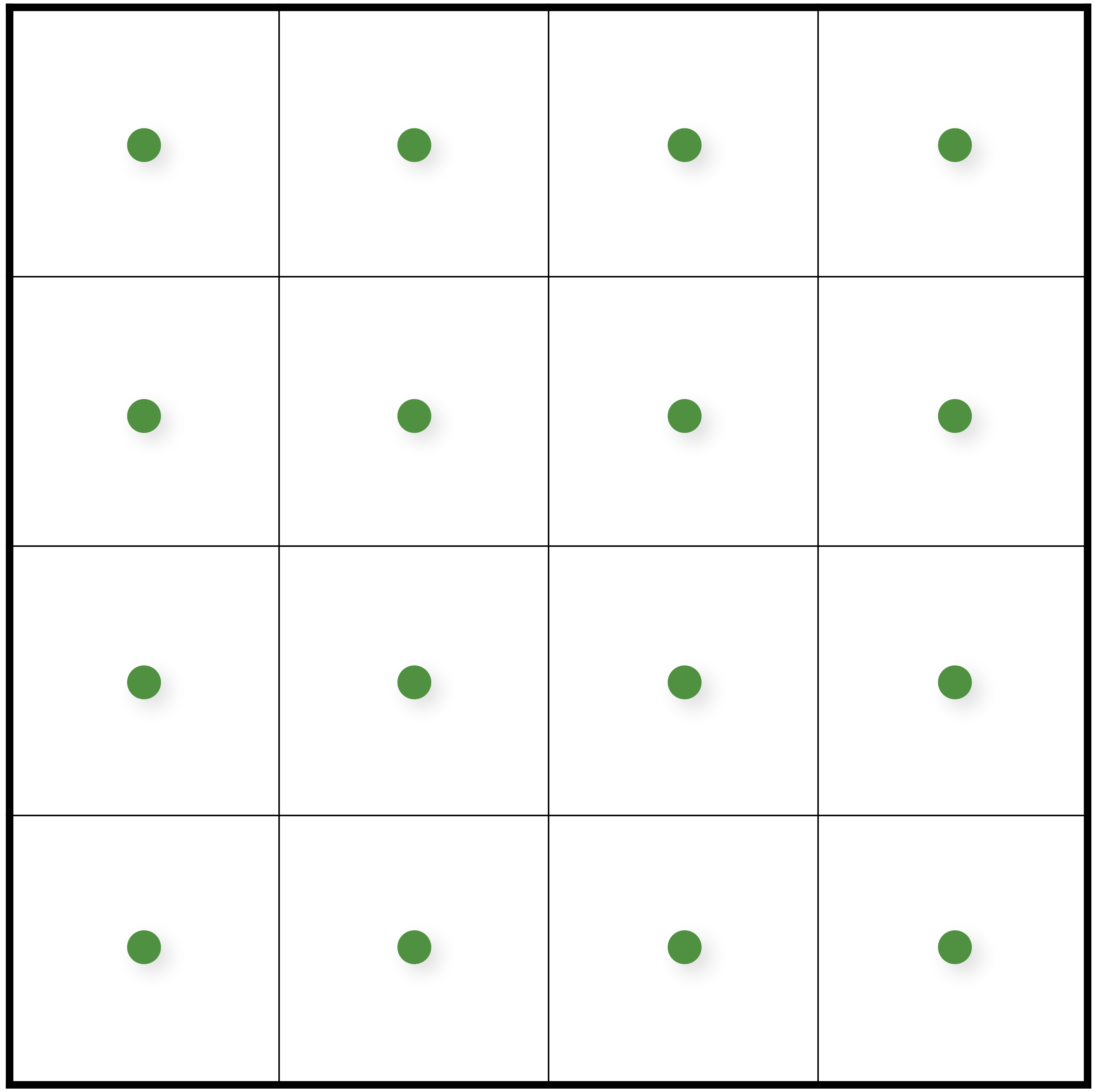

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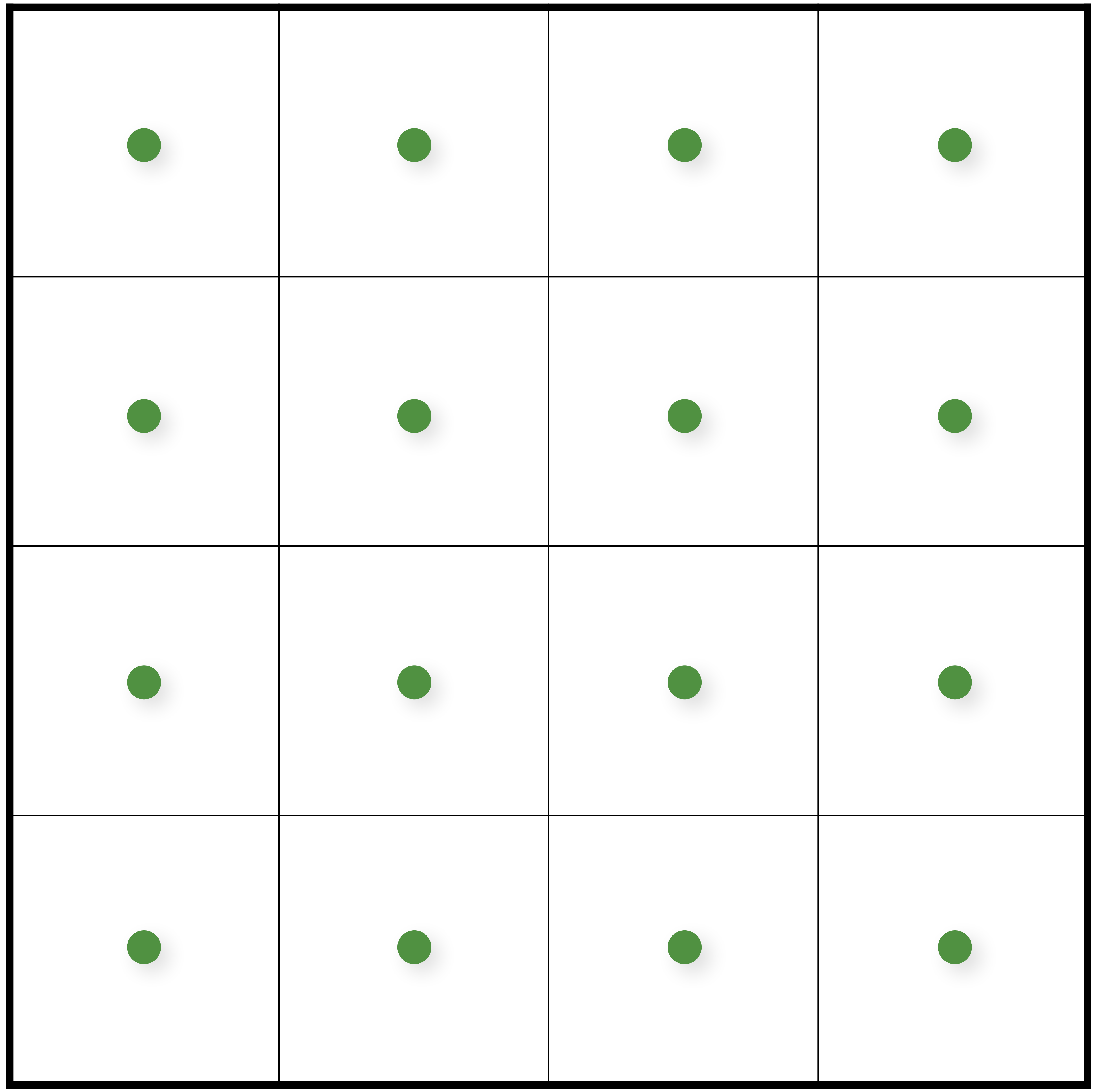
✔ Extends to higher dimensions, but…



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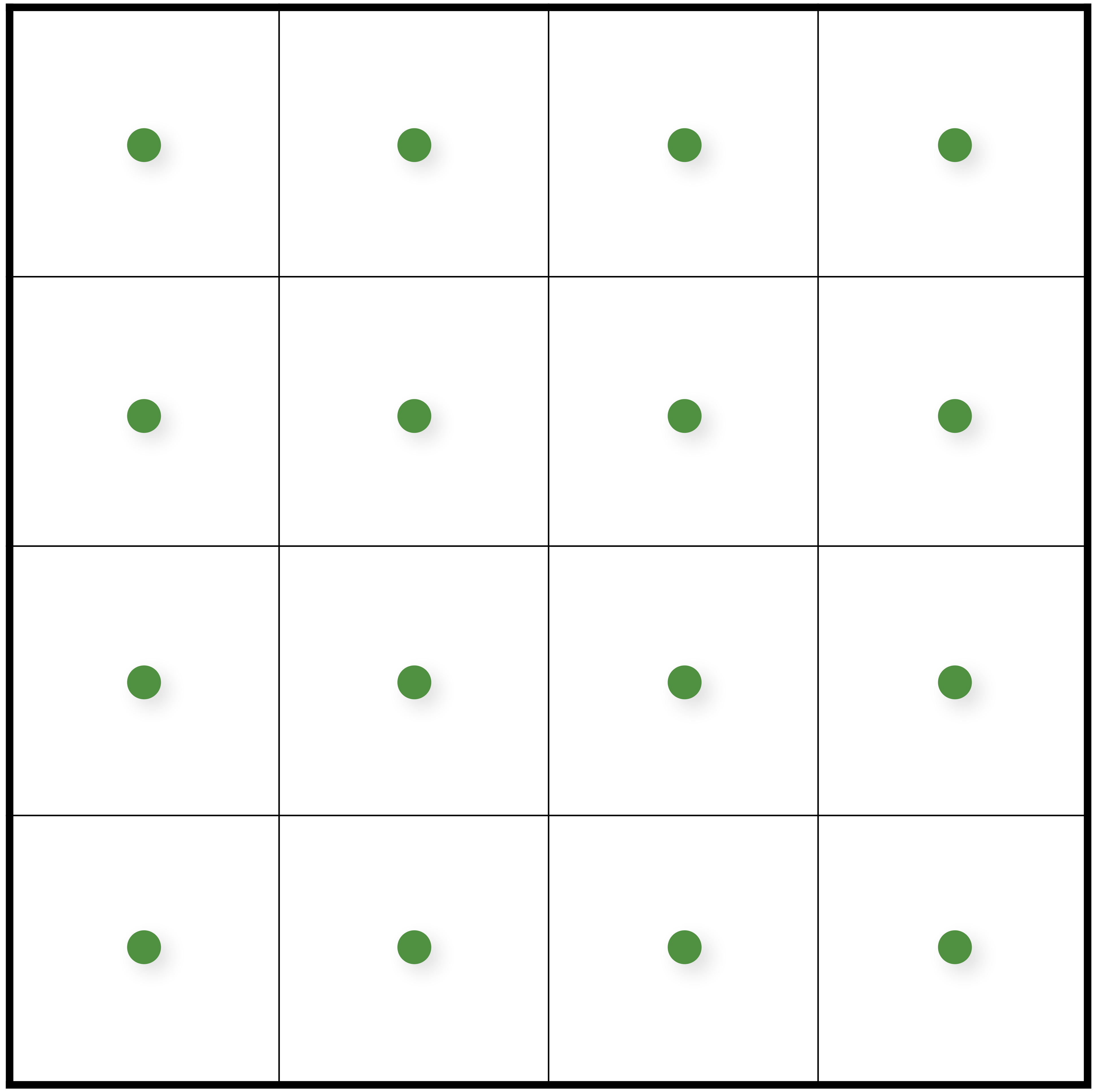

✔ Extends to higher dimensions, but…

✘ Curse of dimensionality

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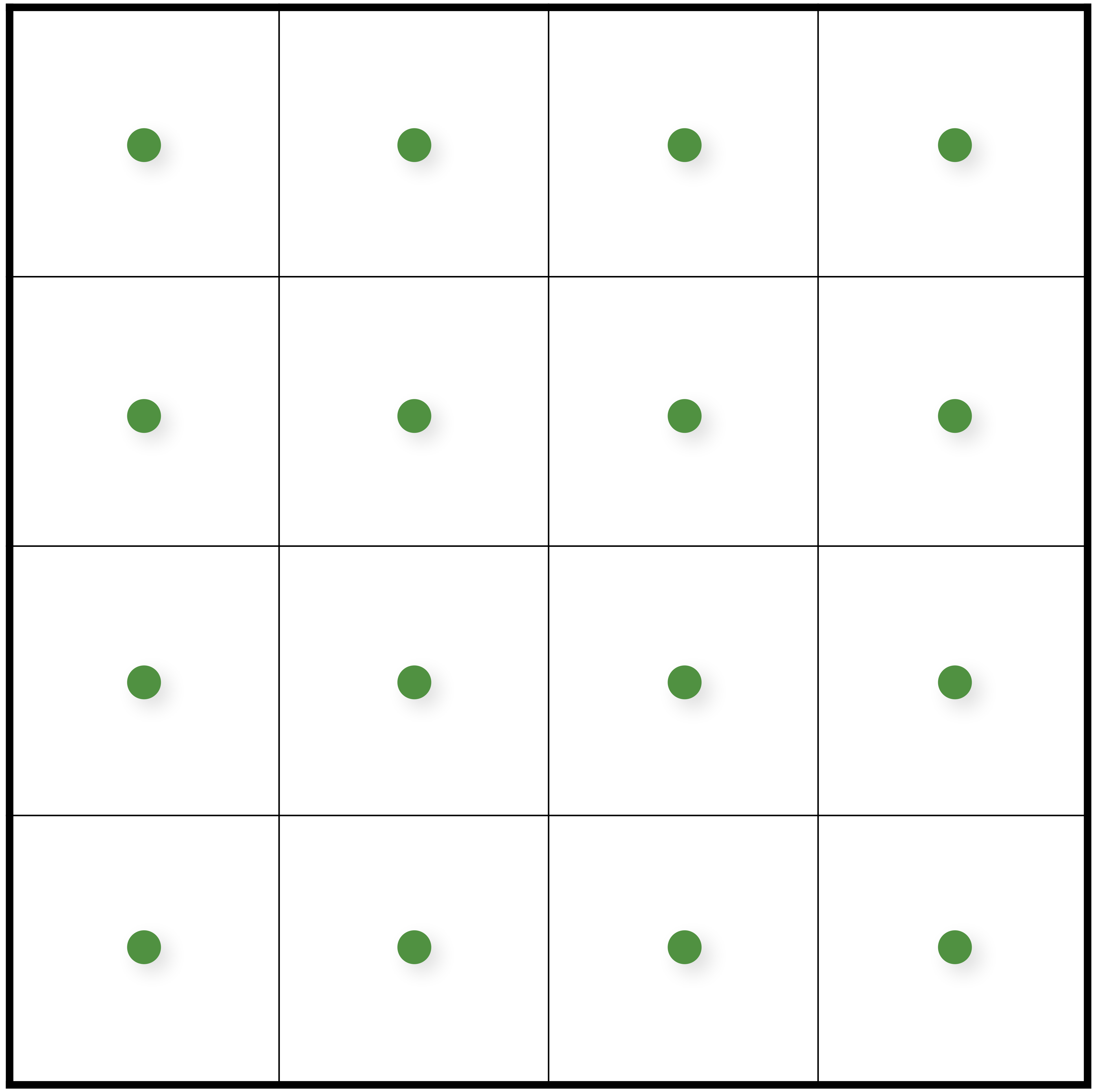

- ✔ Extends to higher dimensions, but…
- ✘ Curse of dimensionality
- ✘ Aliasing

Regular Sampling

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for (uint i = 0; i < numX; i++)
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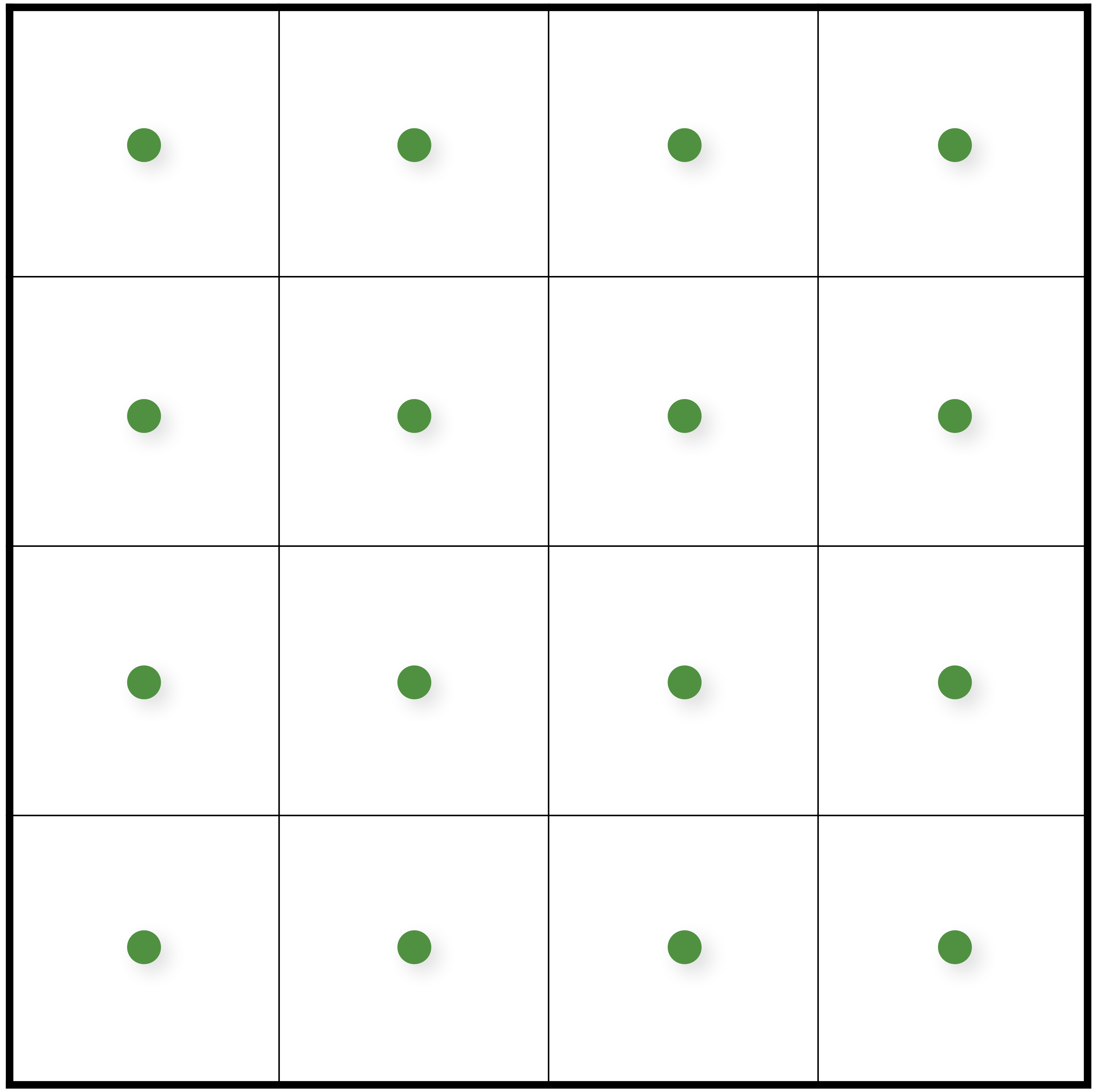

Jittered/Stratified Sampling

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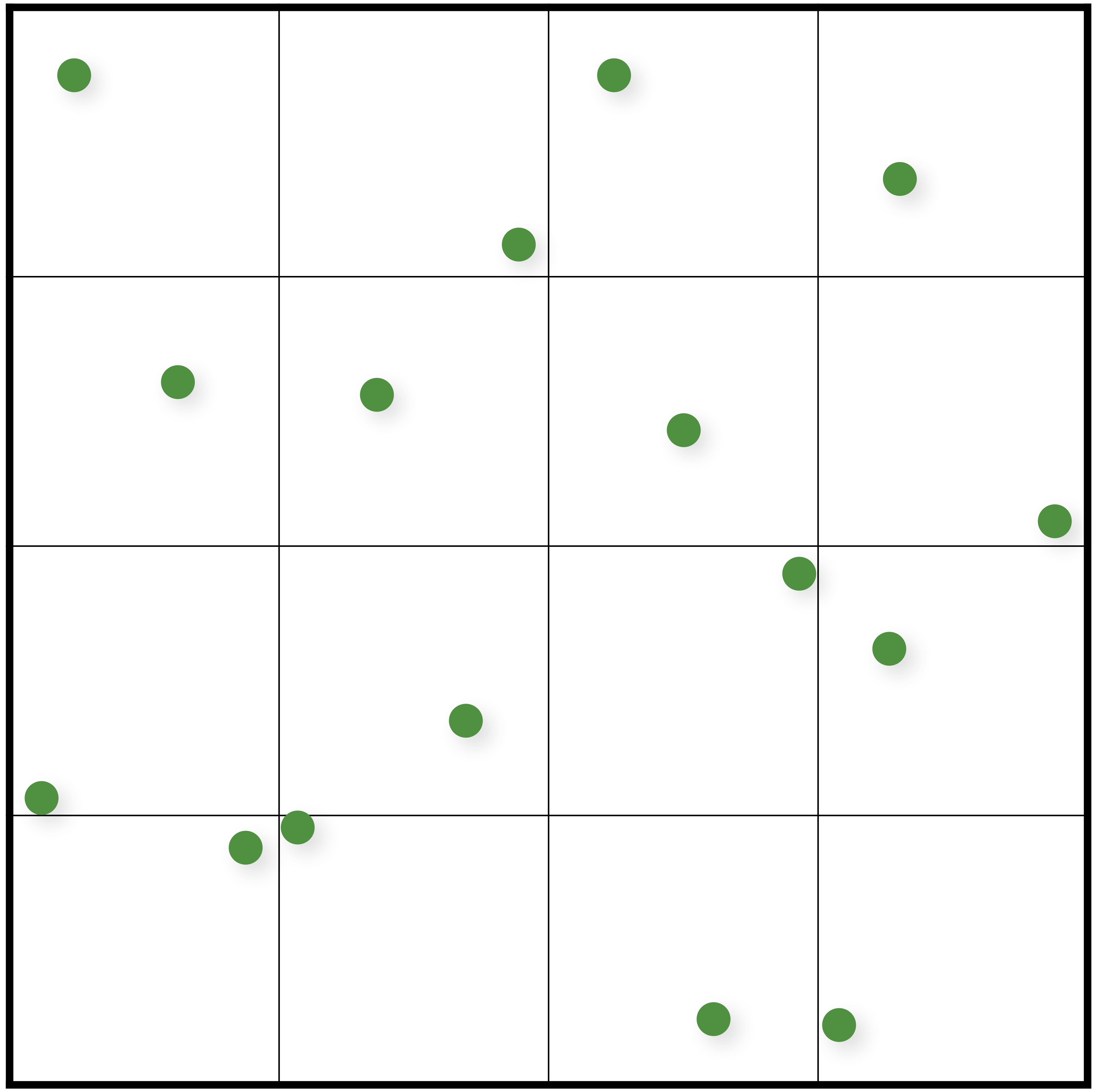

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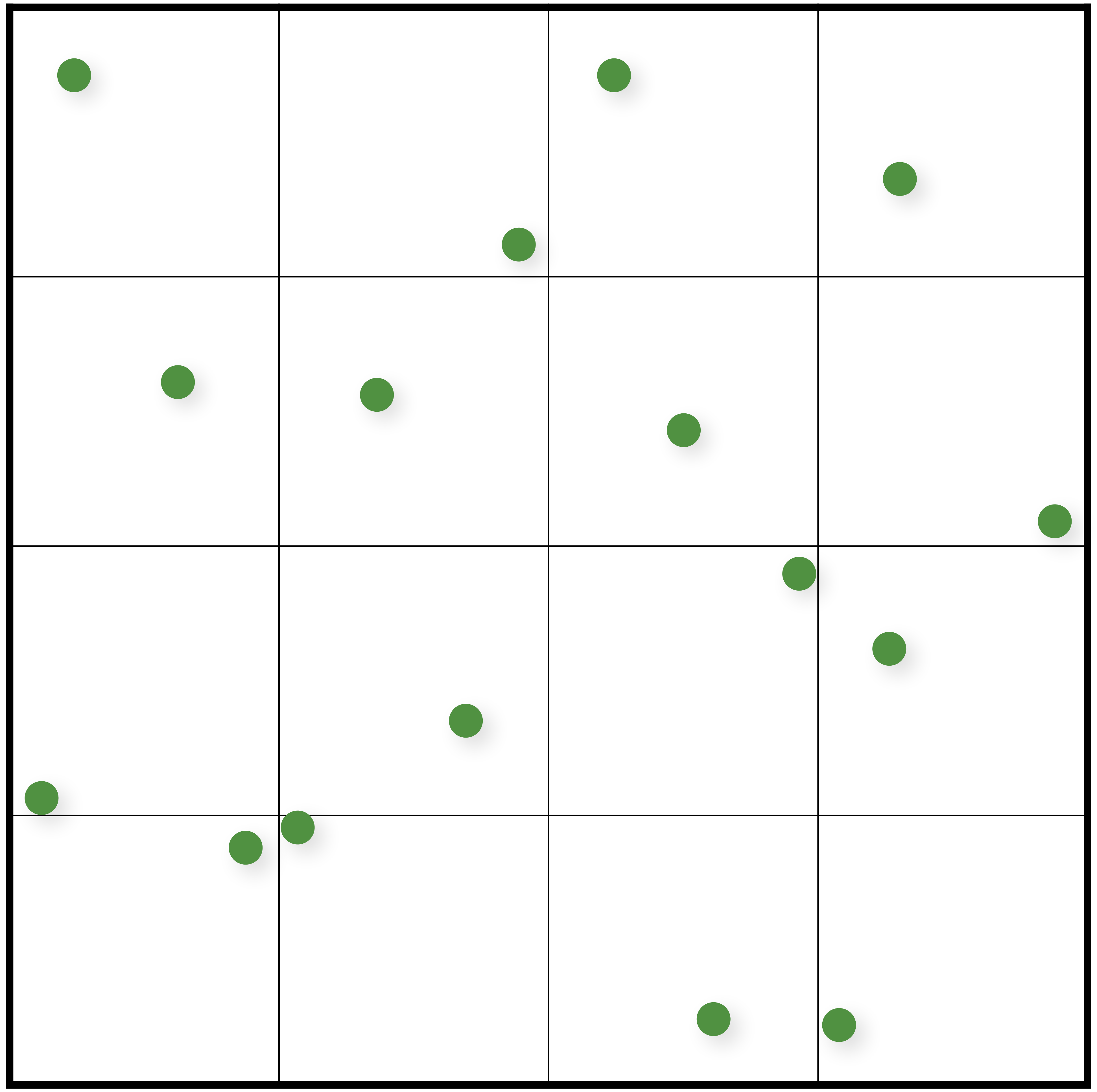
✔ Provably cannot increase variance



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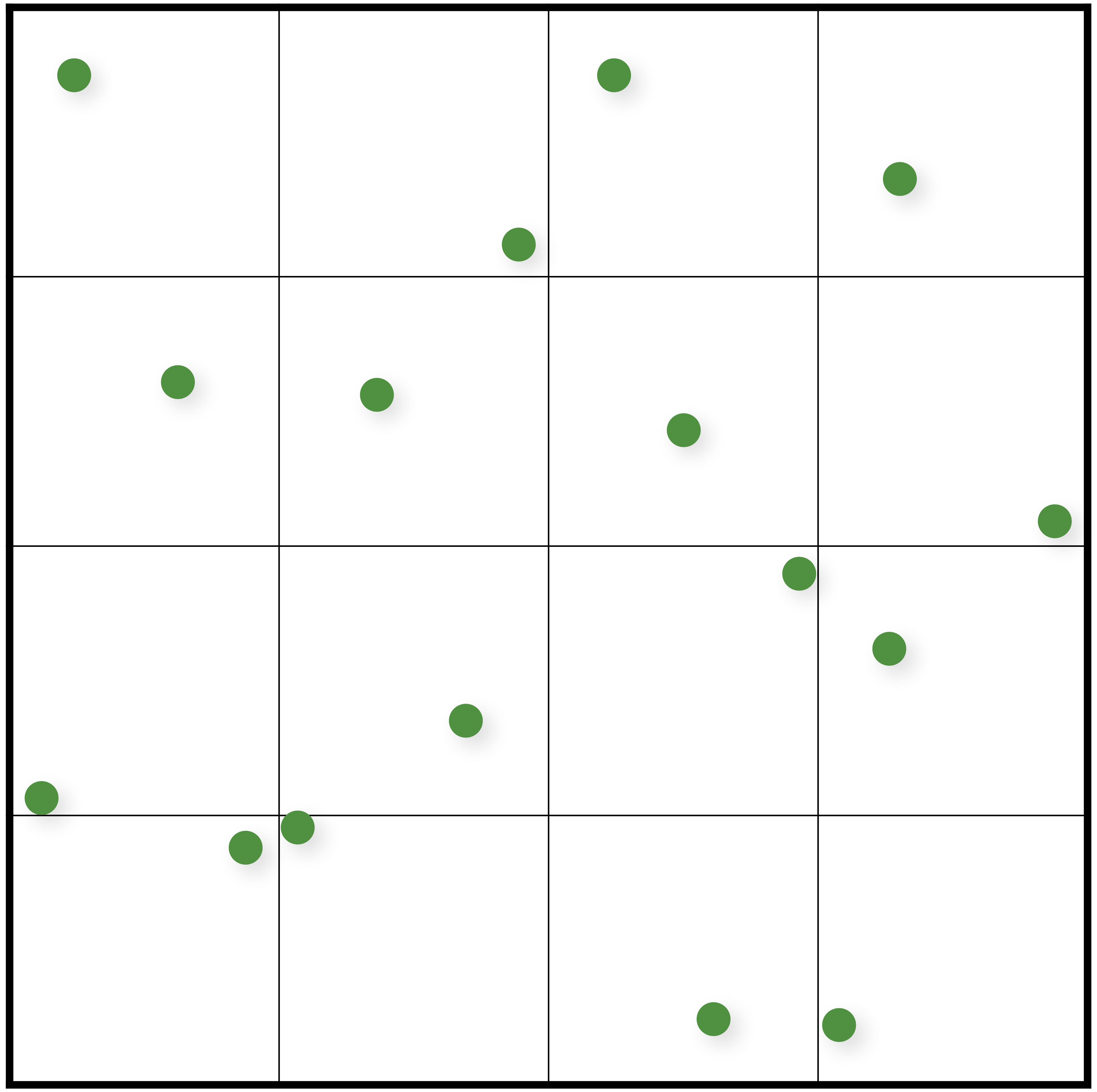

- ✔ Provably cannot increase variance
- ✔ Extends to higher dimensions, but…

Jittered/Stratified Sampling

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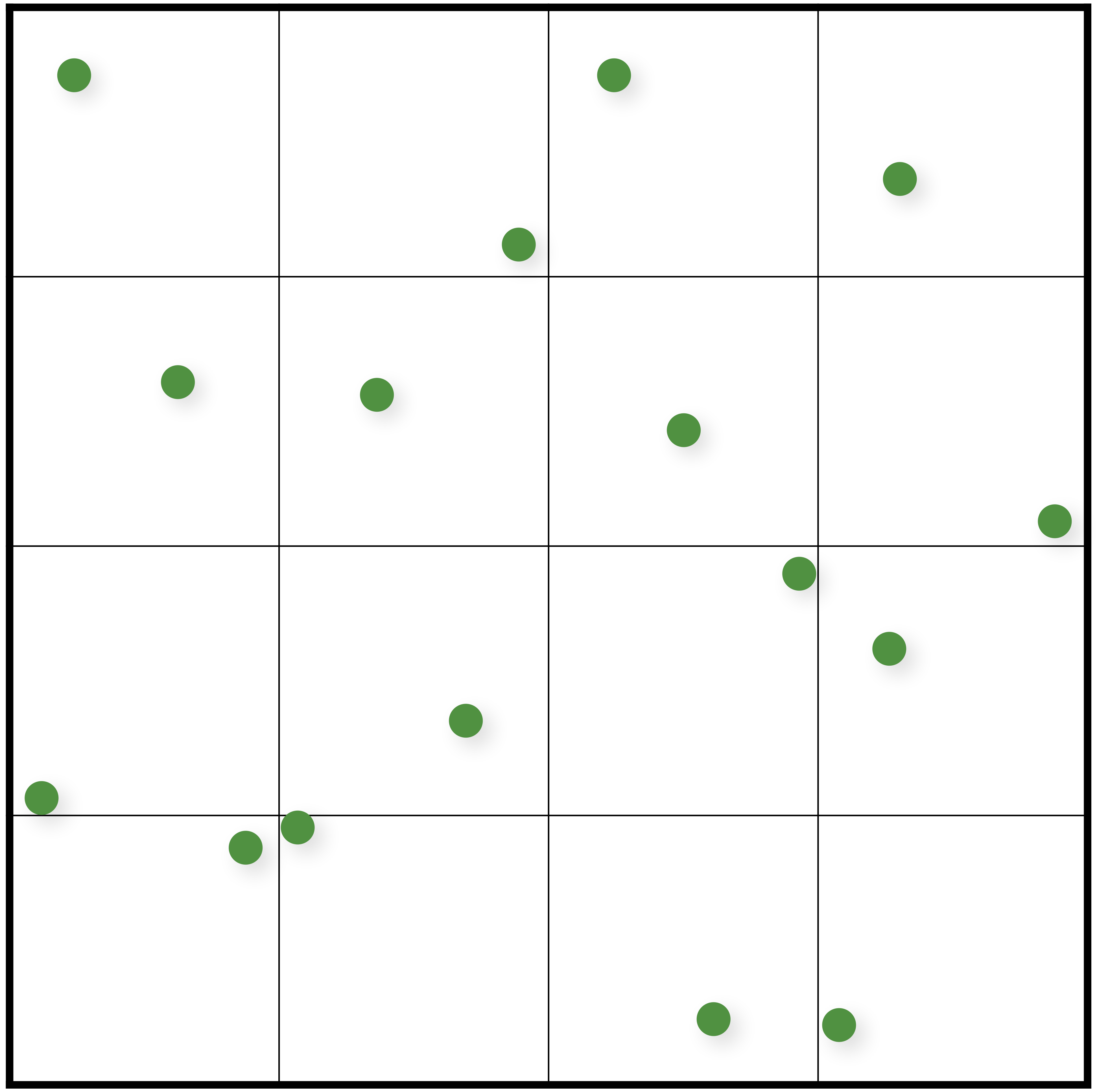

- ✔ Provably cannot increase variance
- ✔ Extends to higher dimensions, but…
- ✘ Curse of dimensionality

Jittered/Stratified Sampling

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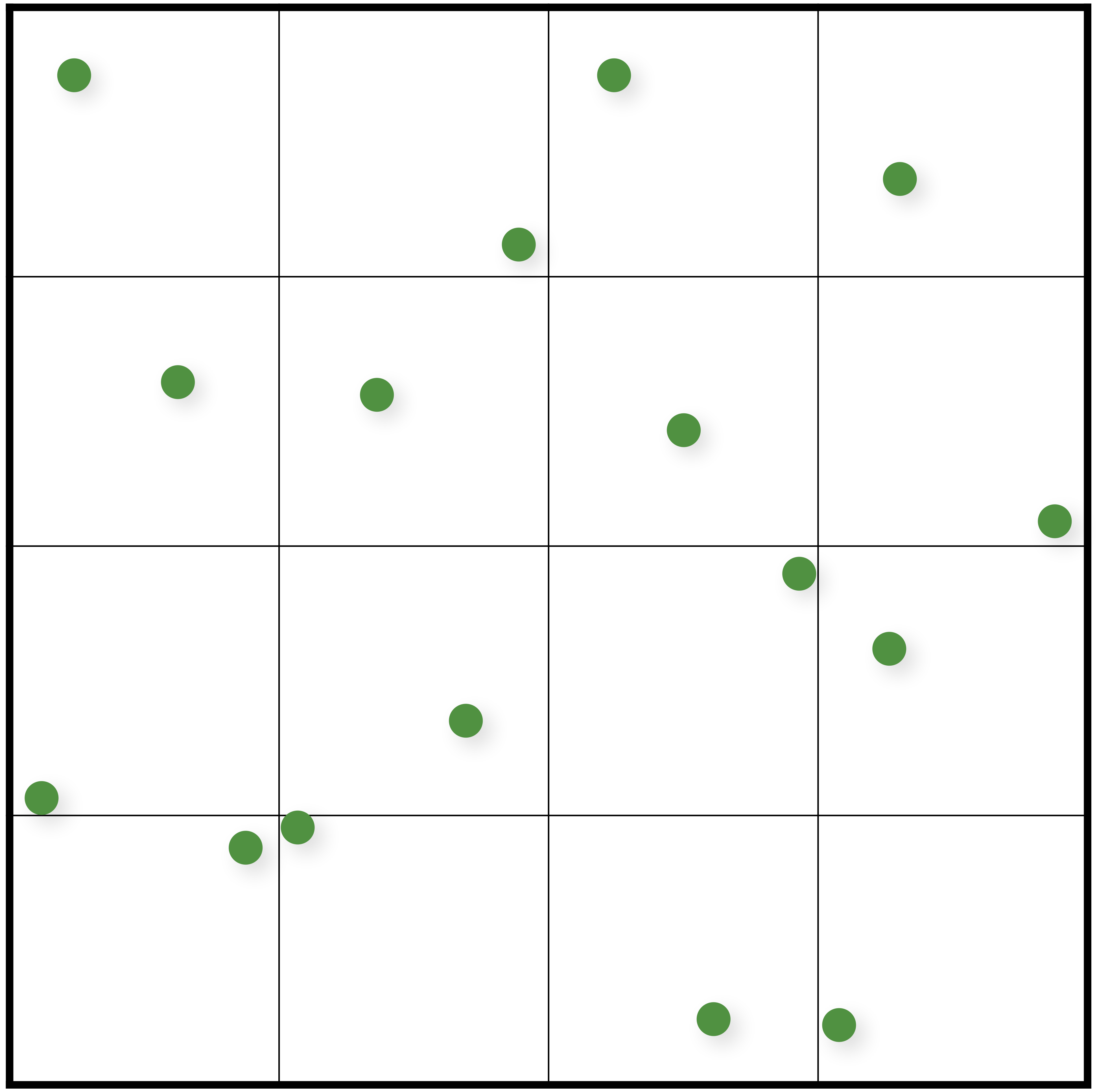
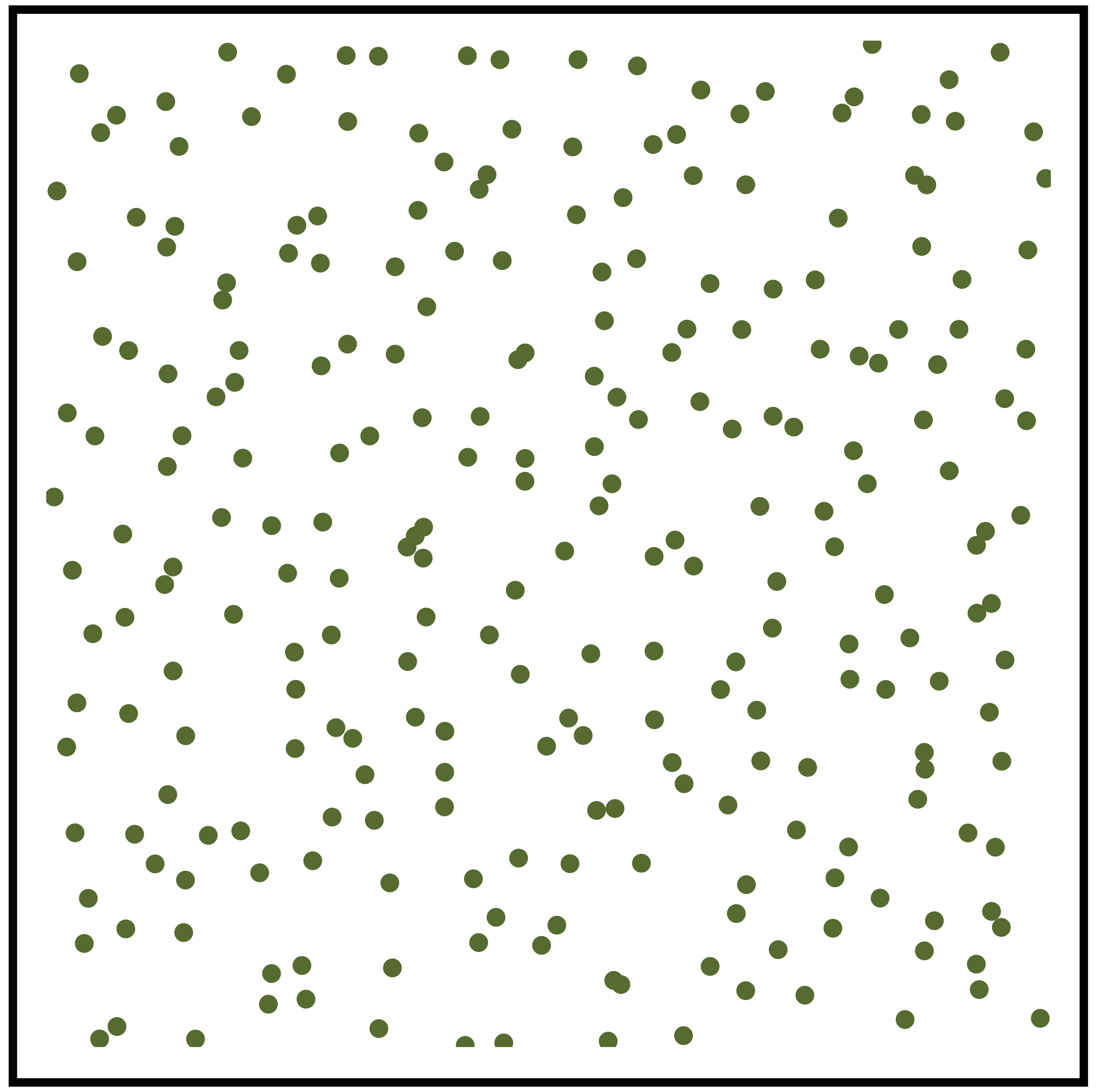


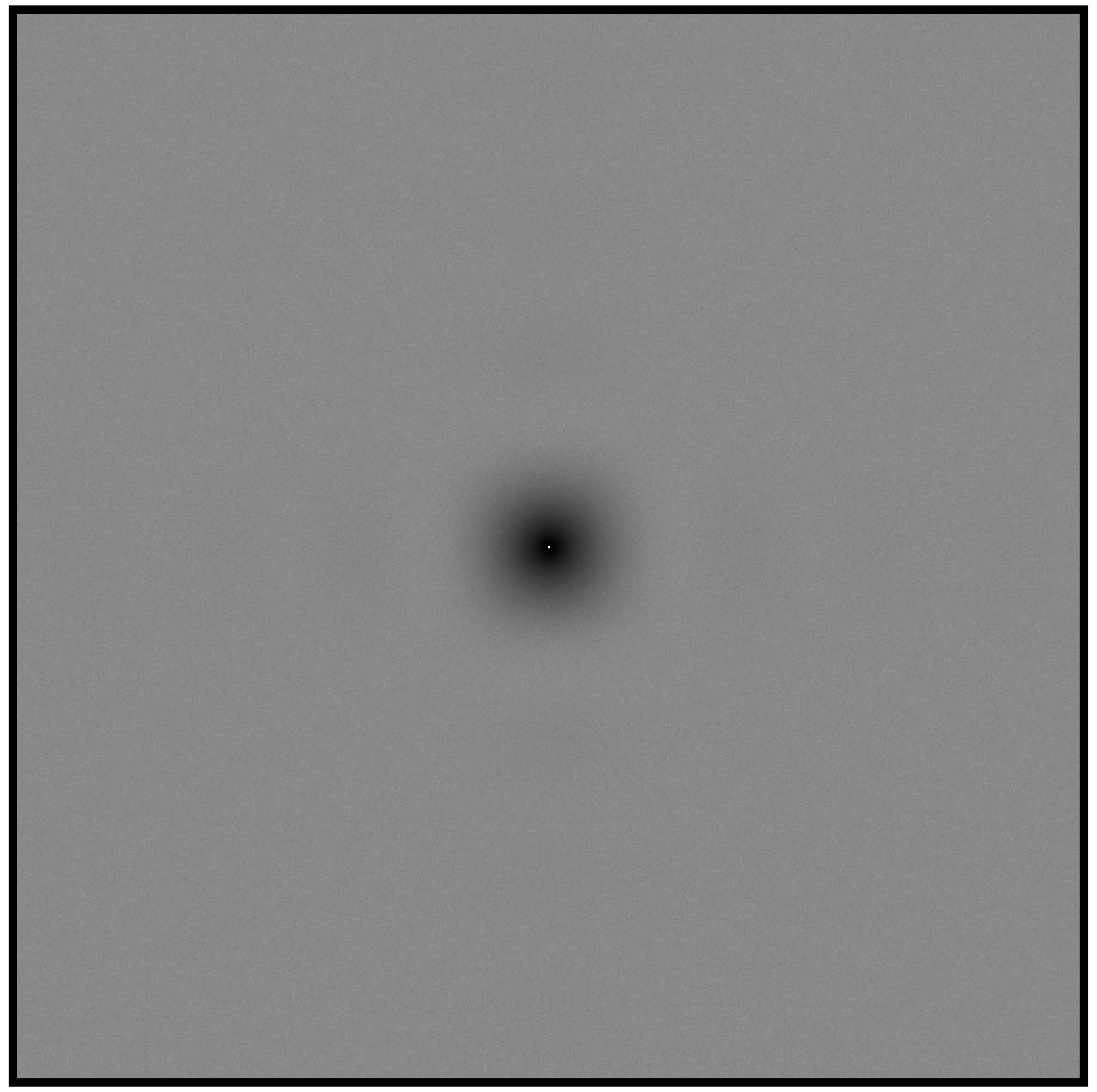
- ✔ Provably cannot increase variance
- ✔ Extends to higher dimensions, but…
- ✘ Curse of dimensionality
- ✘ Not progressive

Jittered Sampling

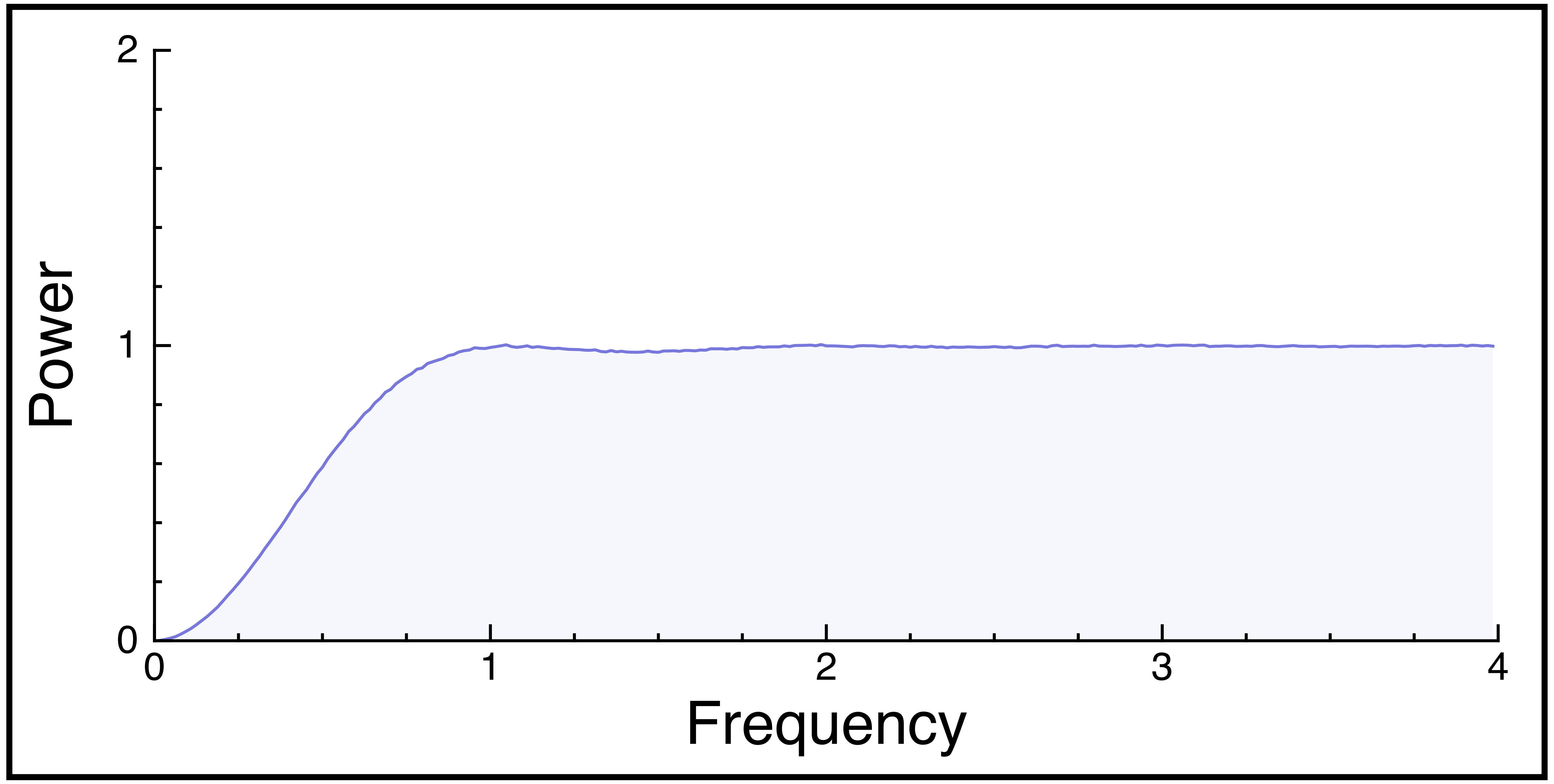
Samples



Expected power spectrum

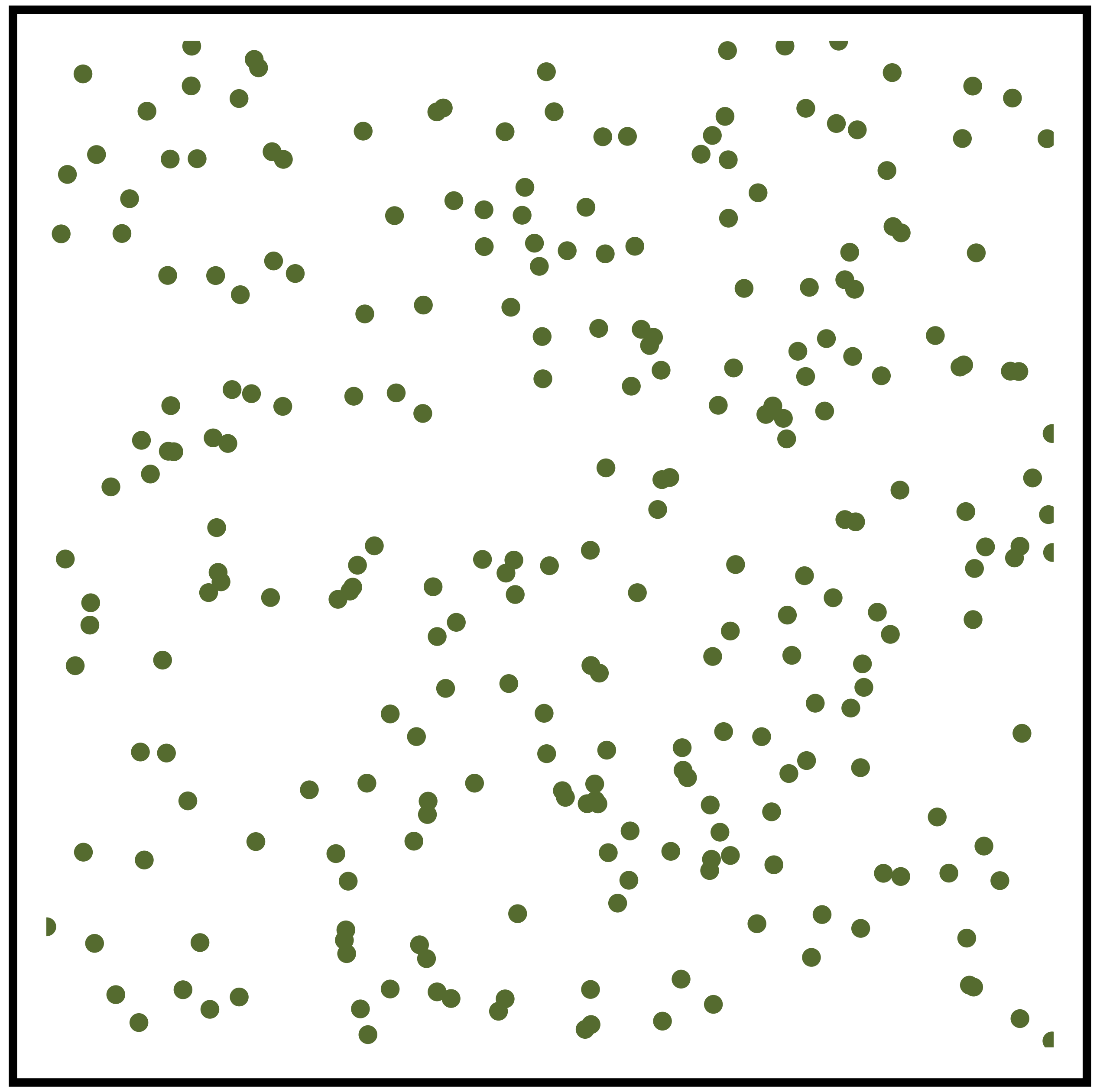


Radial mean

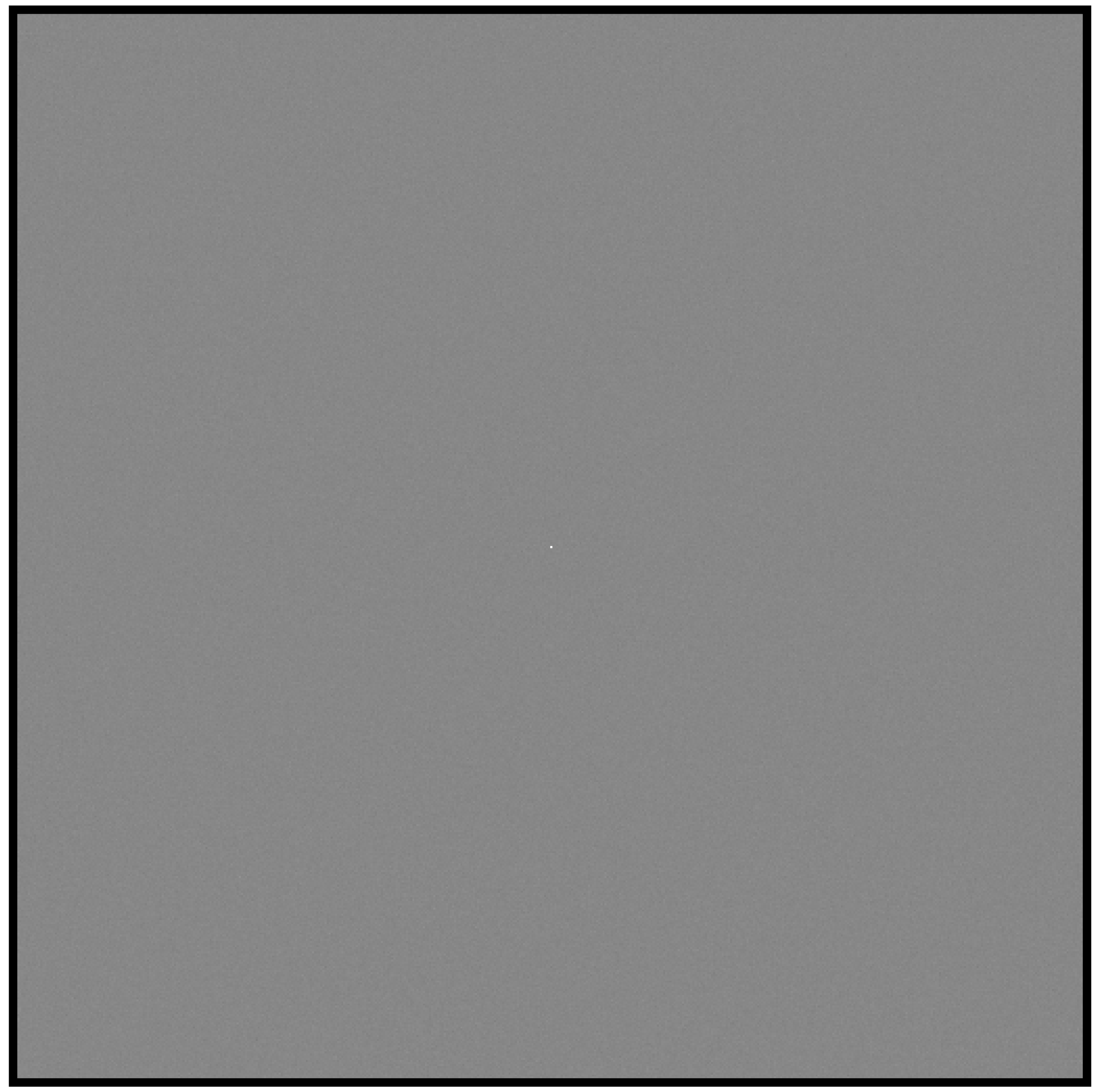


Independent Random Sampling

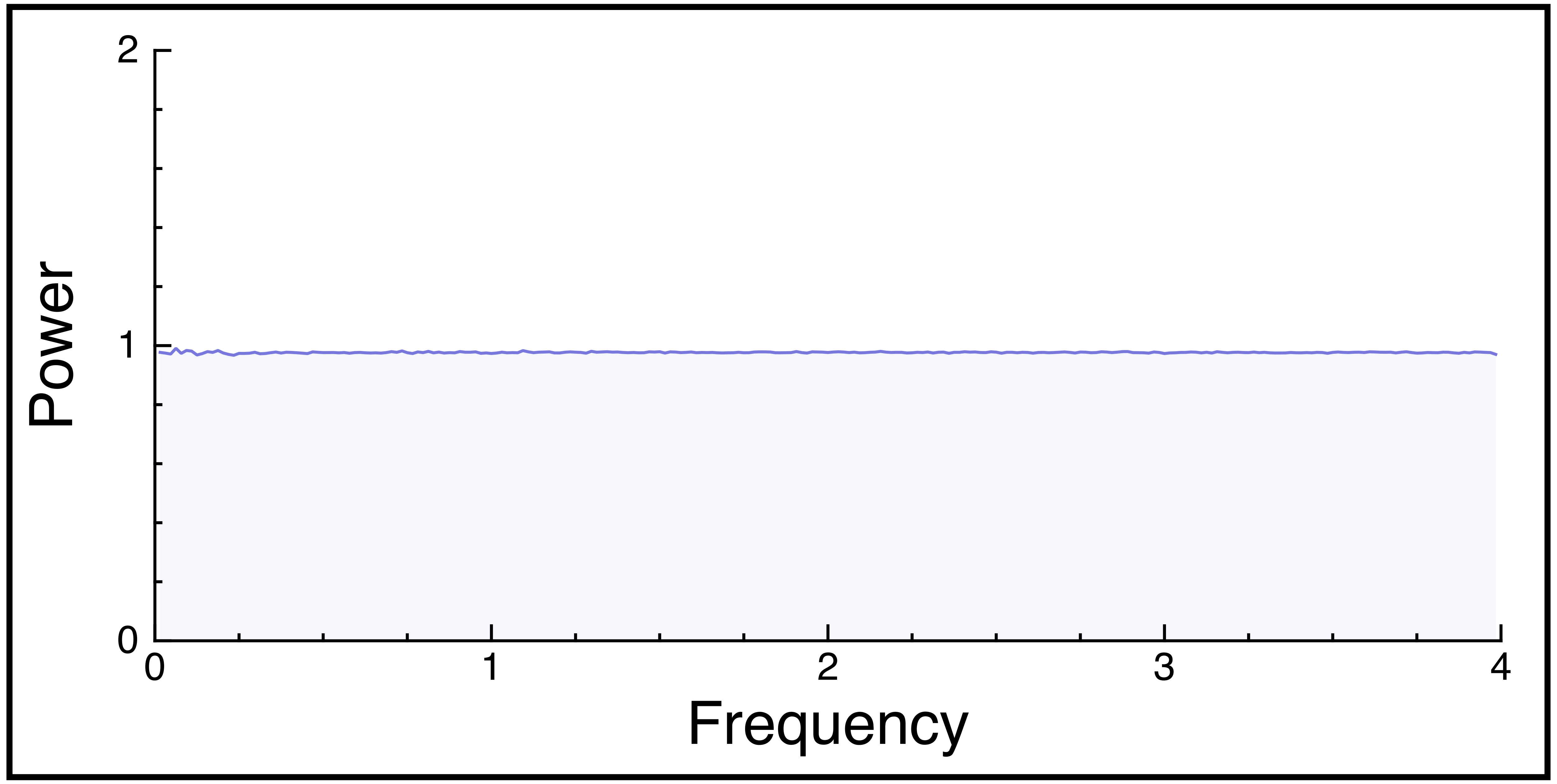
Samples



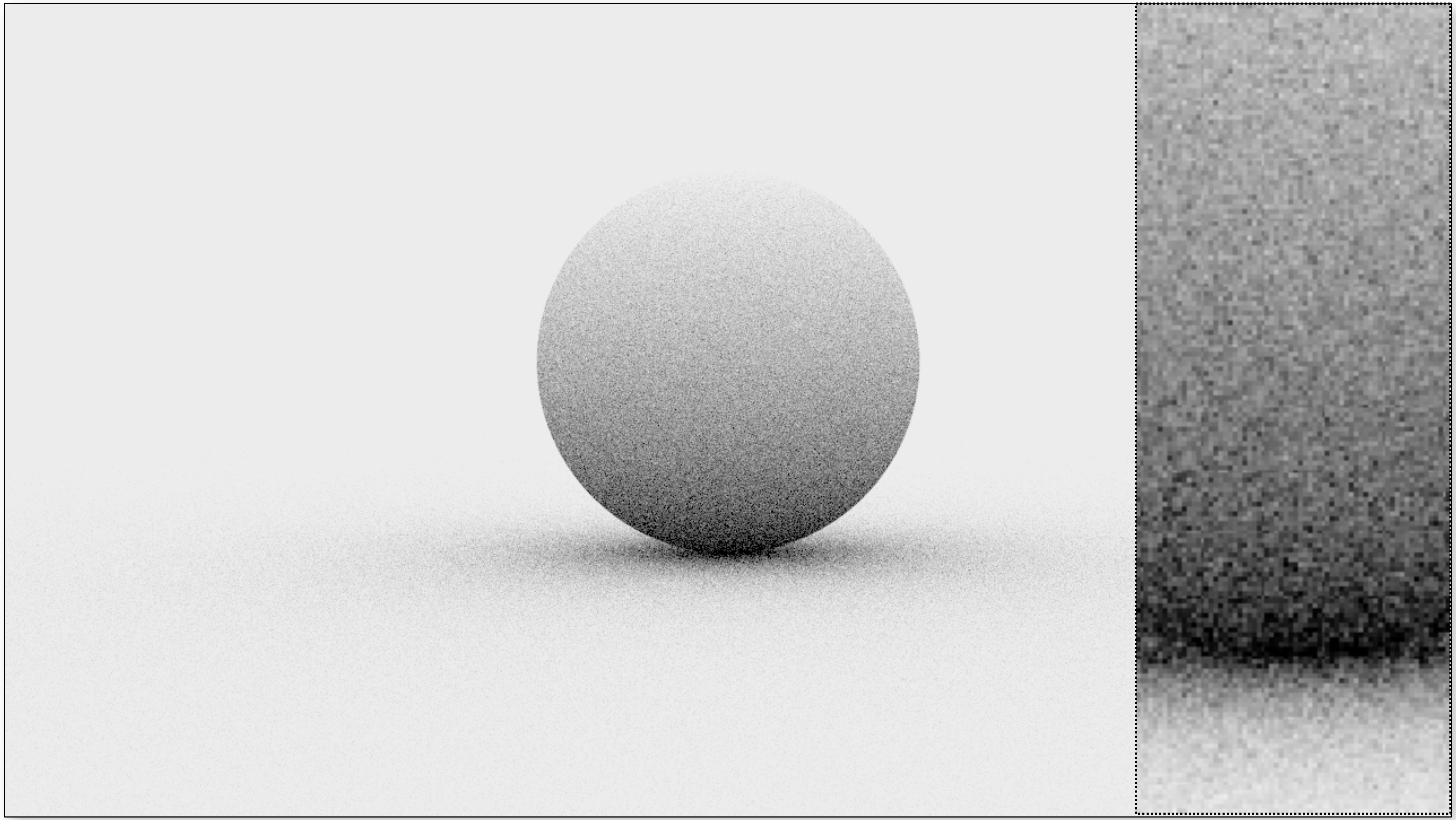
Expected power spectrum



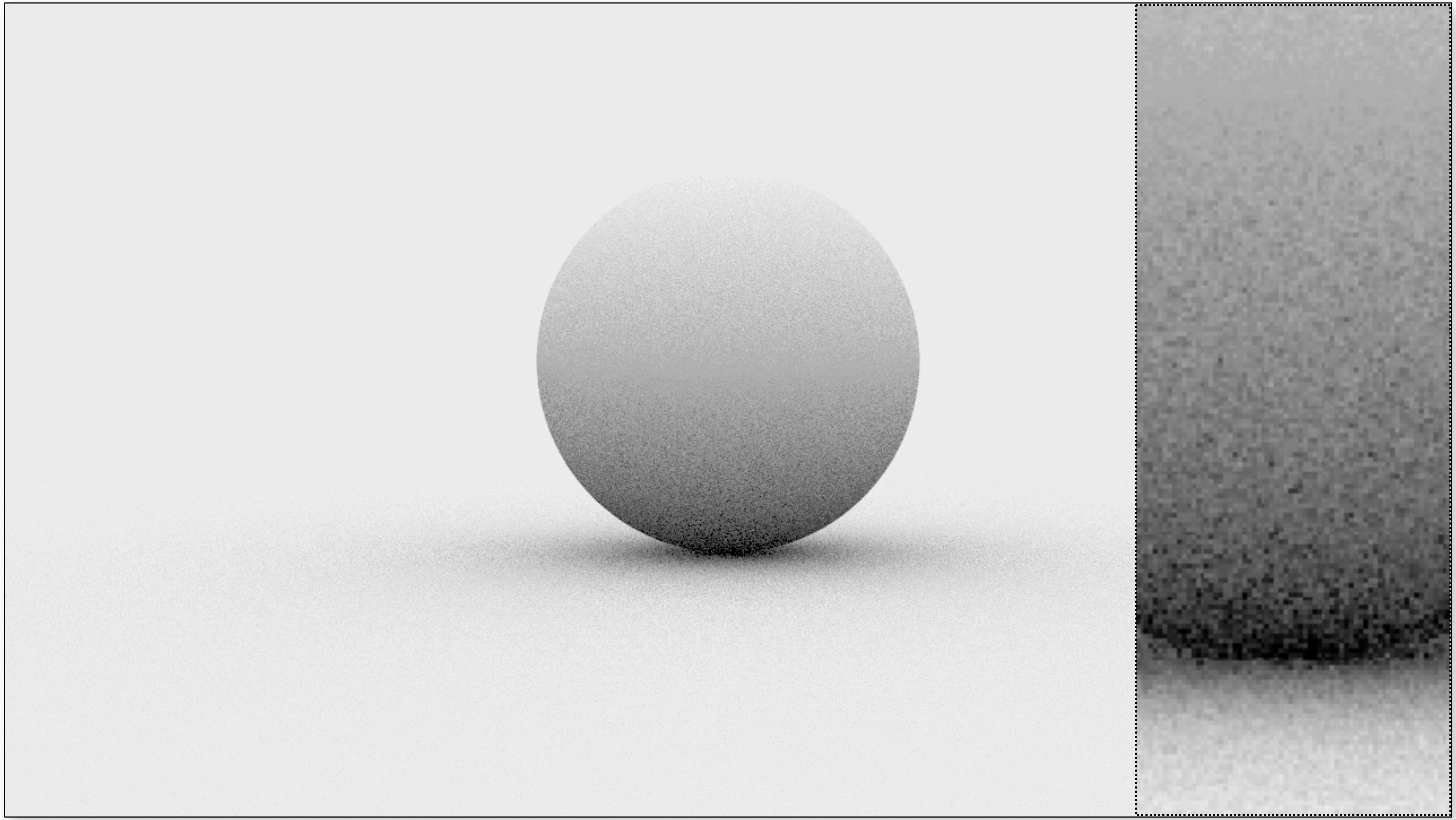
Radial mean



Monte Carlo (16 random samples)



Monte Carlo (16 jittered samples)



Stratifying in Higher Dimensions

Stratification requires O(N^d) samples

- e.g. pixel (2D) + lens (2D) + time (1D) = 5D

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  - splitting 2 times in 5D = 2^5 = 32 samples
  - splitting 3 times in 5D = 3^5 = 243 samples!

Stratifying in Higher Dimensions

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  - splitting 2 times in 5D = 2^5 = 32 samples
  - splitting 3 times in 5D = 3^5 = 243 samples!

Inconvenient for large d

- cannot select sample count with fine granularity

Uncorrelated Jitter [Cook et al. 84]

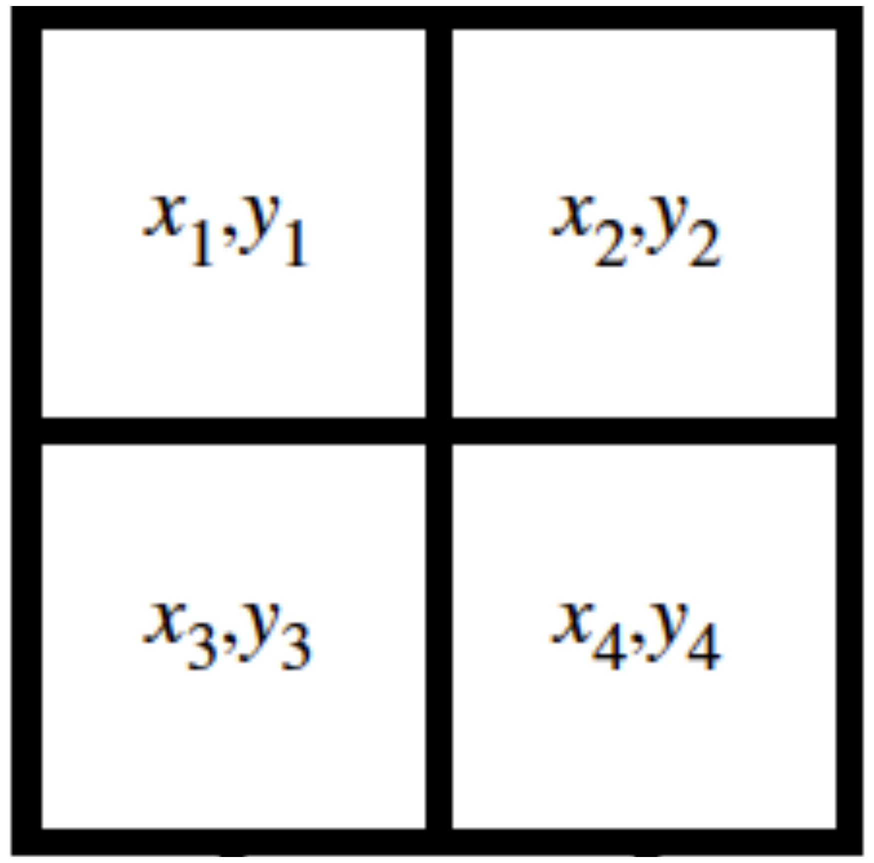
Uncorrelated Jitter [Cook et al. 84]

Compute stratified samples in sub-dimensions

Uncorrelated Jitter [Cook et al. 84]

Compute stratified samples in sub-dimensions

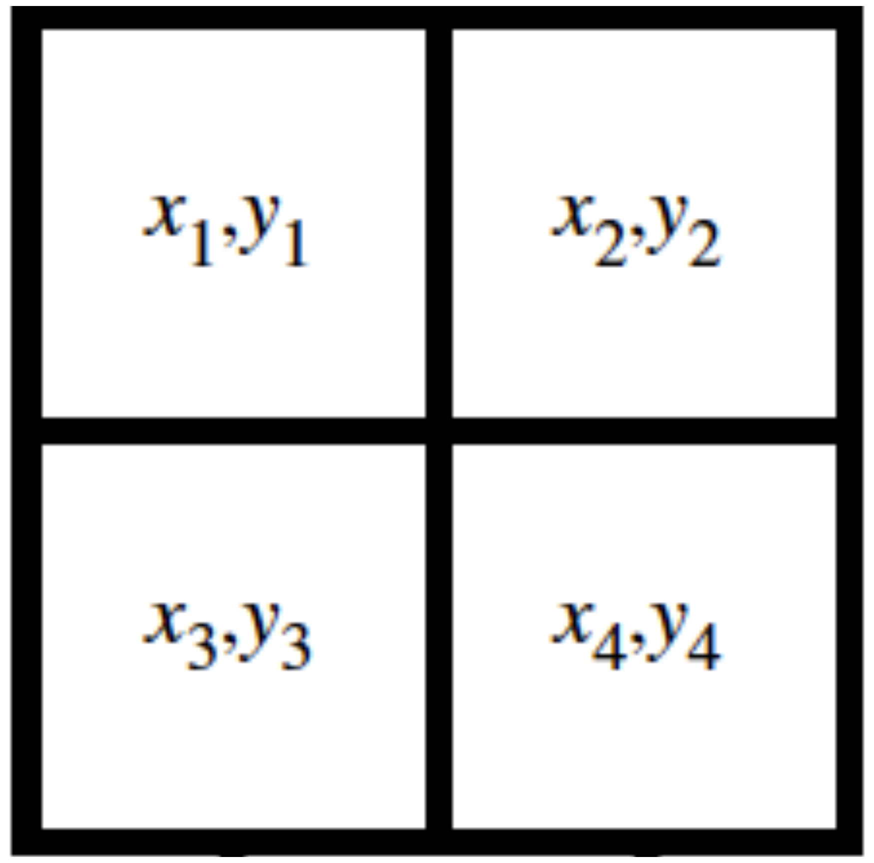
- 2D jittered (x,y) for pixel

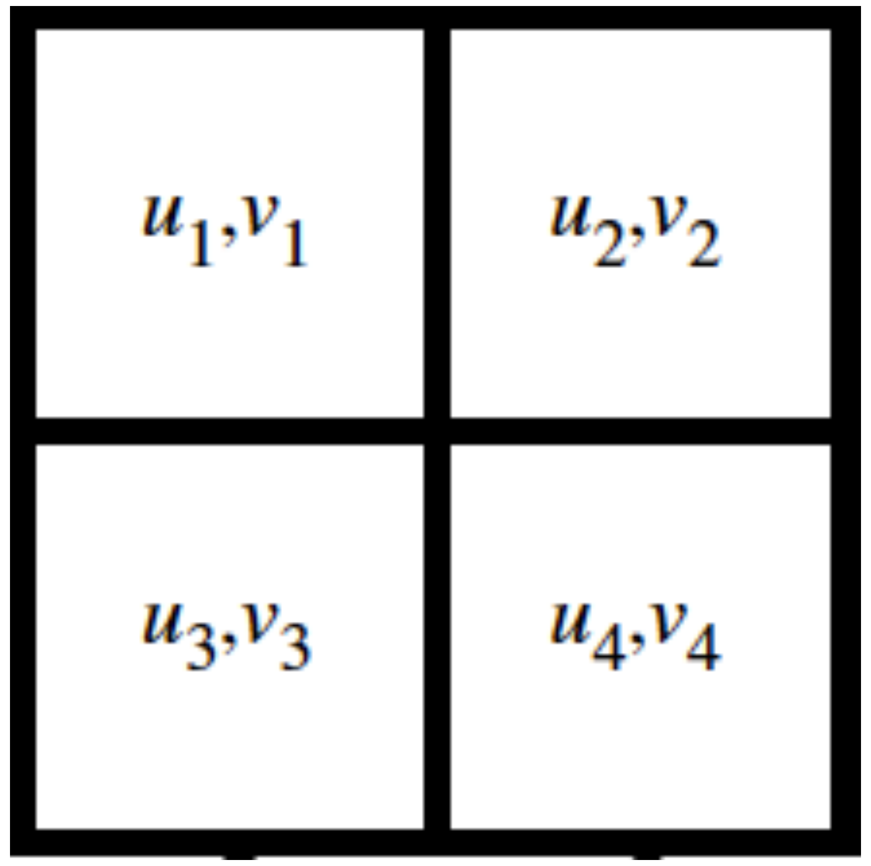


Uncorrelated Jitter [Cook et al. 84]

Compute stratified samples in sub-dimensions

- 2D jittered (x,y) for pixel
- 2D jittered (u,v) for lens

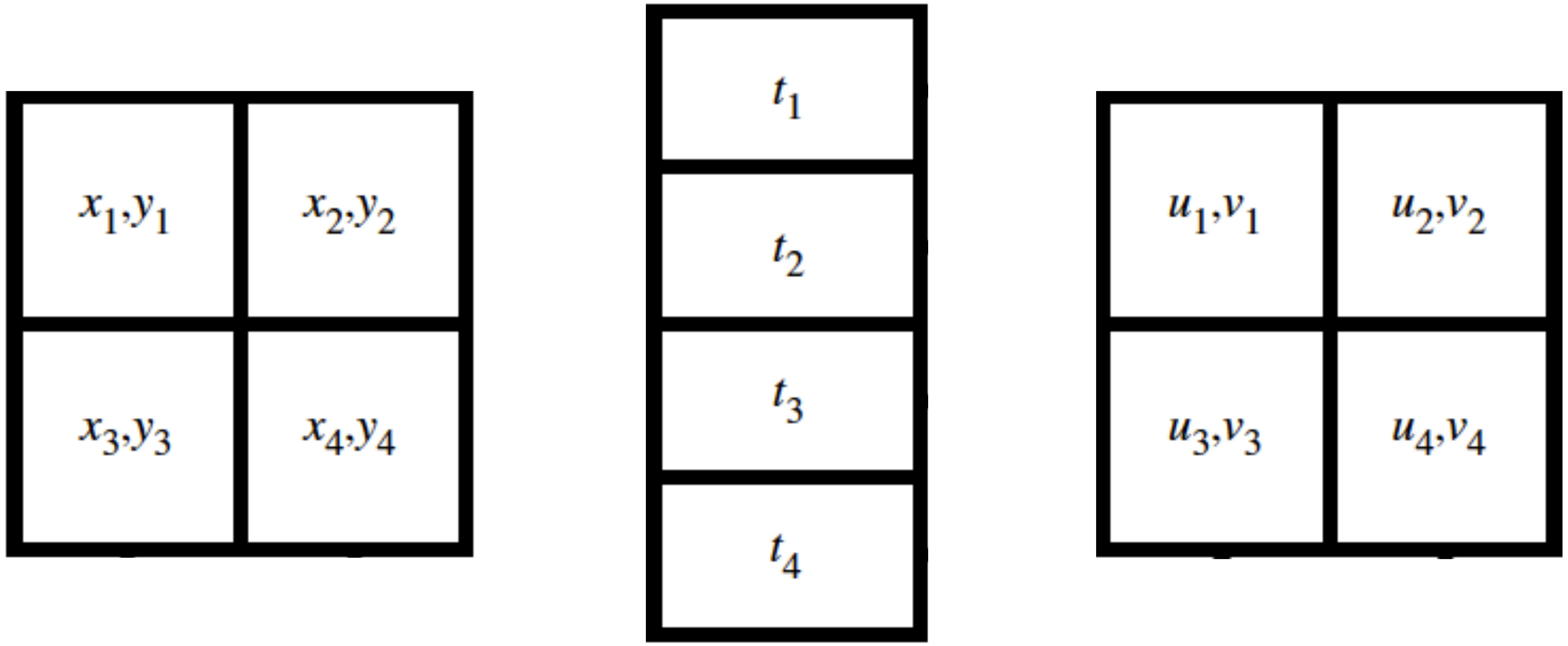




Uncorrelated Jitter [Cook et al. 84]

Compute stratified samples in sub-dimensions

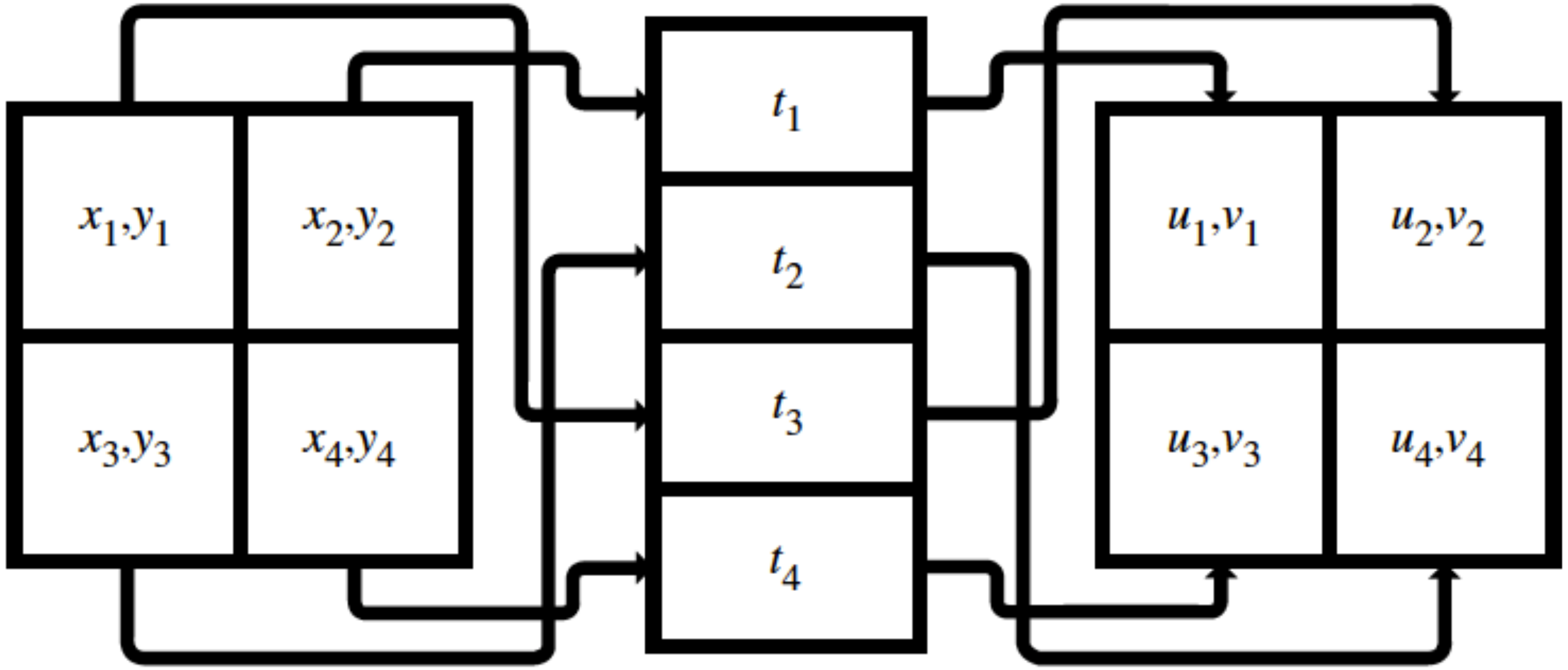
- 2D jittered (x,y) for pixel
- 2D jittered (u,v) for lens
- 1D jittered (t) for time



Uncorrelated Jitter [Cook et al. 84]

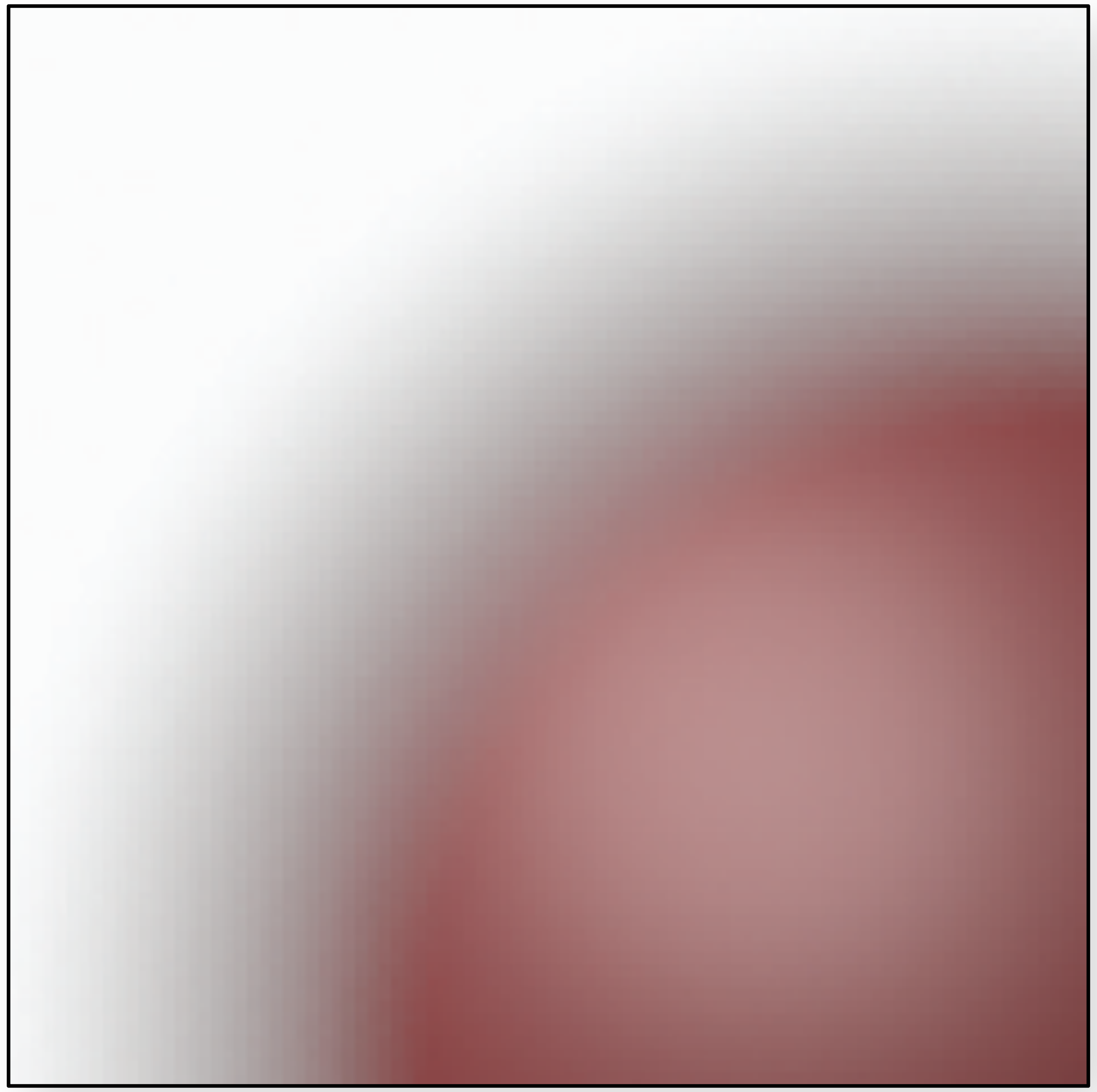
Compute stratified samples in sub-dimensions

- 2D jittered (x,y) for pixel
- 2D jittered (u,v) for lens
- 1D jittered (t) for time
- combine dimensions in random order

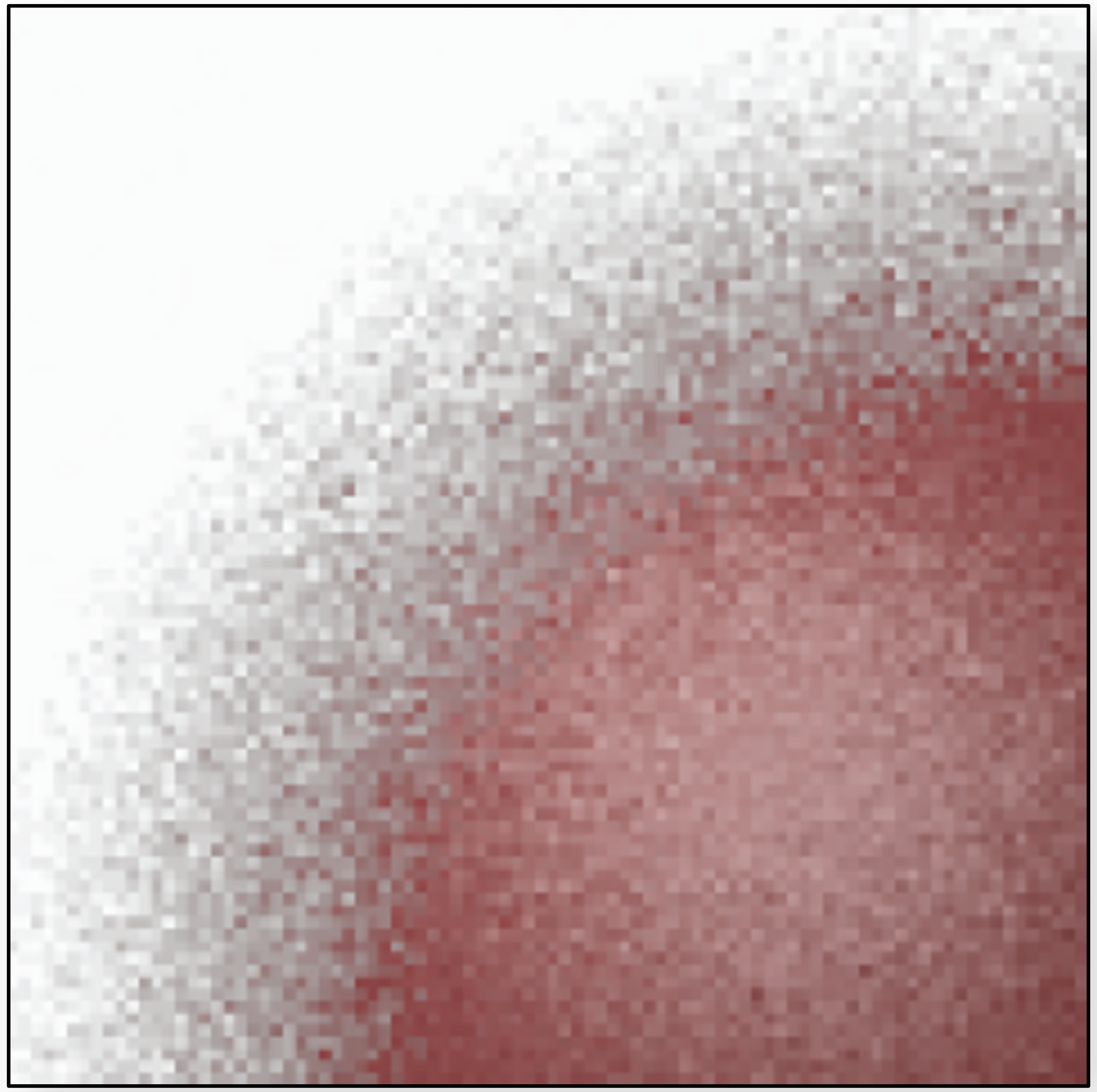


Depth of Field (4D)

Reference



Random Sampling



Uncorrelated Jitter

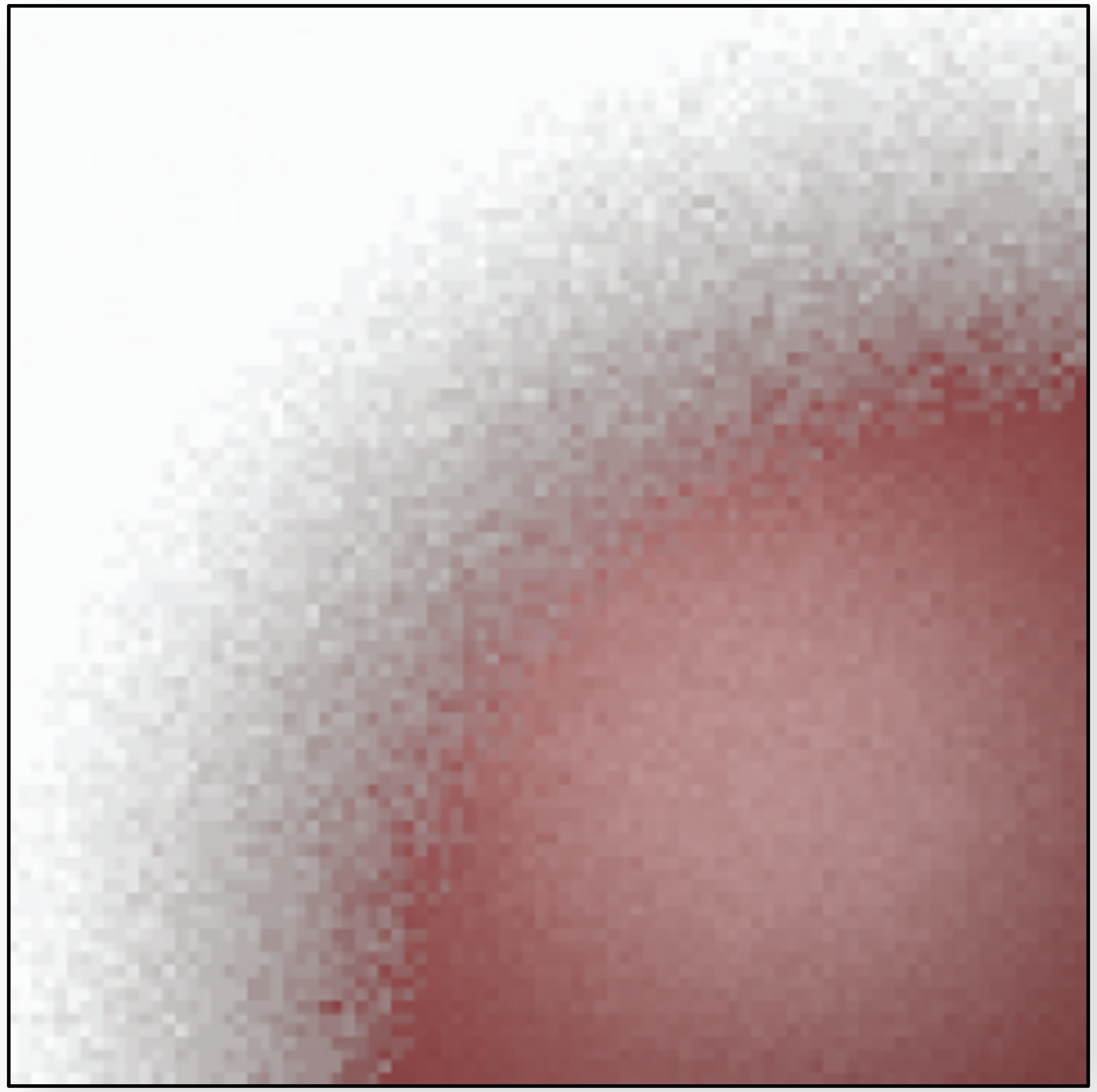


Image source: PBRTe2 [Pharr & Humphreys 2010]

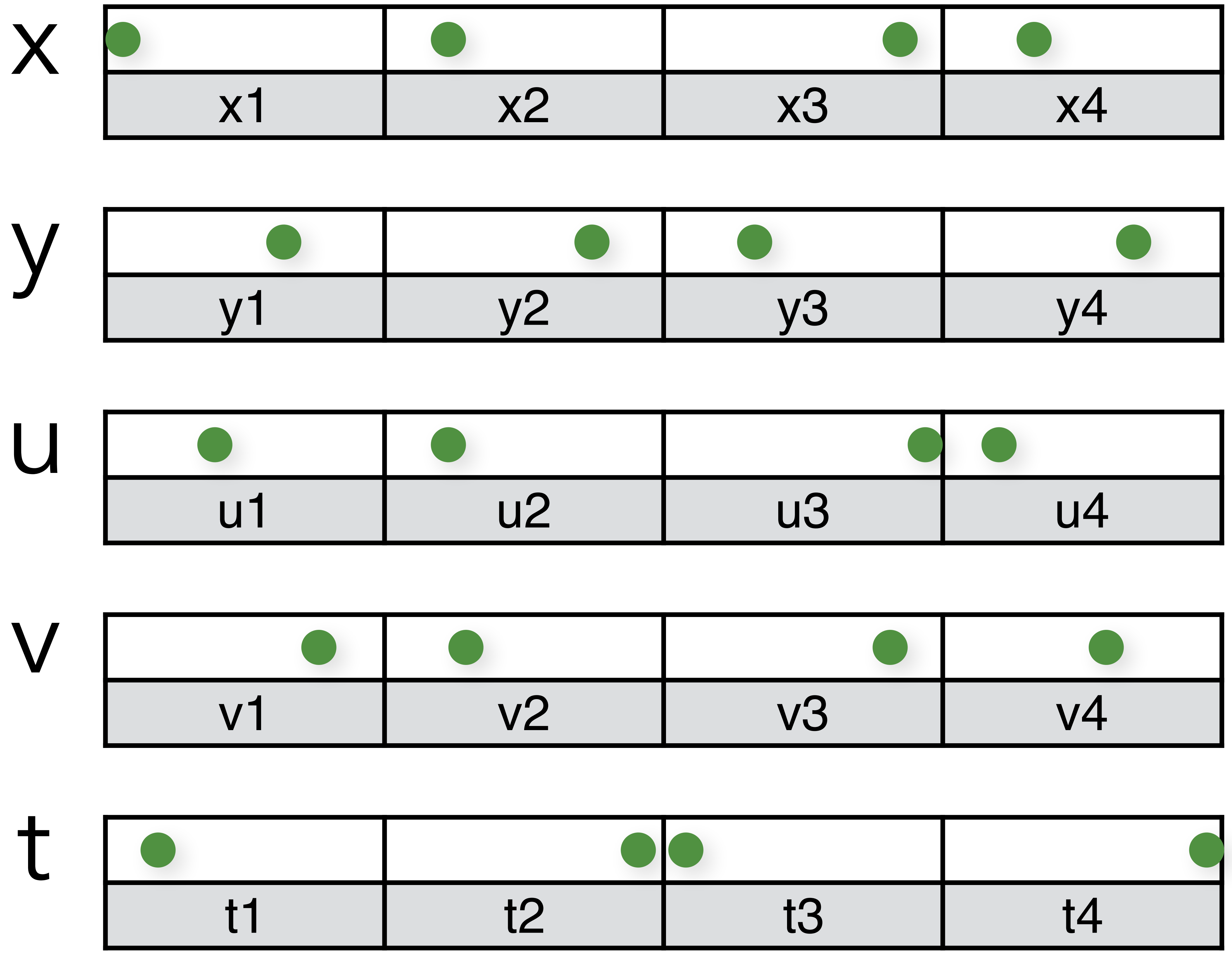
Uncorrelated Jitter → Latin Hypercube

Stratify samples in each dimension separately

Uncorrelated Jitter → Latin Hypercube

Stratify samples in each dimension separately

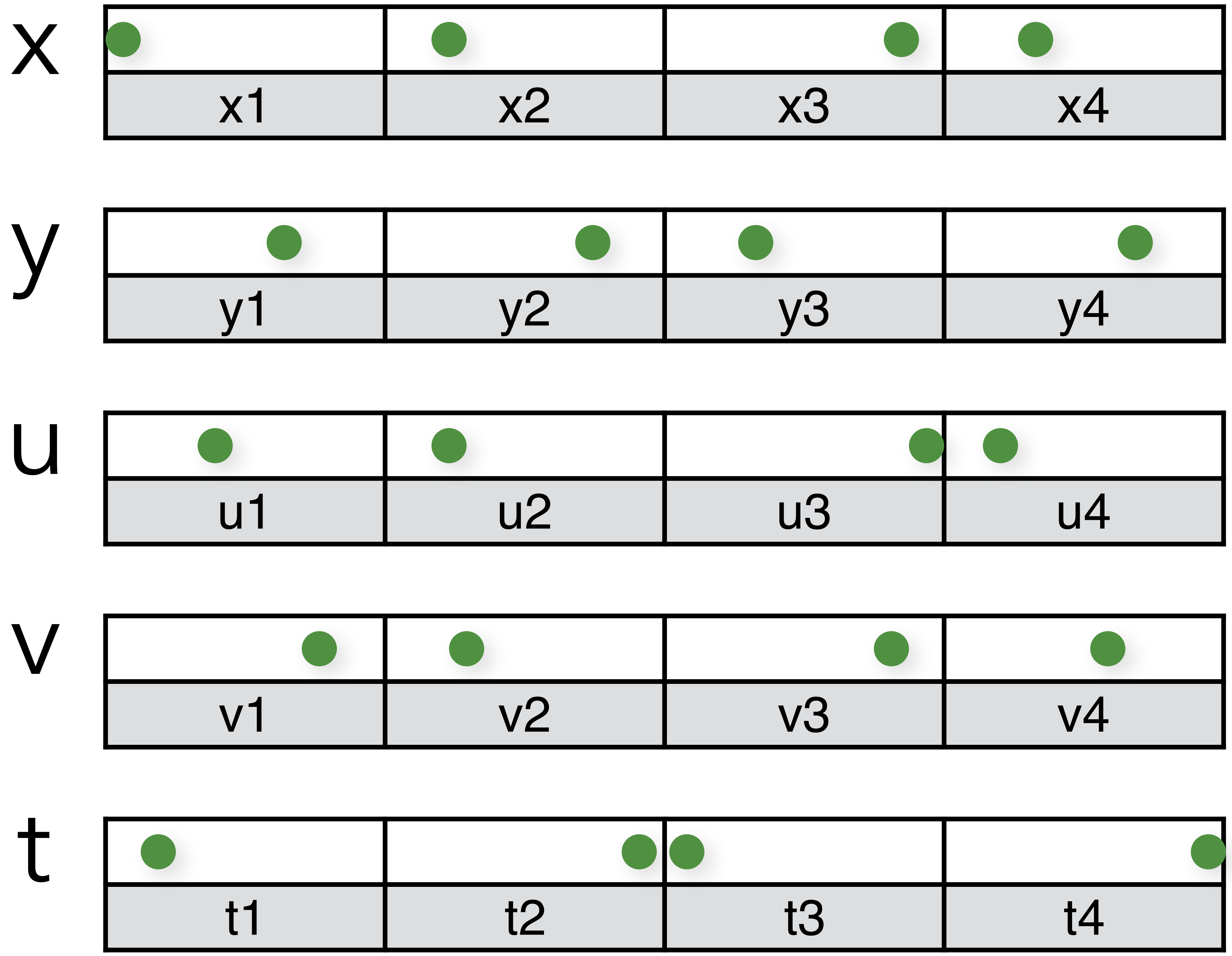
- for 5D: 5 separate 1D jittered point sets



Uncorrelated Jitter → Latin Hypercube

Stratify samples in each dimension separately

- for 5D: 5 separate 1D jittered point sets
- combine dimensions in random order



Uncorrelated Jitter → Latin Hypercube

Stratify samples in each dimension separately

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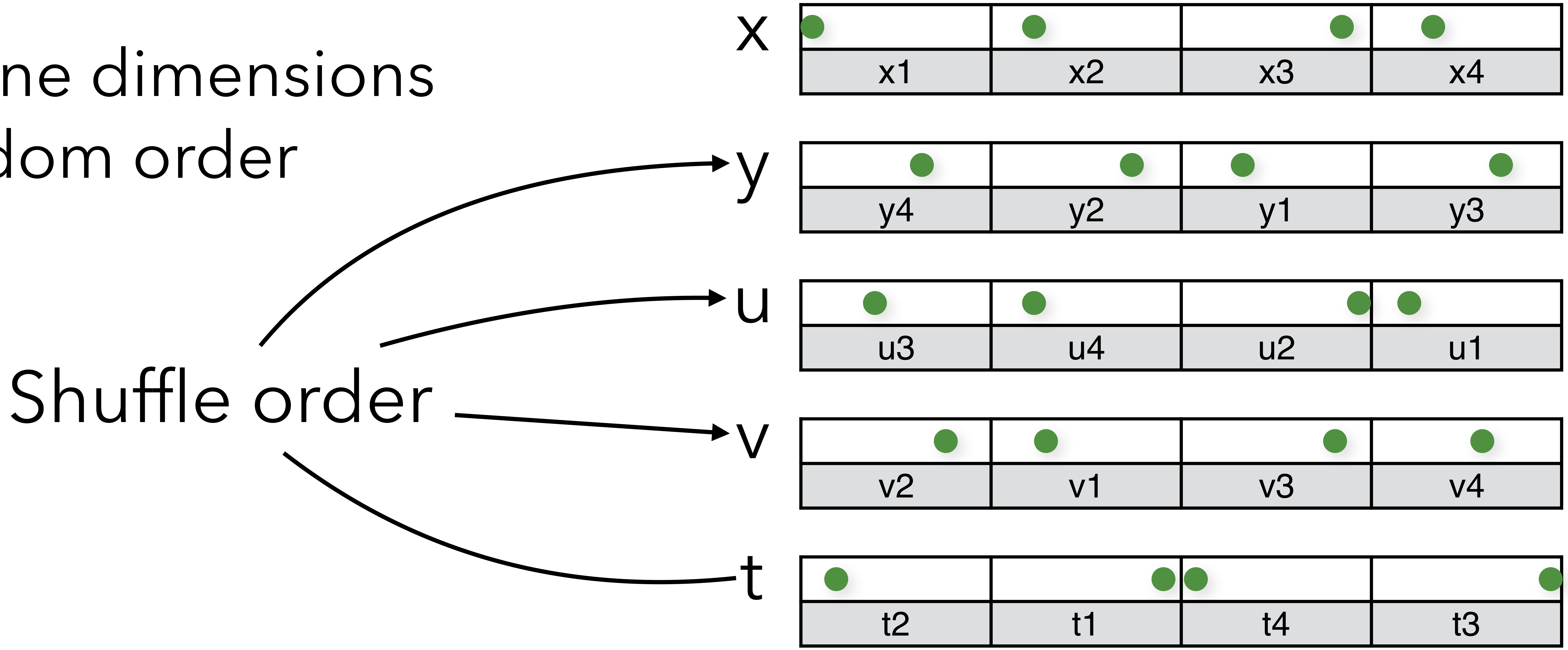
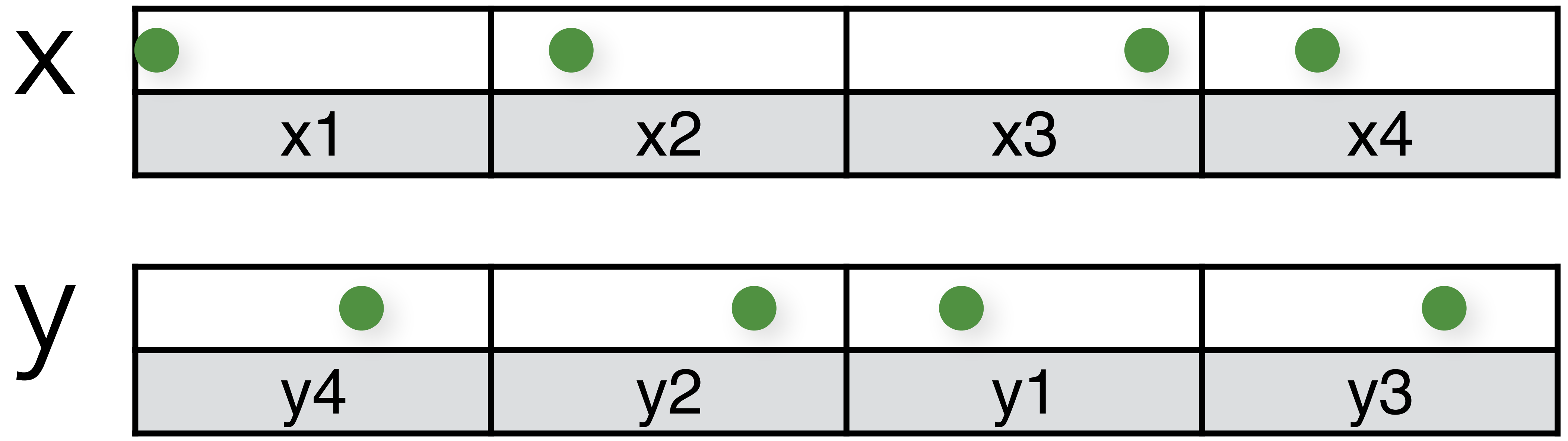


N-Rooks = 2D Latin Hypercube [Shirley 91]

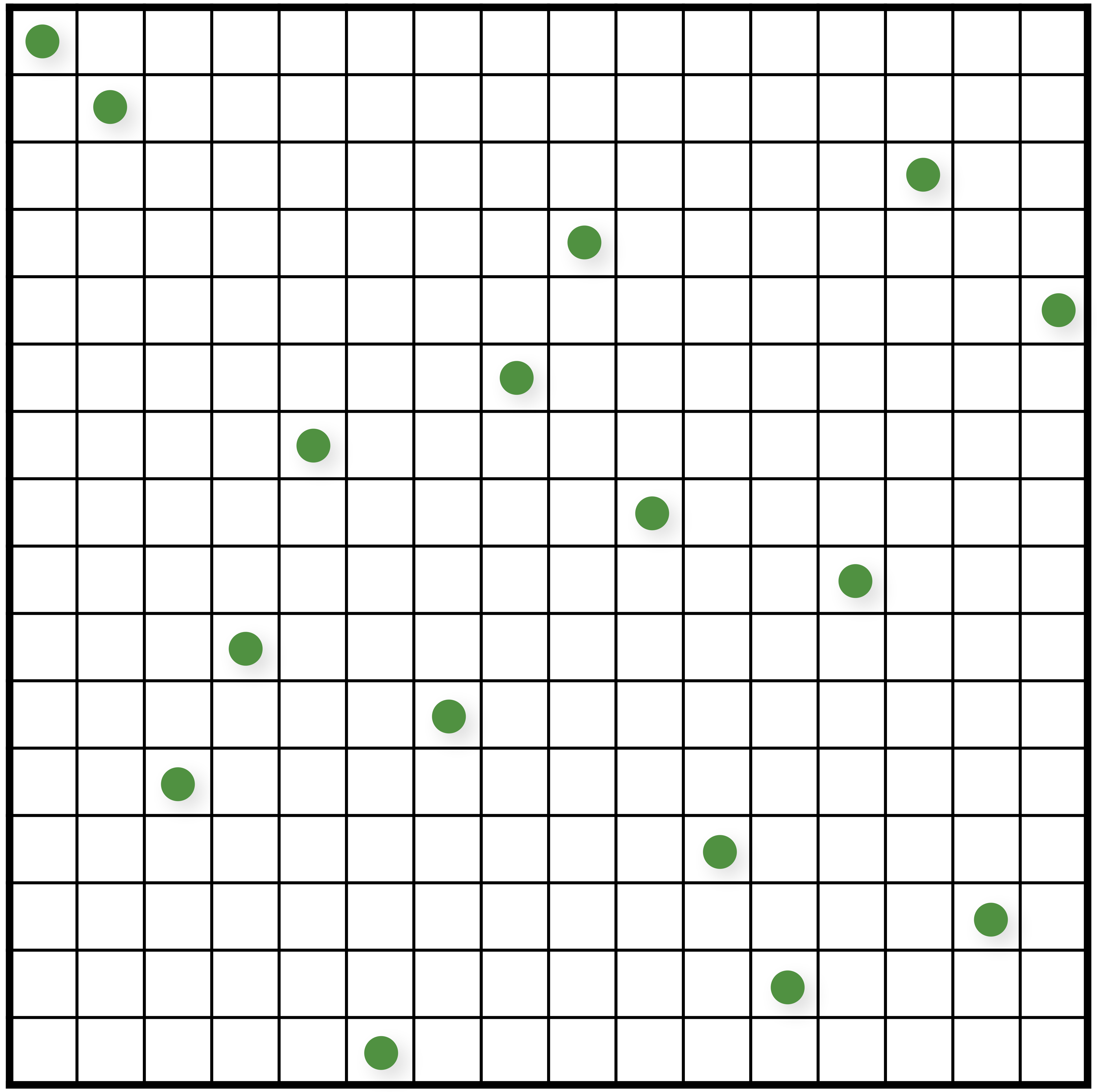
Stratify samples in each dimension separately

- for **2D**: **2** separate 1D jittered point sets
- combine dimensions in random order



Latin Hypercube (N-Rooks) Sampling

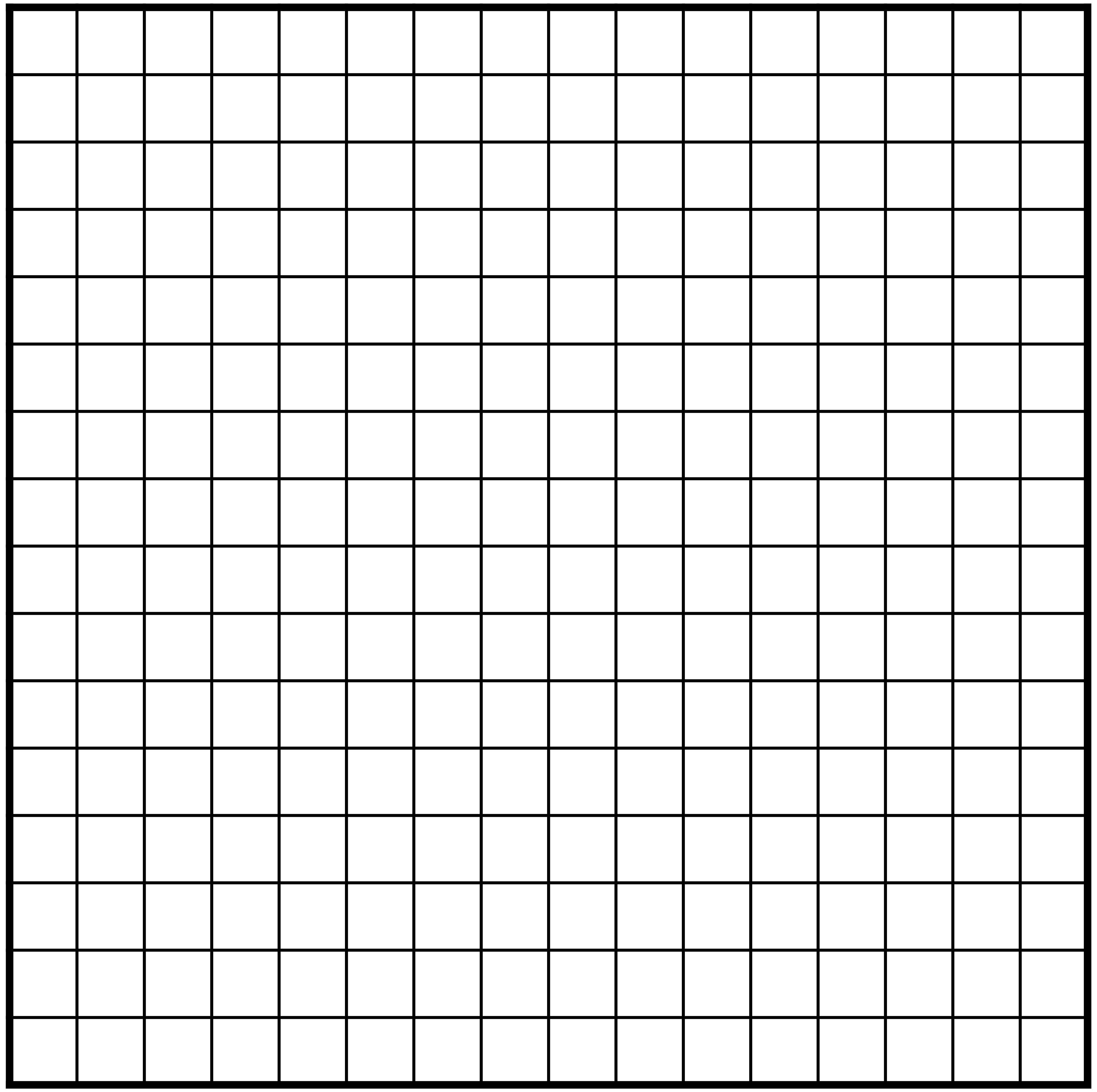
[Shirley 91]



Latin Hypercube (N-Rooks) Sampling

```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
        for (uint i = 0; i < numS; i++)
                samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
        shuffle(samples(d,:));
```



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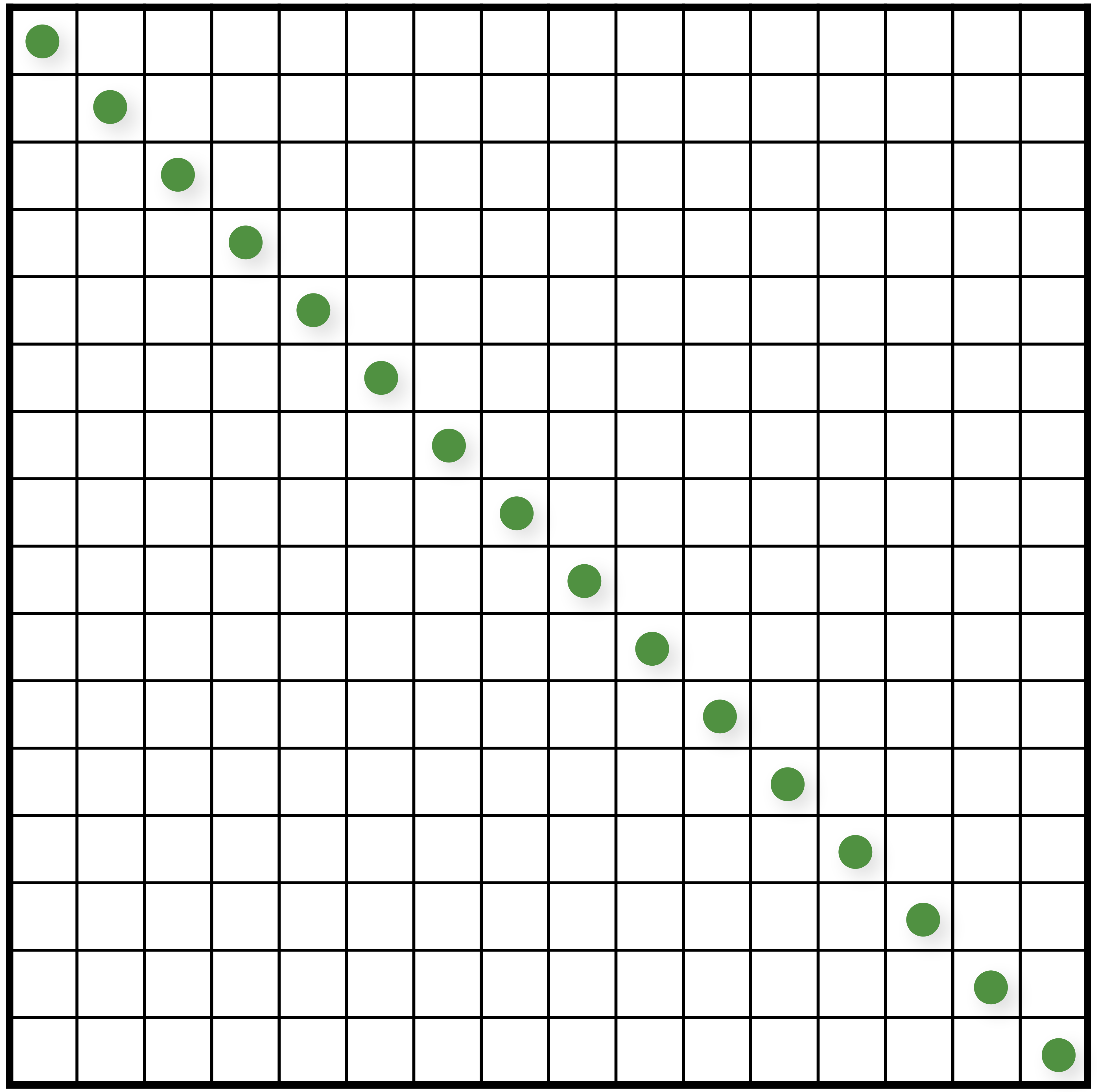

Initialize

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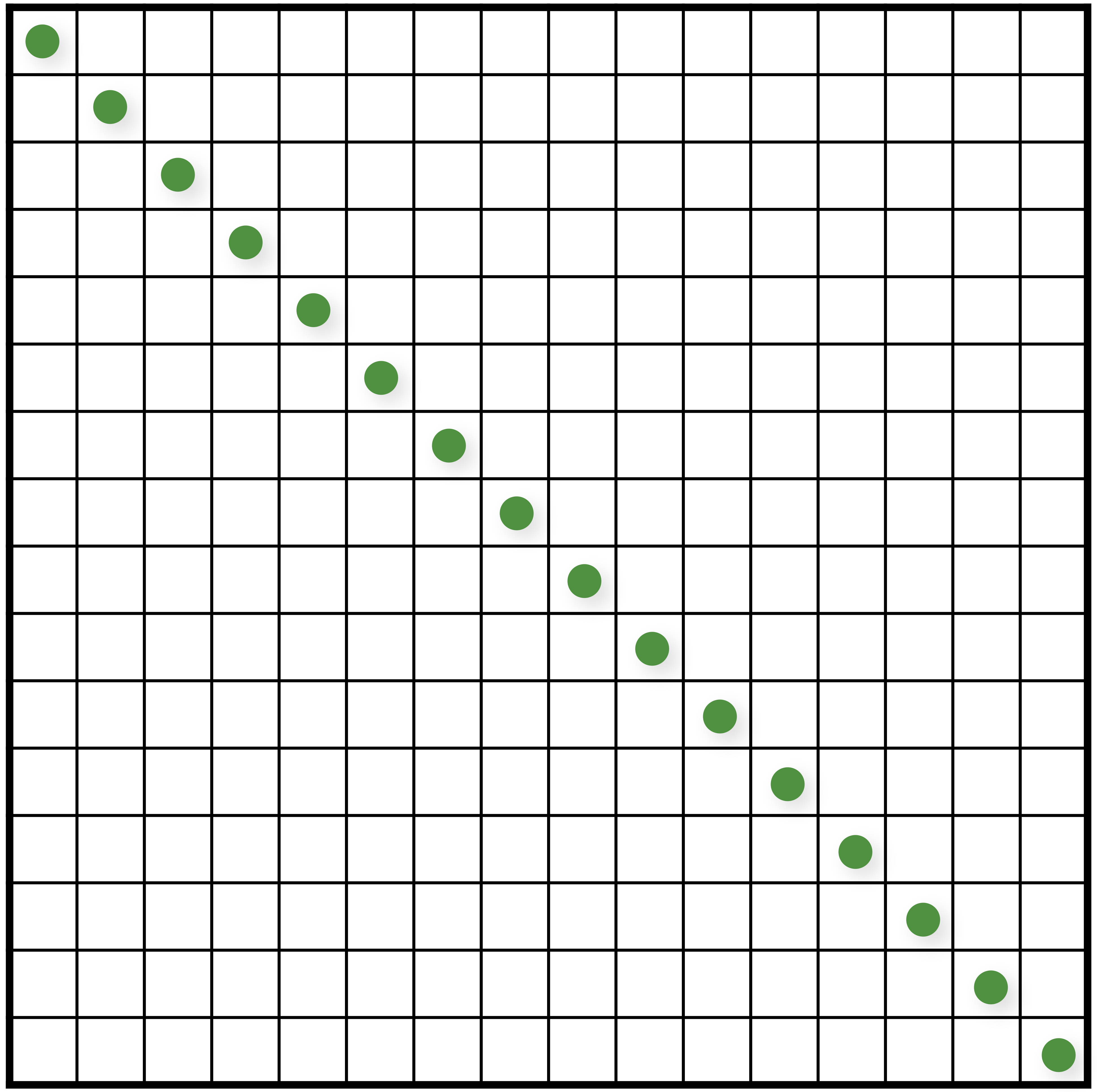

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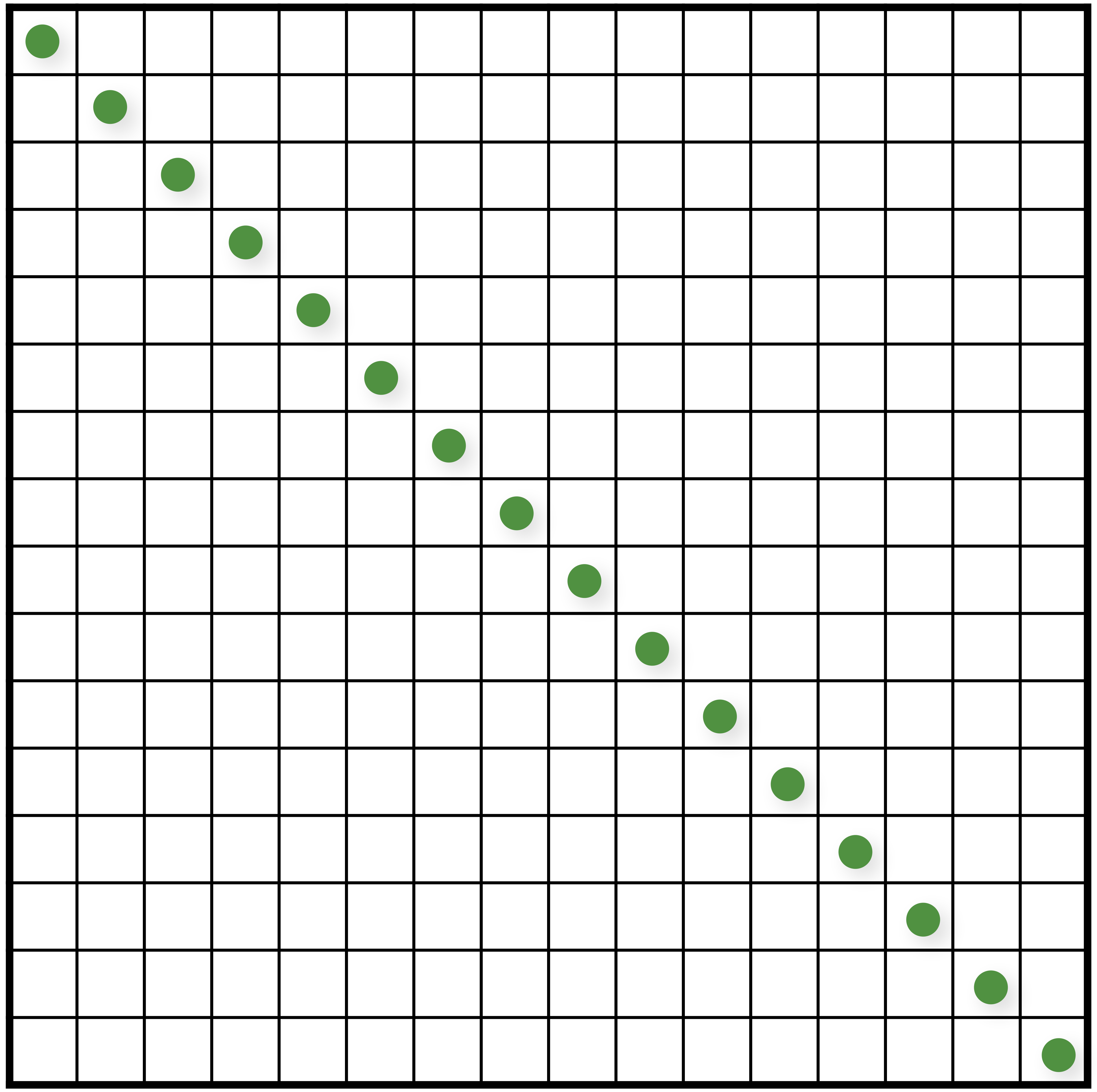

Shuffle rows

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// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
        shuffle(samples(d,:));
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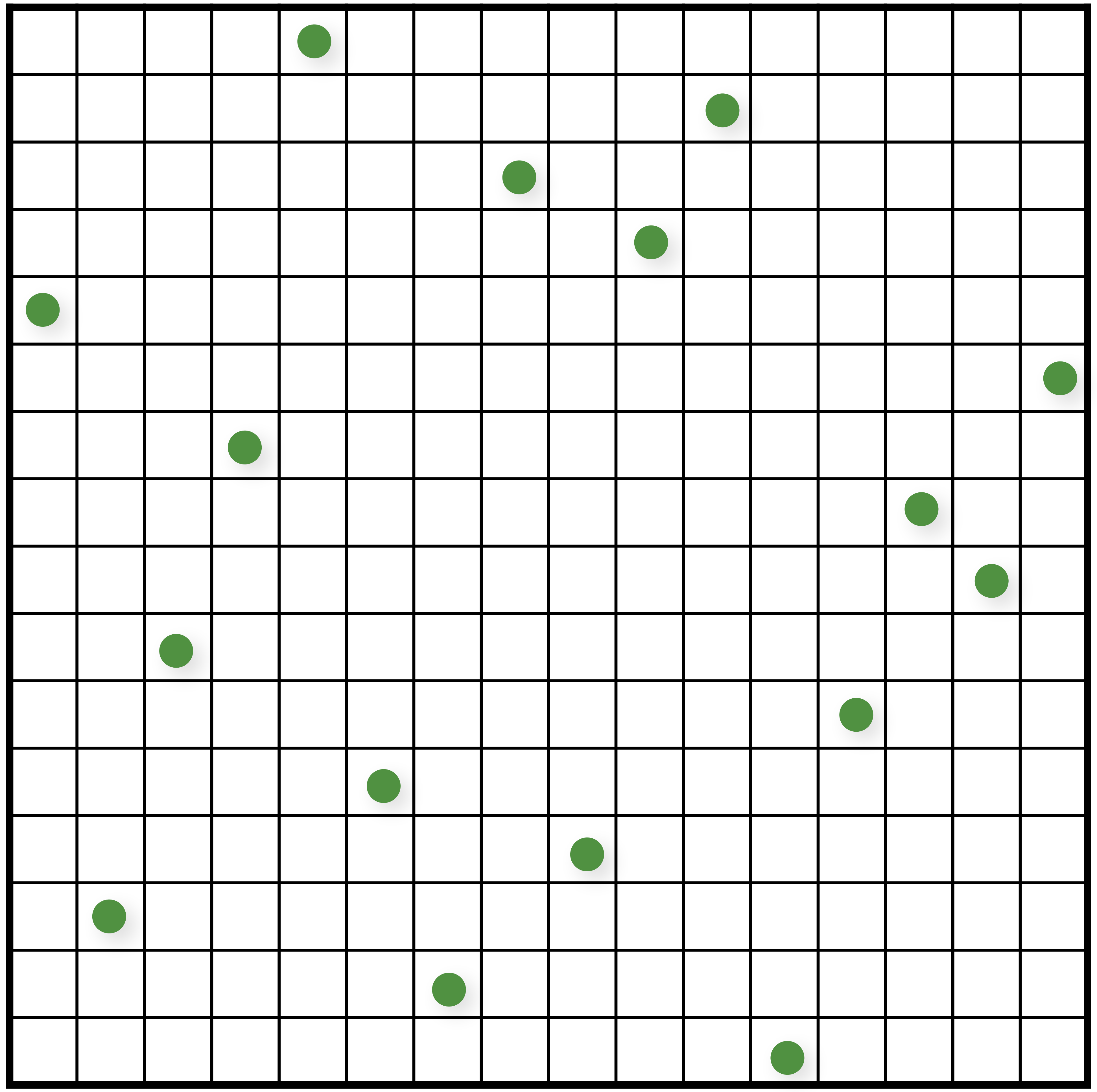

Shuffle rows

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// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
        for (uint i = 0; i < numS; i++)
                samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
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```

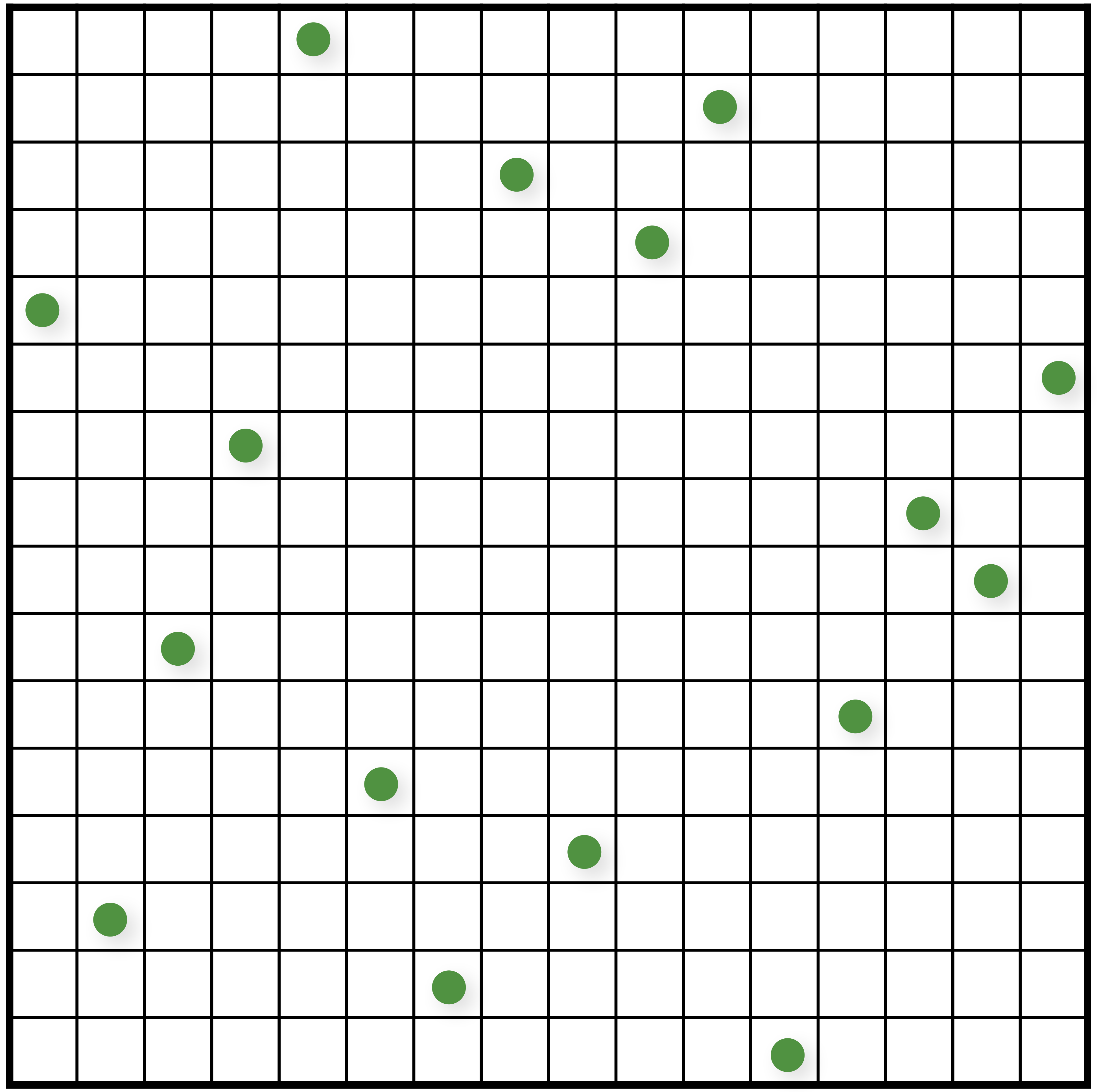

Shuffle rows

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// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
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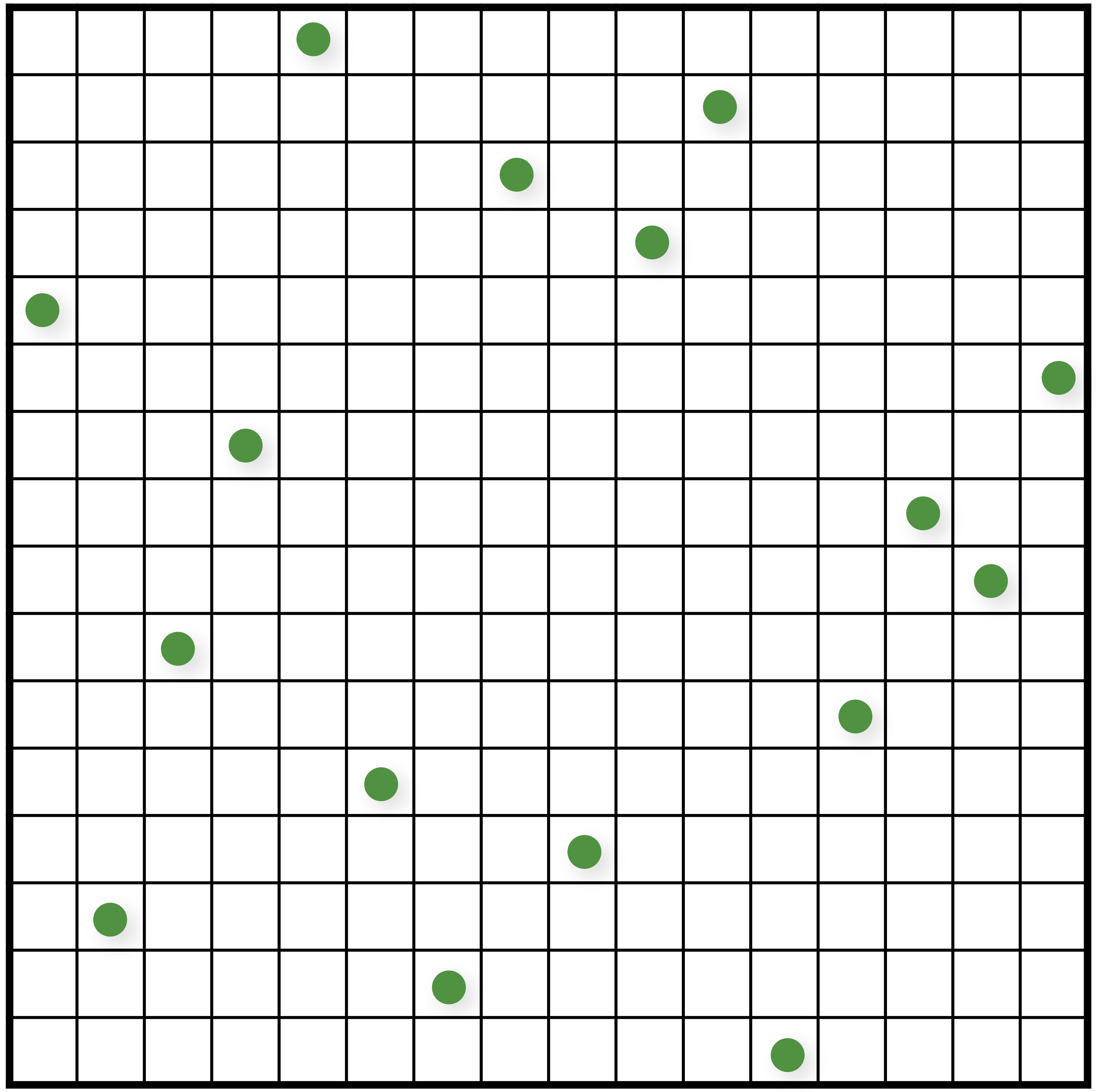

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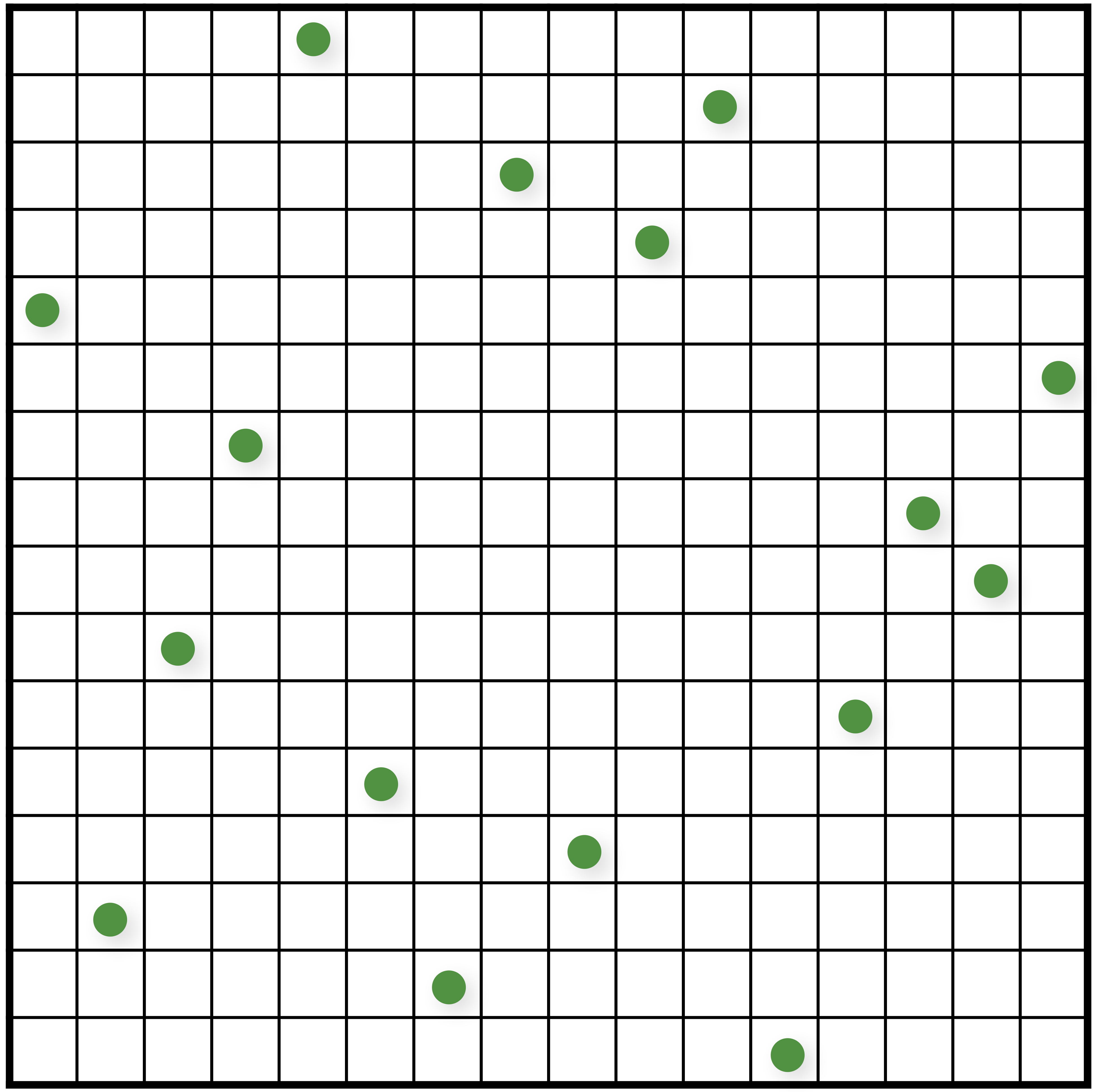
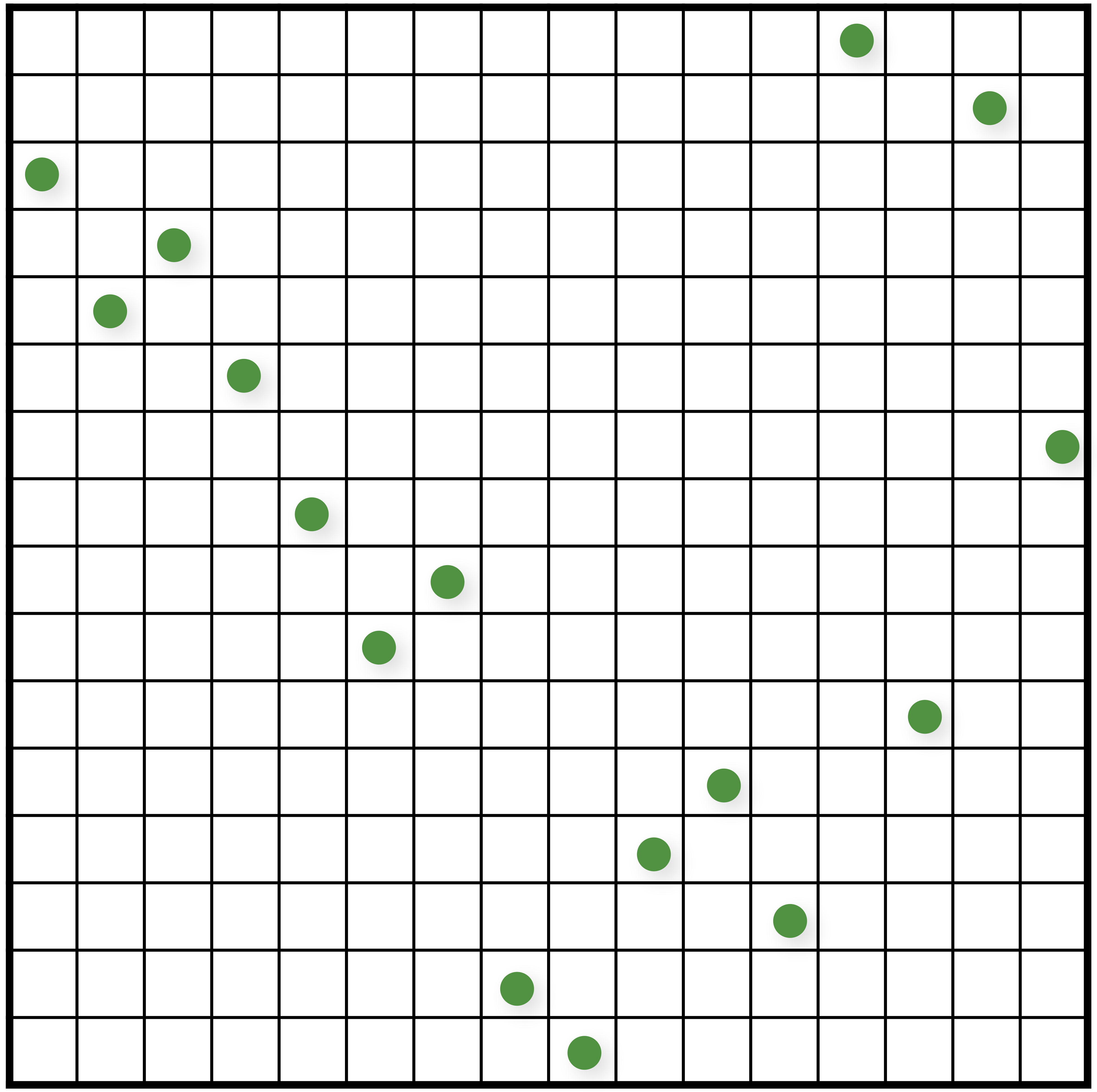


Shuffle columns

Latin Hypercube (N-Rooks) Sampling

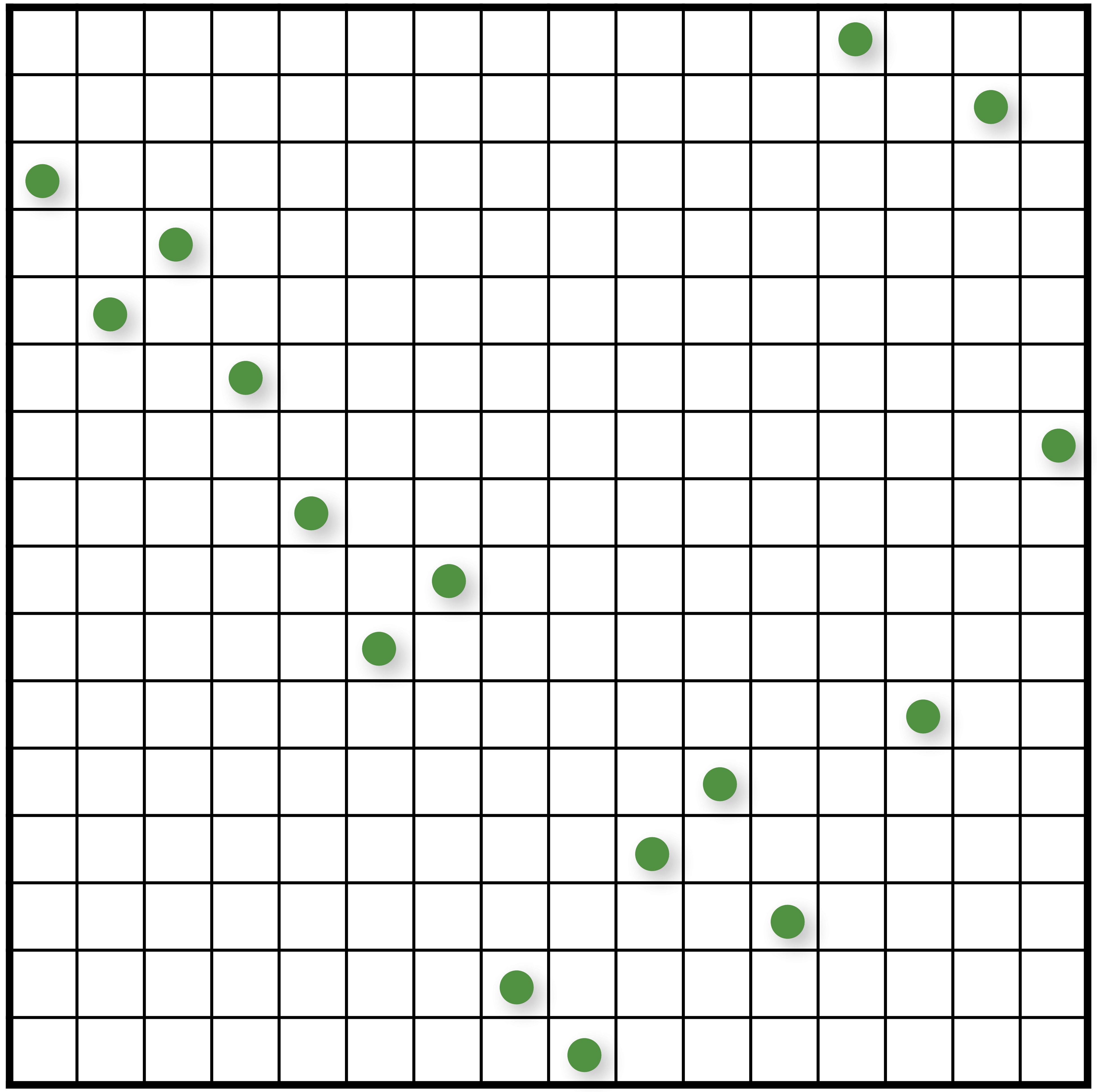
```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
        for (uint i = 0; i < numS; i++)
                samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
        shuffle(samples(d,:));
```



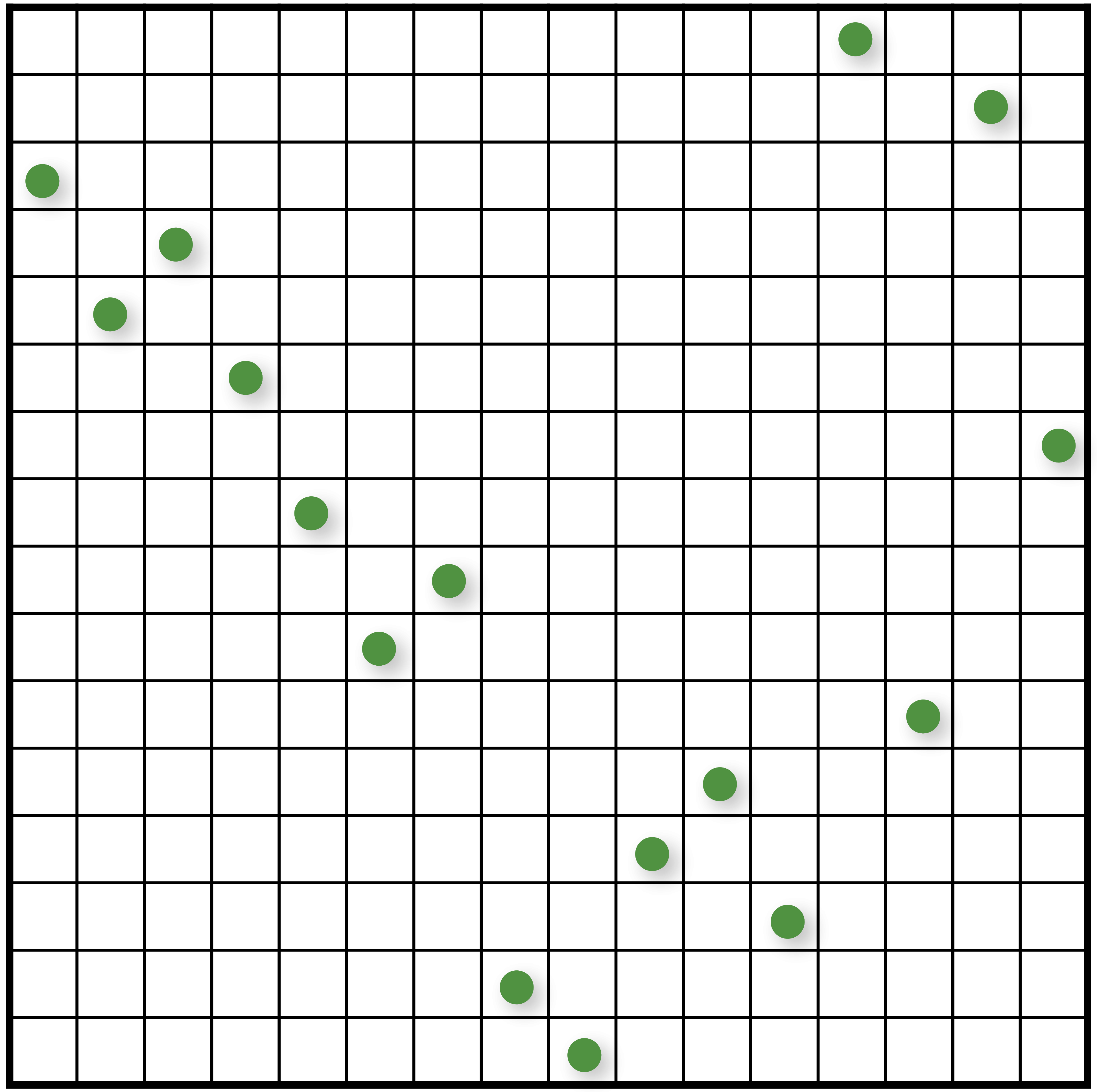
Shuffle columns

Latin Hypercube (N-Rooks) Sampling

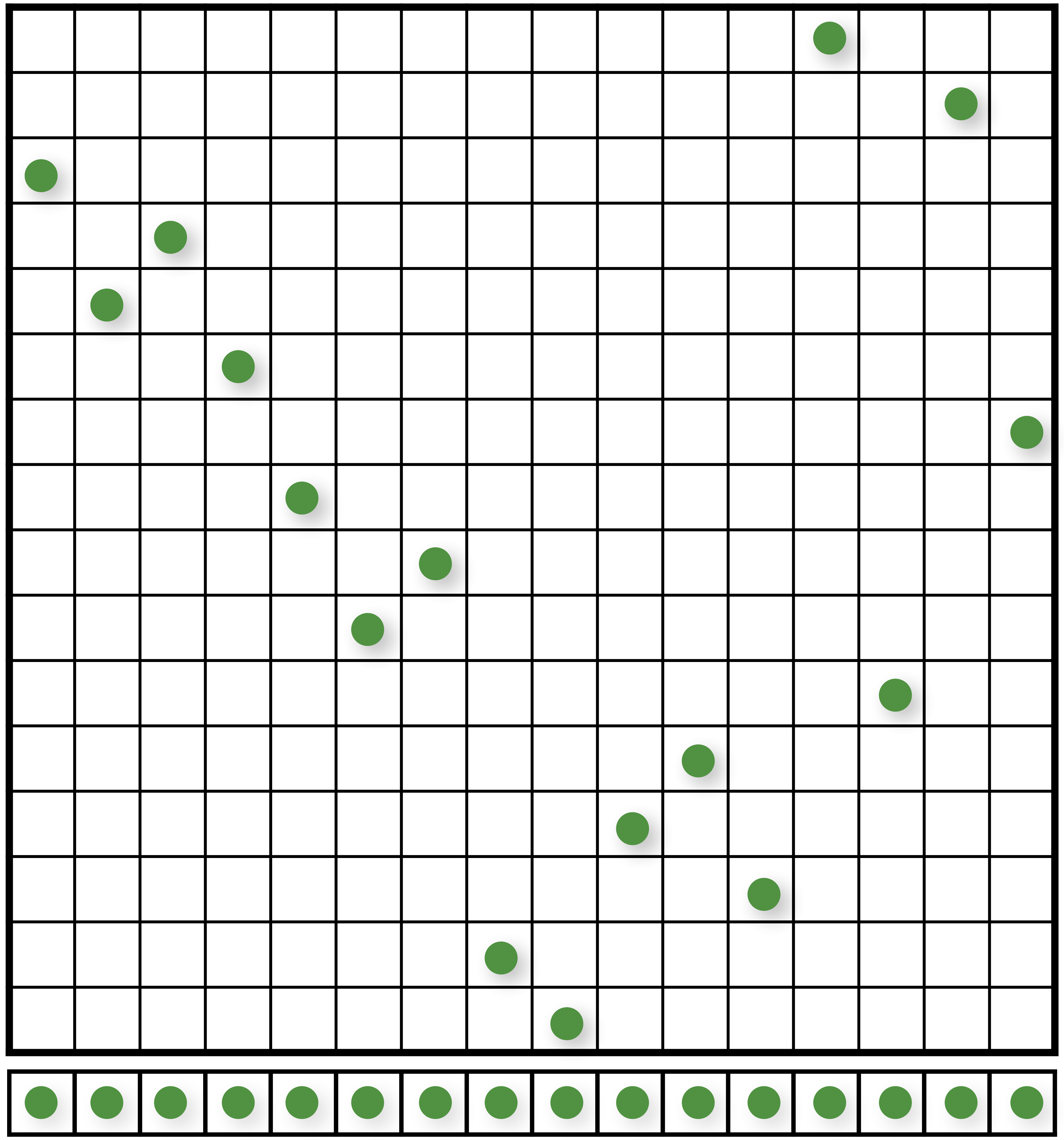
```
// initialize the diagonal
for (uint d = 0; d < numDimensions; d++)
	for (uint i = 0; i < numS; i++)
		samples(d,i) = (i + randf())/numS;

// shuffle each dimension independently
for (uint d = 0; d < numDimensions; d++)
	shuffle(samples(d,:));
```



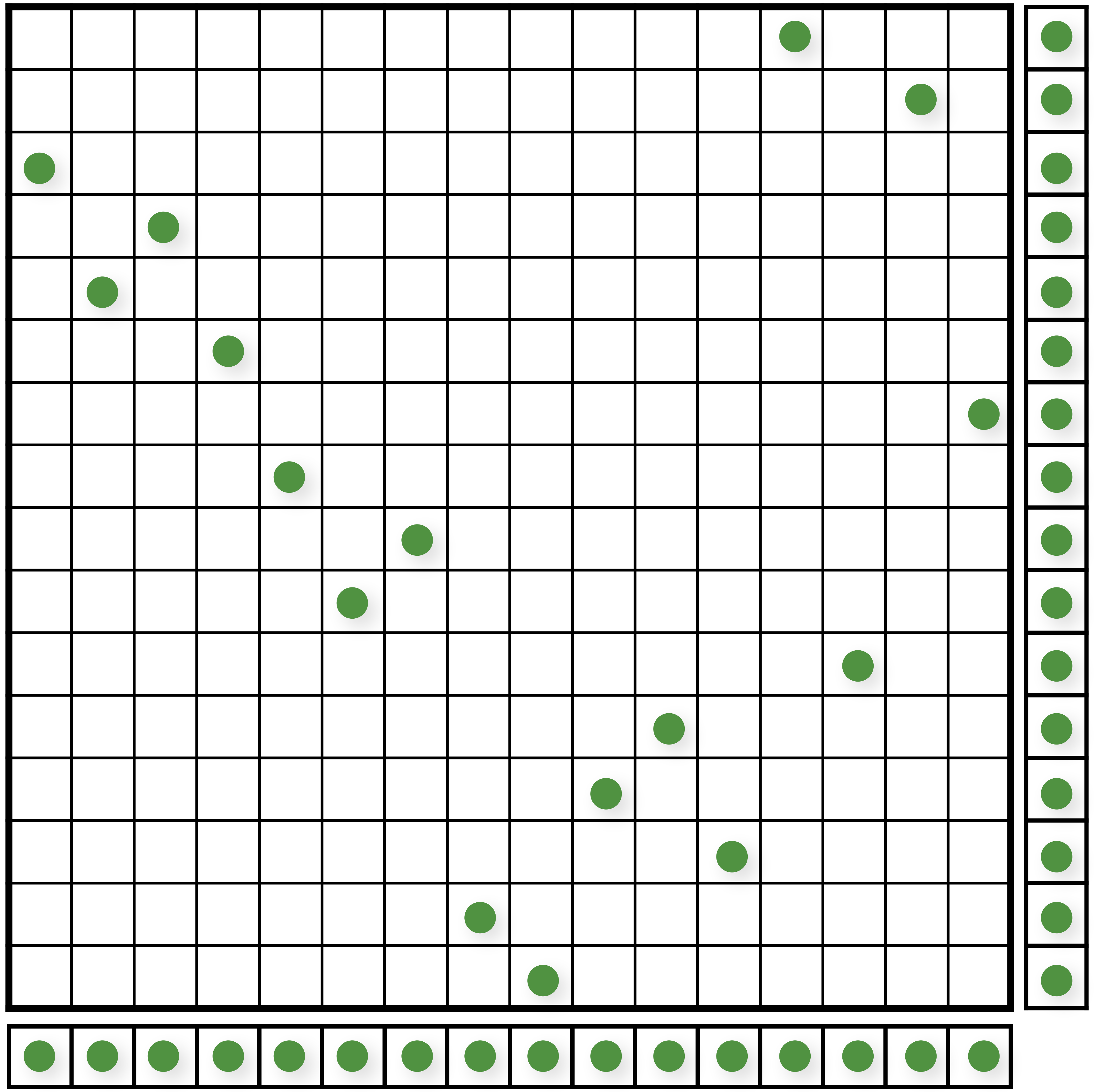
Latin Hypercube (N-Rooks) Sampling



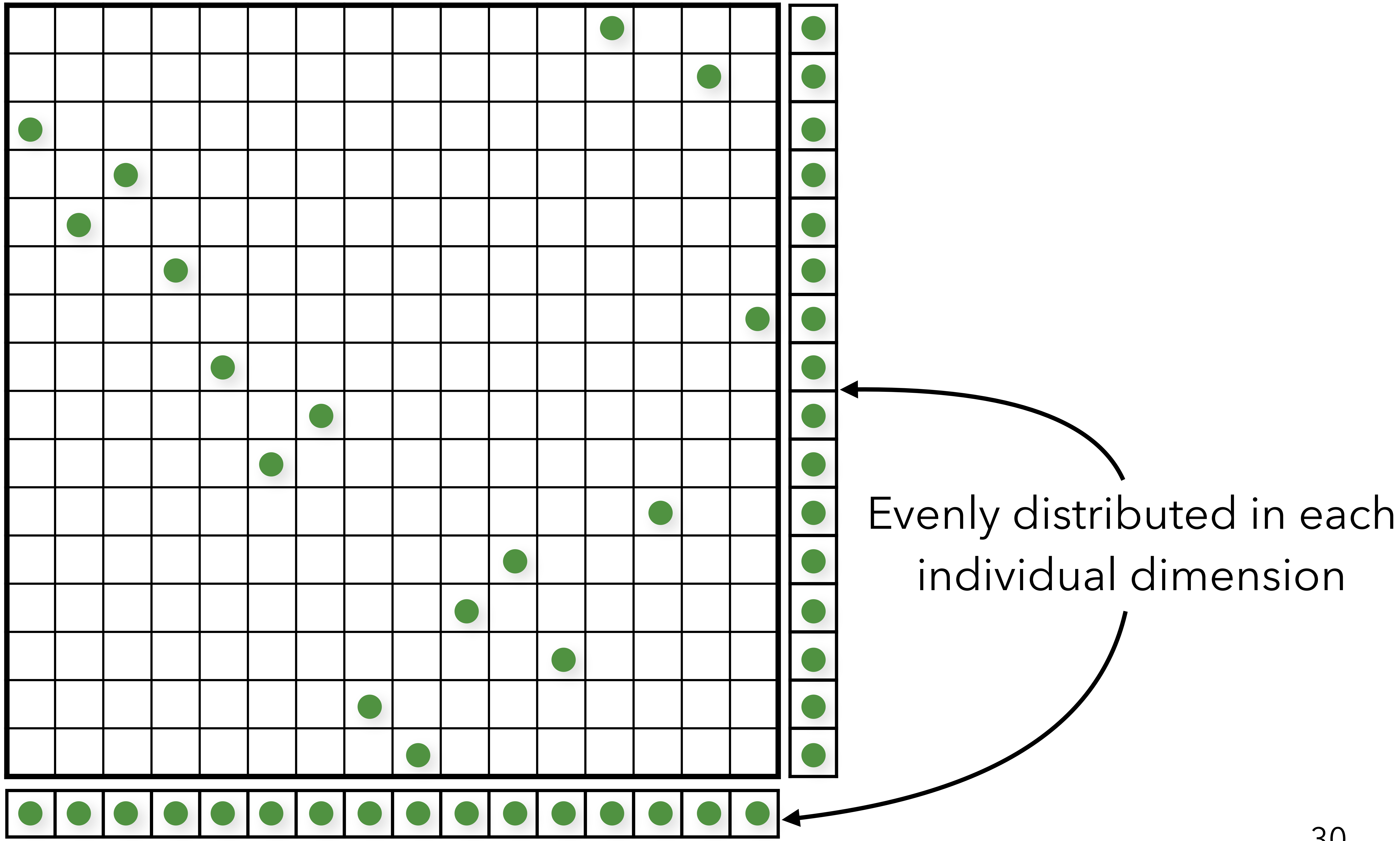
Latin Hypercube (N-Rooks) Sampling



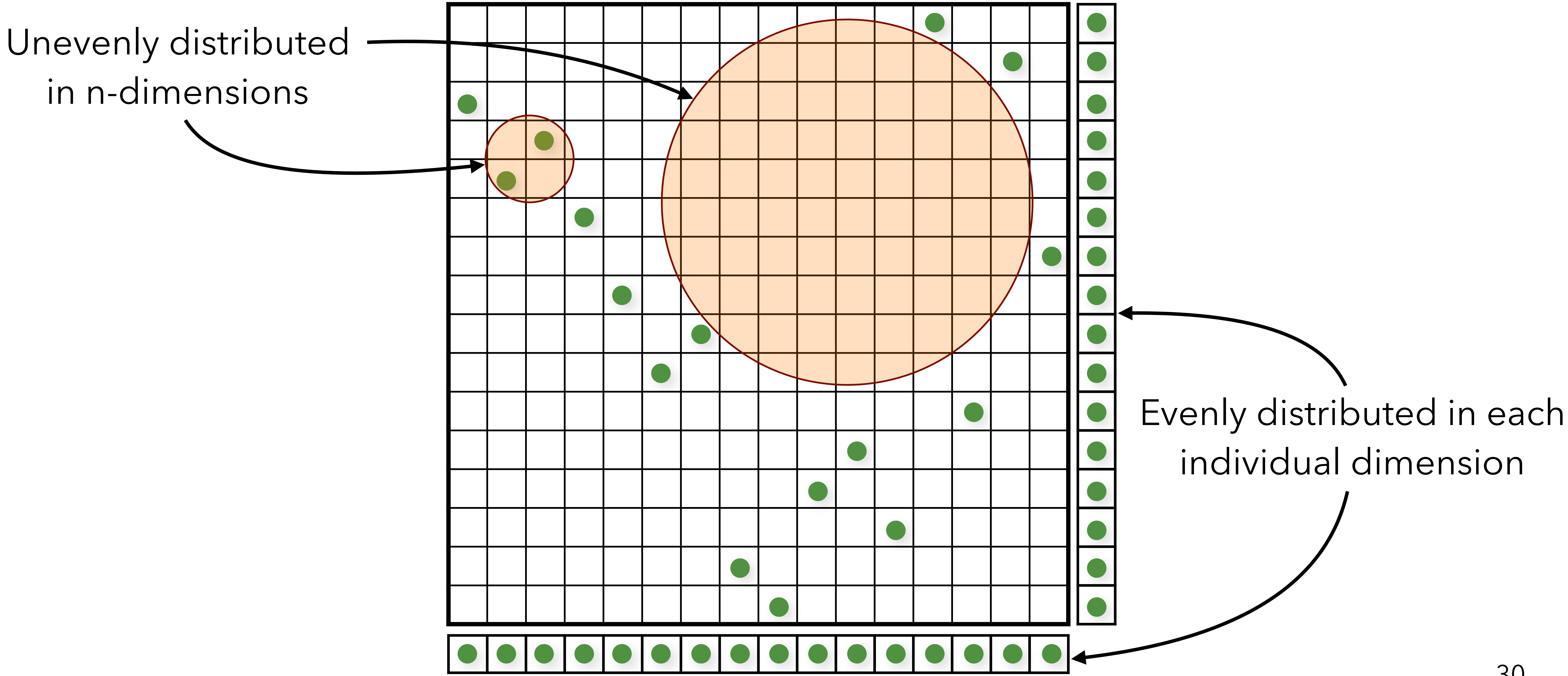
Latin Hypercube (N-Rooks) Sampling



Latin Hypercube (N-Rooks) Sampling

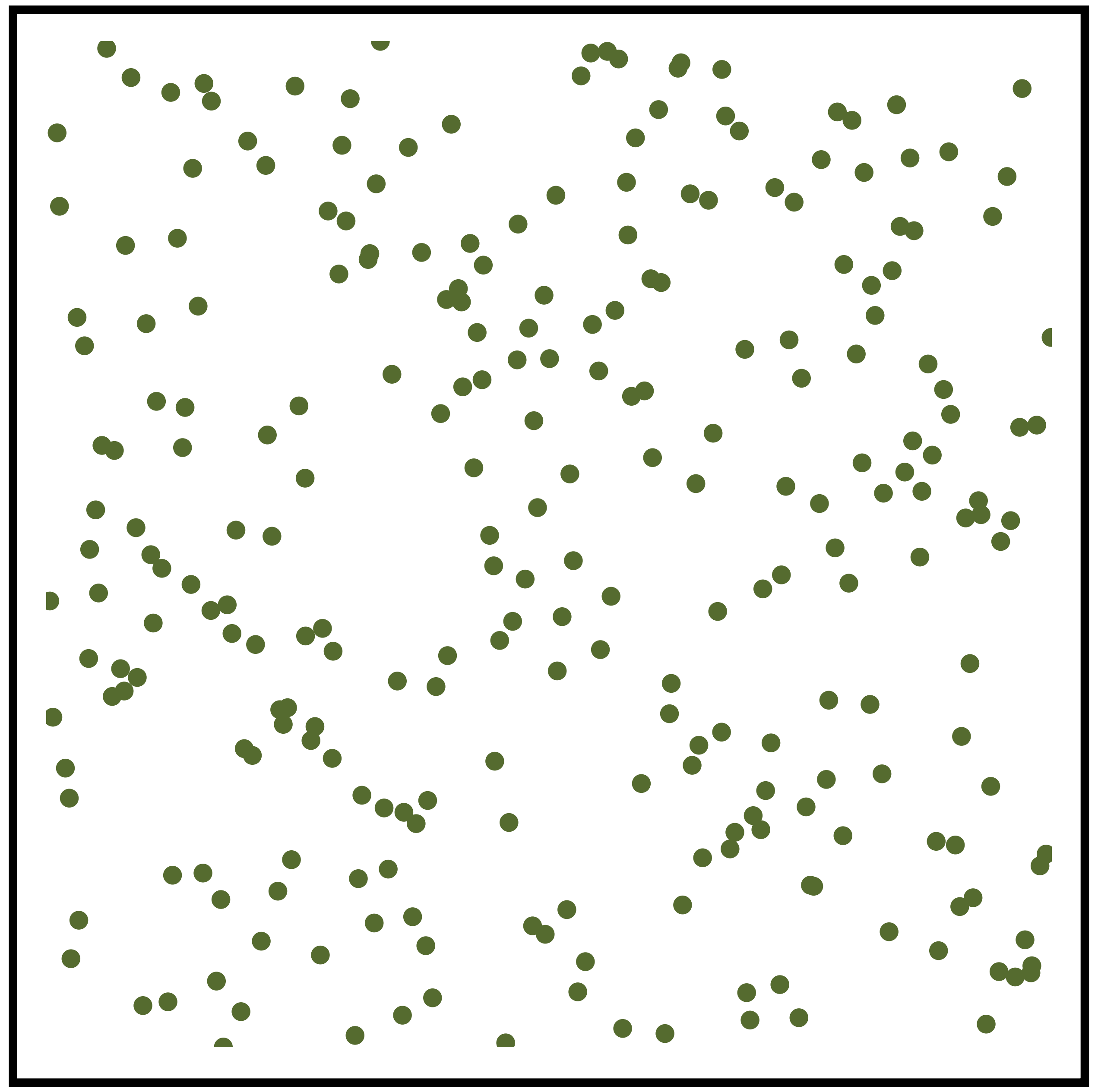


Latin Hypercube (N-Rooks) Sampling

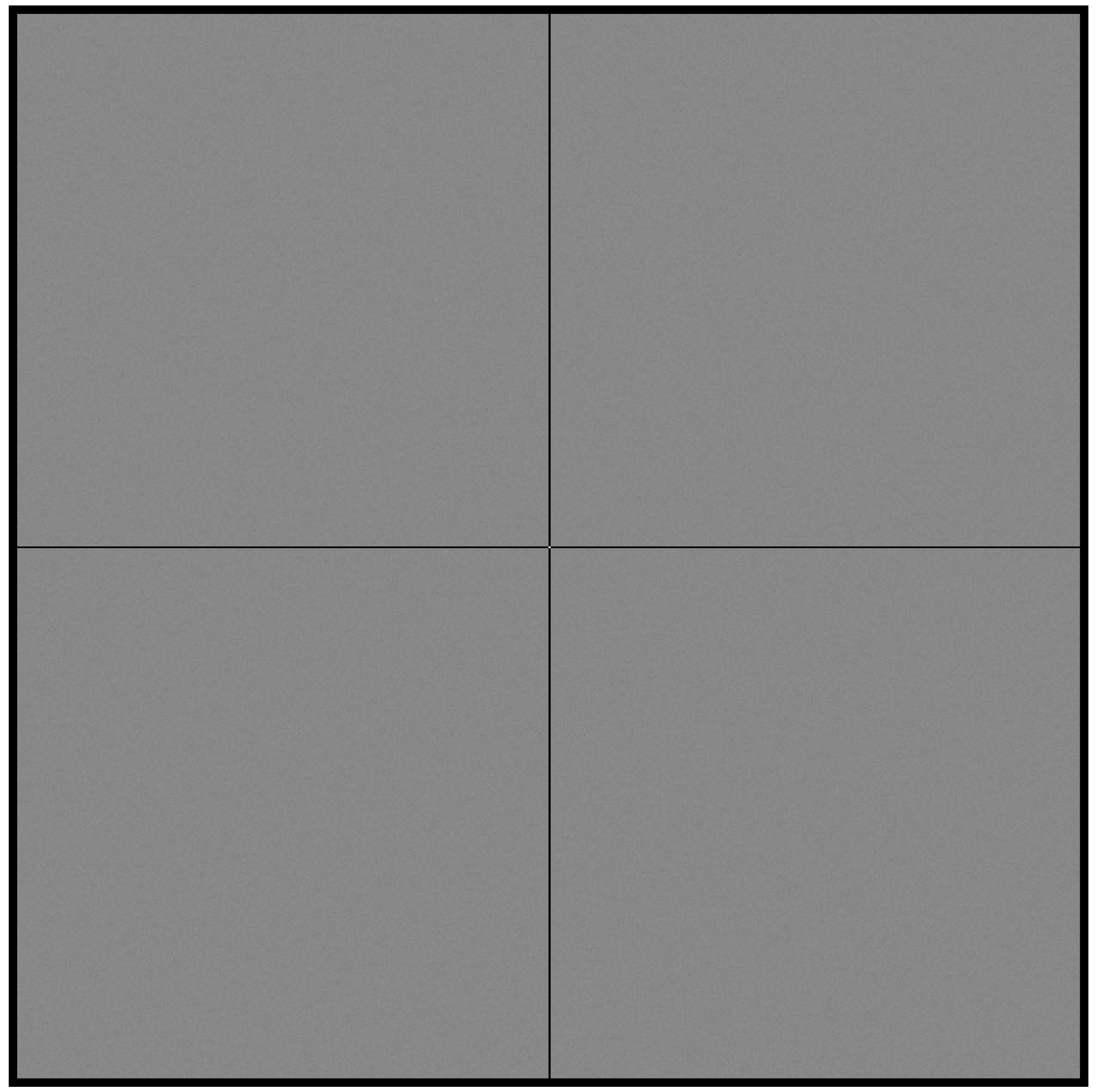


N-Rooks Sampling

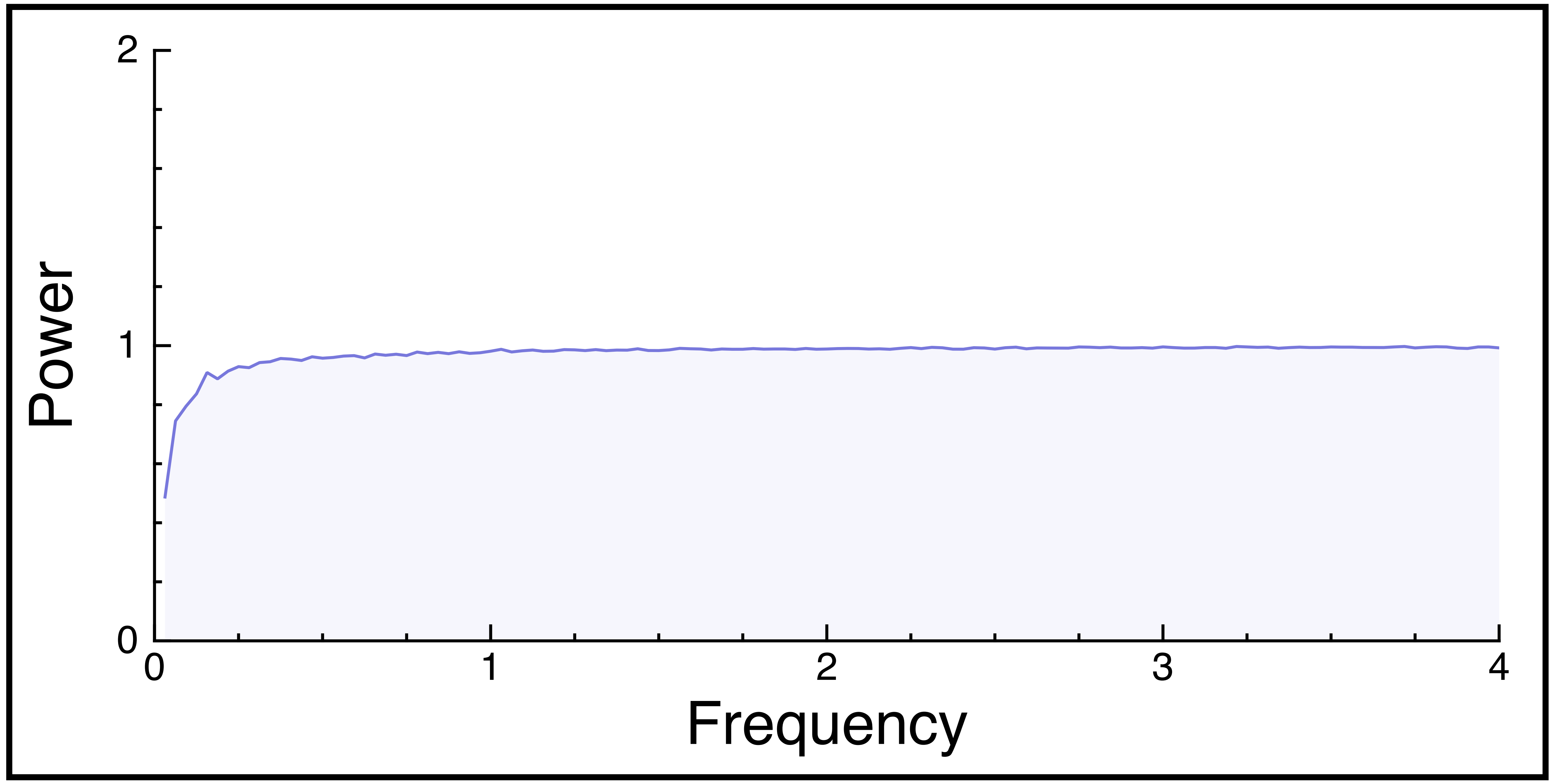
Samples



Expected power spectrum



Radial mean

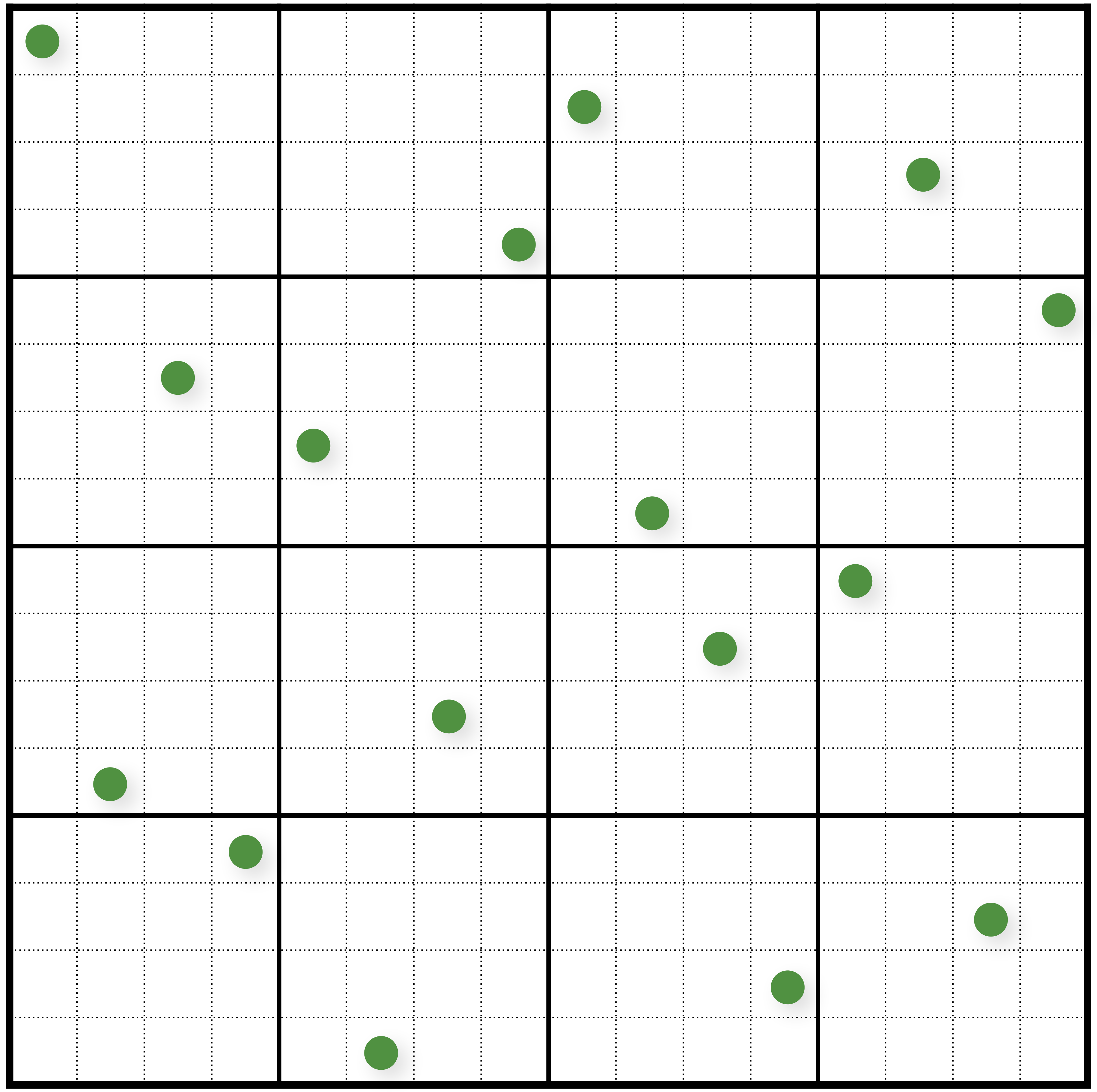


Multi-Jittered Sampling

Kenneth Chiu, Peter Shirley, and Changyaw Wang. "Multi-jittered sampling." In *Graphics Gems IV*, pp. 370–374. Academic Press, May 1994.

- combine N-Rooks and Jittered stratification constraints

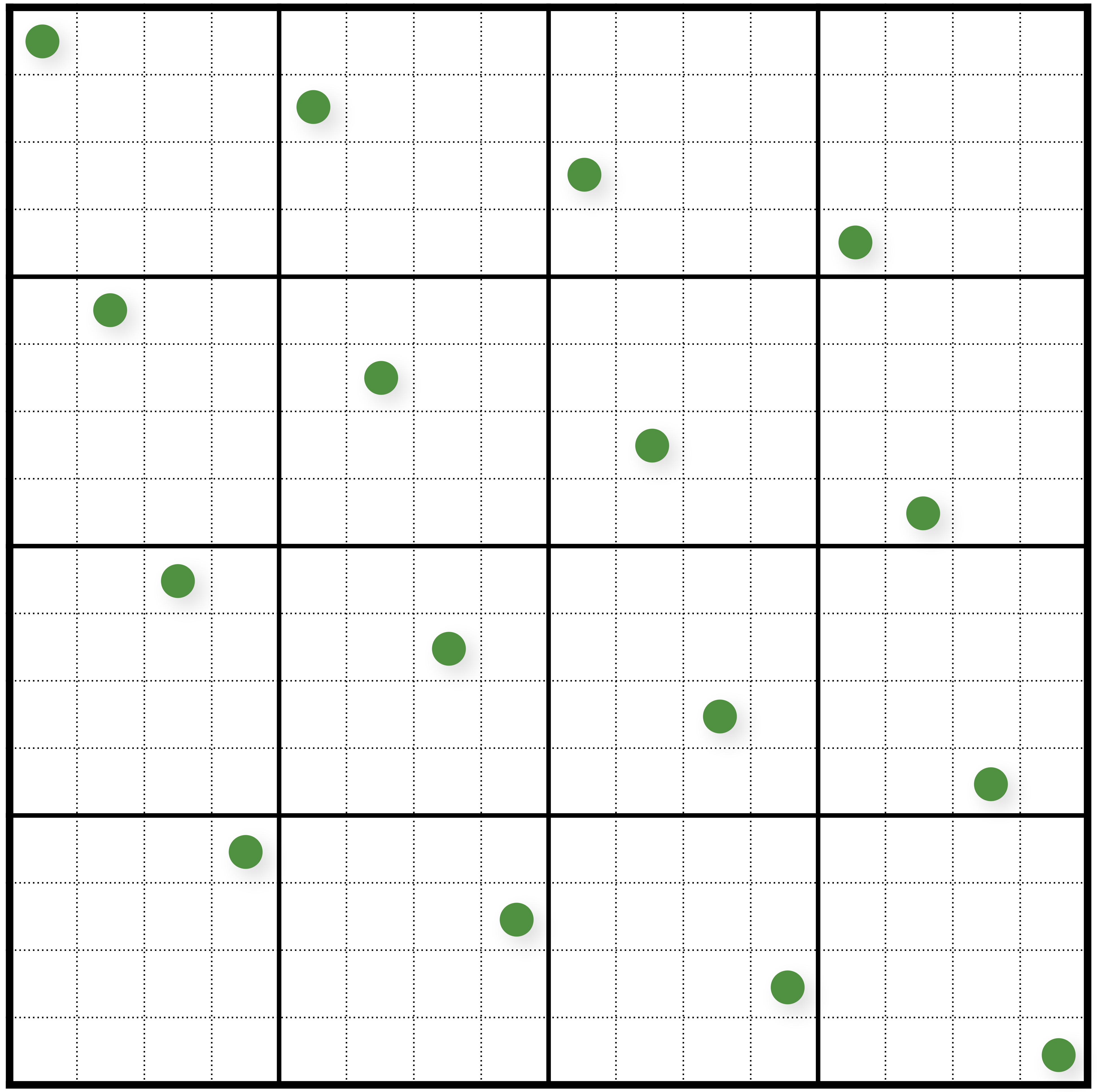
Multi-Jittered Sampling



Multi-Jittered Sampling

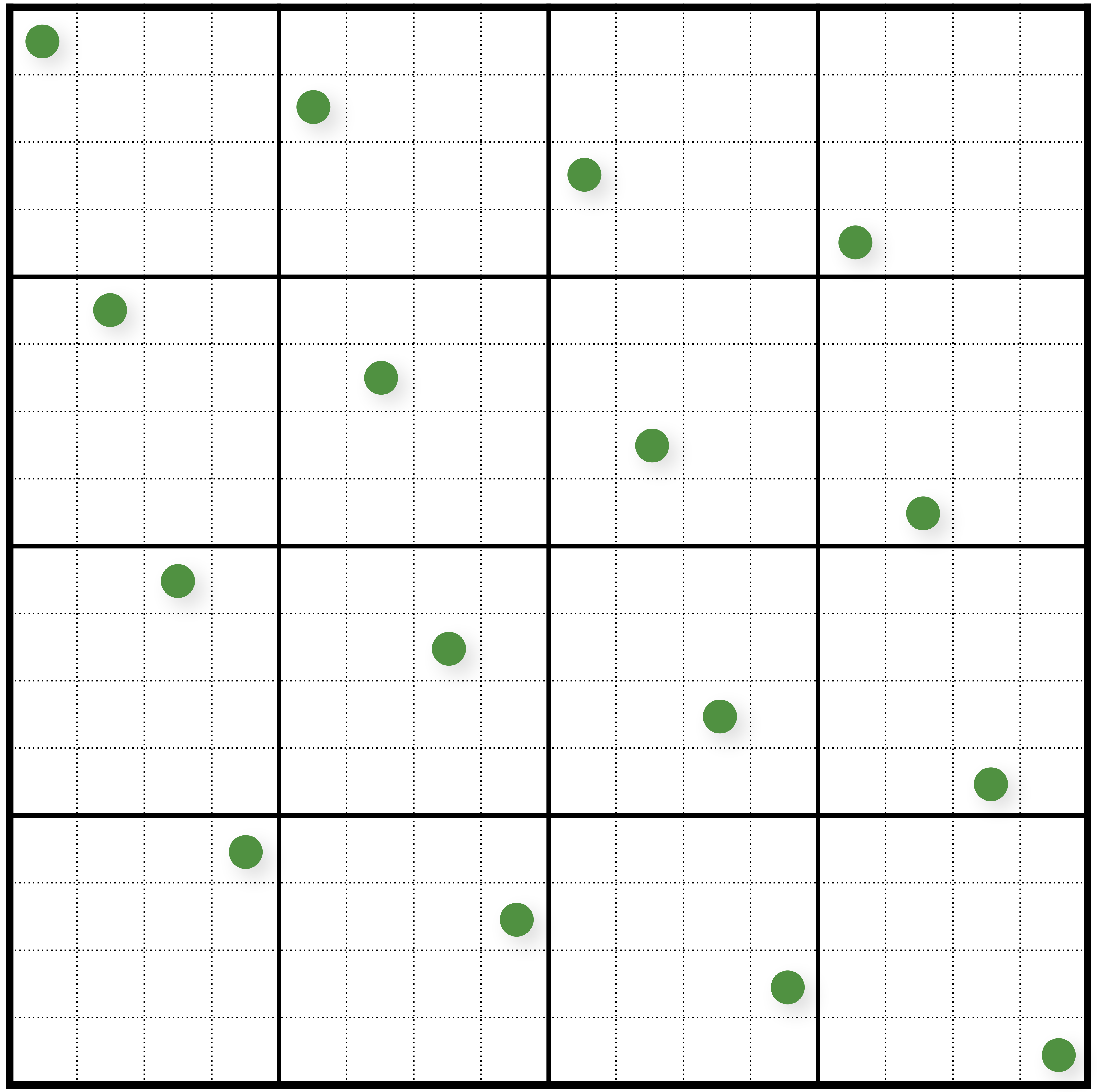
```
// initialize
float cellSize = 1.0 / (resX*resY);
for (uint i = 0; i < resX; i++)
	for (uint j = 0; j < resY; j++)
	{
		samples(i,j).x = i/resX + (j+randf()) / (resX*resY);
		samples(i,j).y = j/resY + (i+randf()) / (resX*resY);
	}

// shuffle x coordinates within each column of cells
for (uint i = 0; i < resX; i++)
	for (uint j = resY-1; j >= 1; j--)
		swap(samples(i, j).x, samples(i, randi(0, j)).x);

// shuffle y coordinates within each row of cells
for (unsigned j = 0; j < resY; j++)
	for (unsigned i = resX-1; i >= 1; i--)
		swap(samples(i, j).y, samples(randi(0, i), j).y);
```

Multi-Jittered Sampling



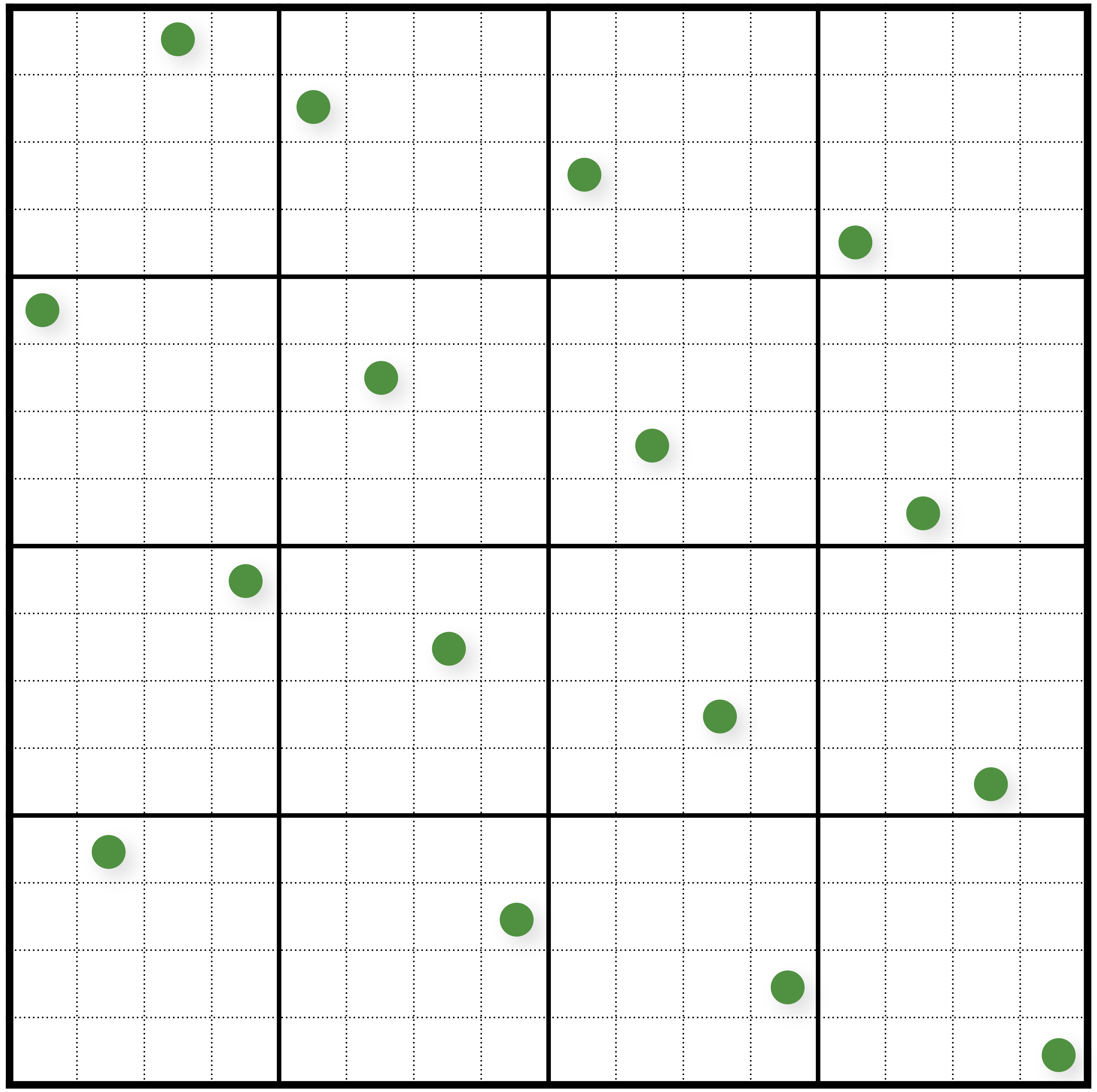
Initialize

Multi-Jittered Sampling



Shuffle x-coords

Multi-Jittered Sampling



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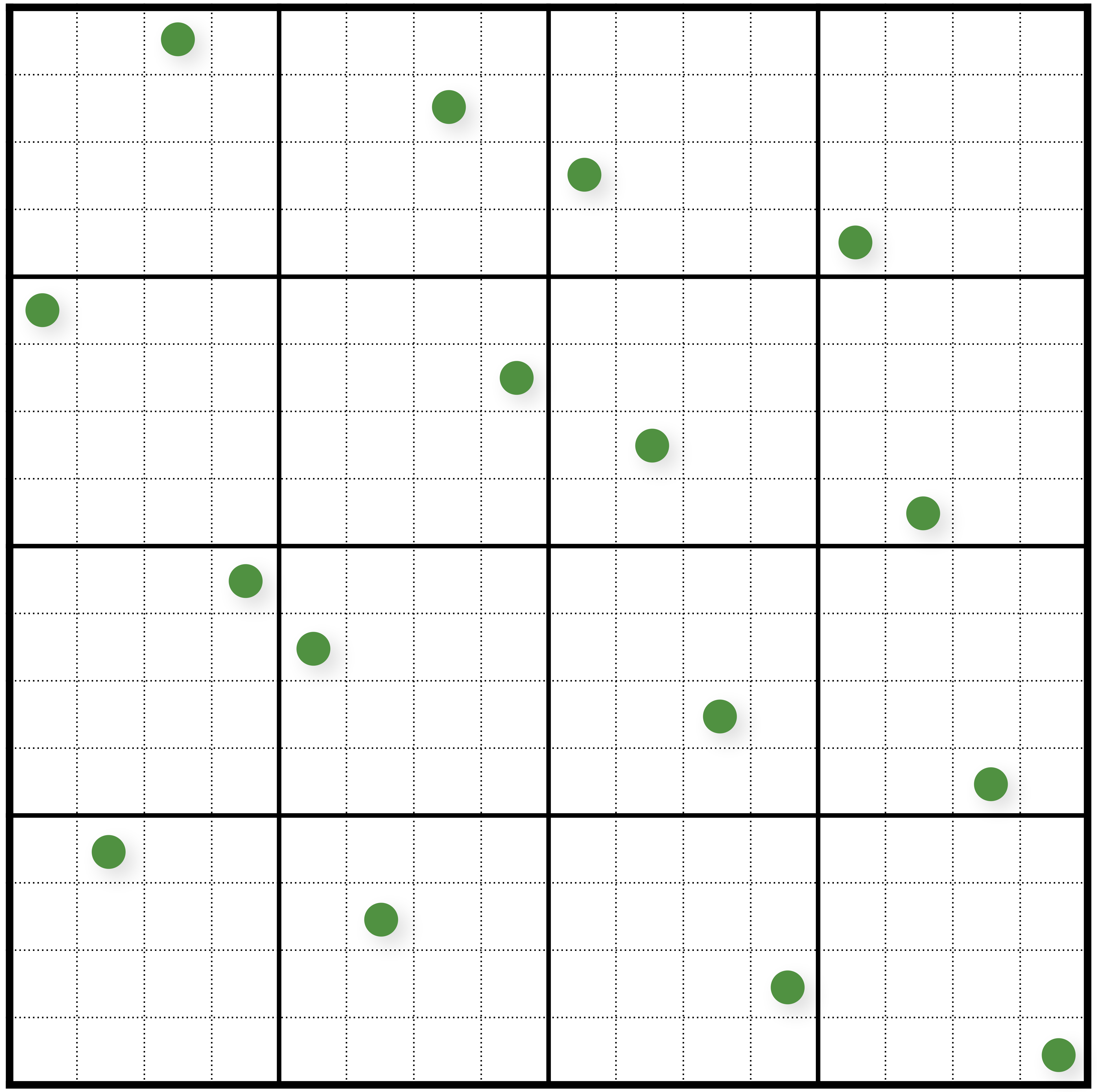
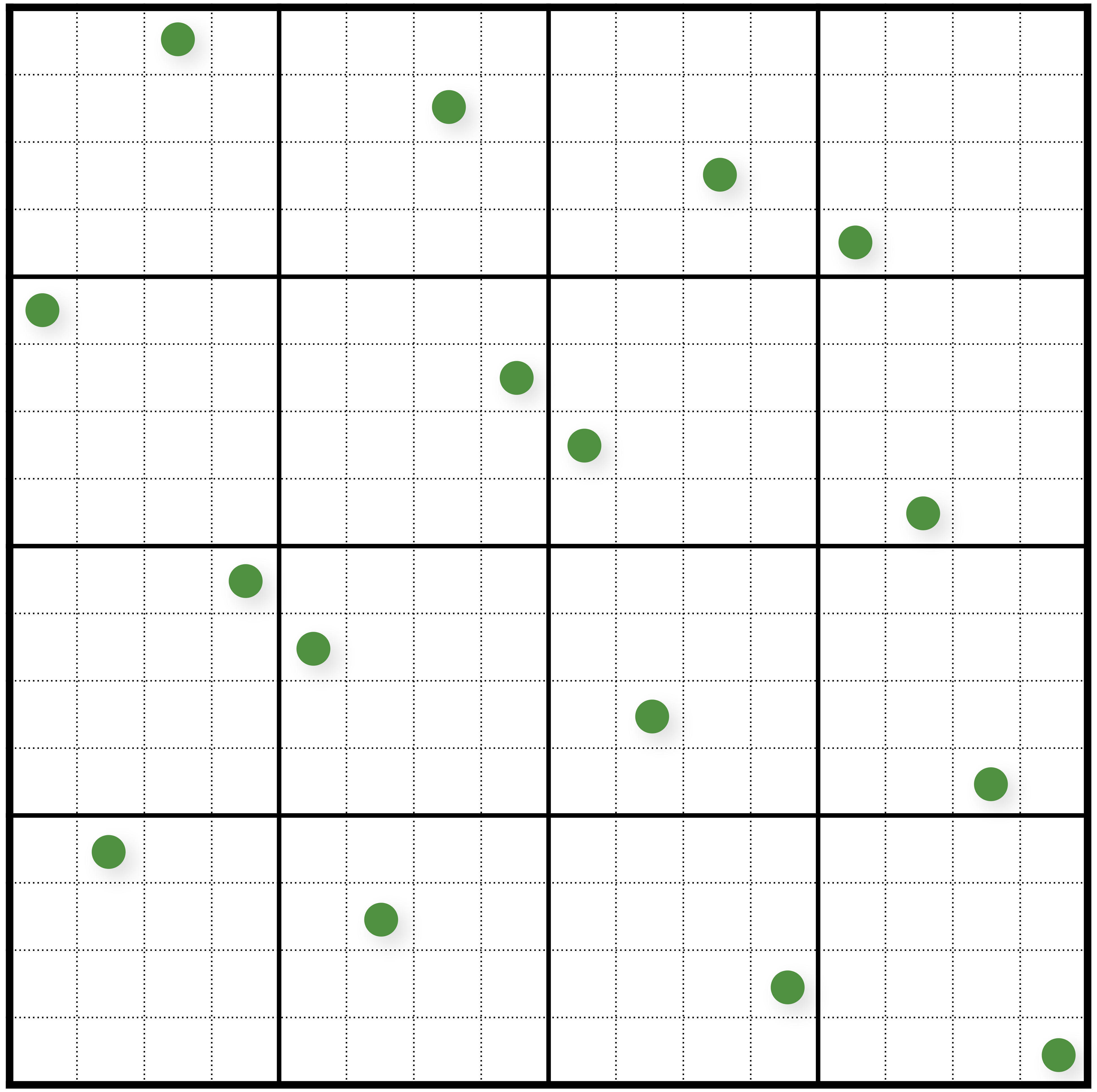
Multi-Jittered Sampling



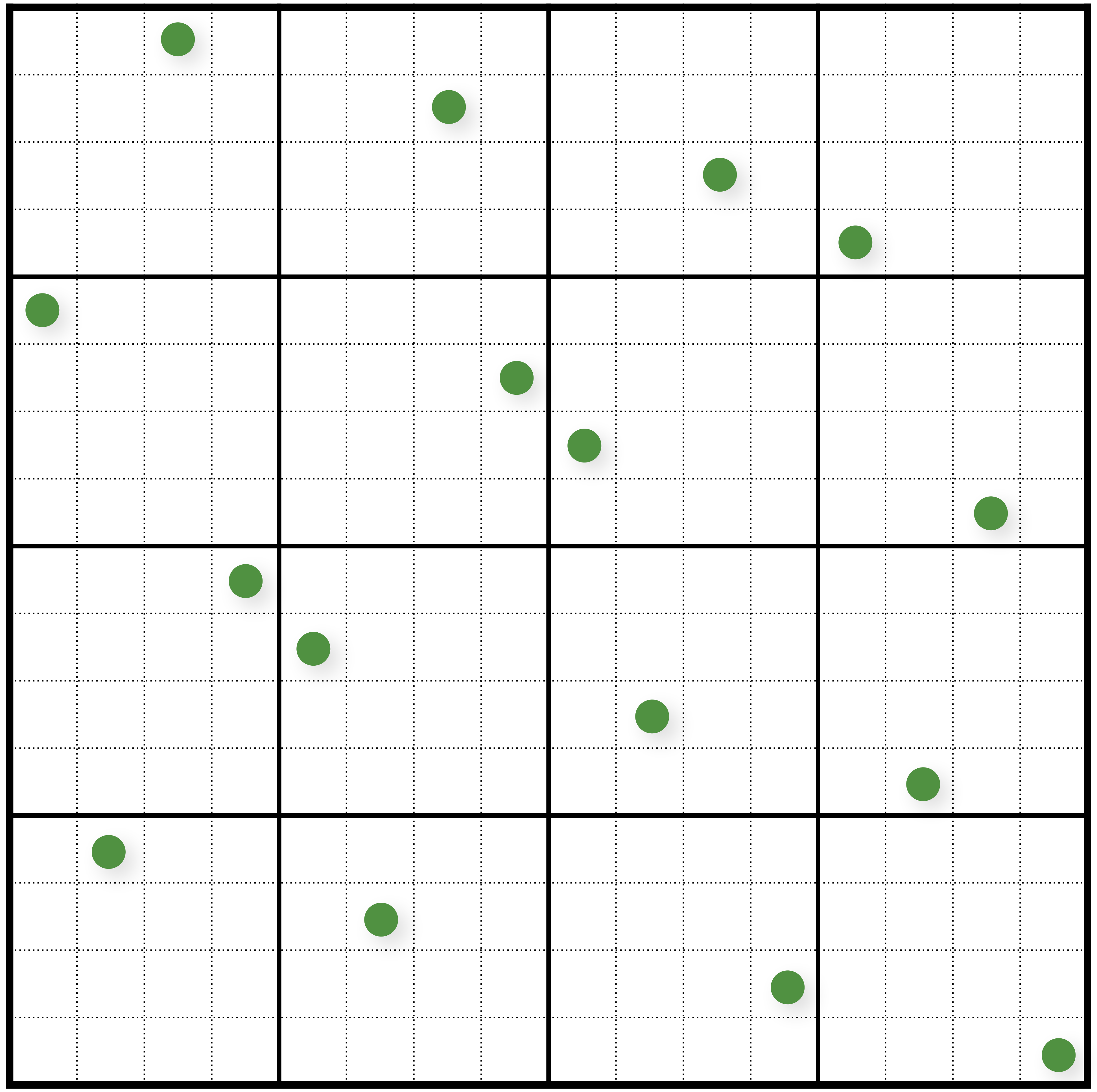
Shuffle x-coords

Multi-Jittered Sampling



Shuffle x-coords

Multi-Jittered Sampling



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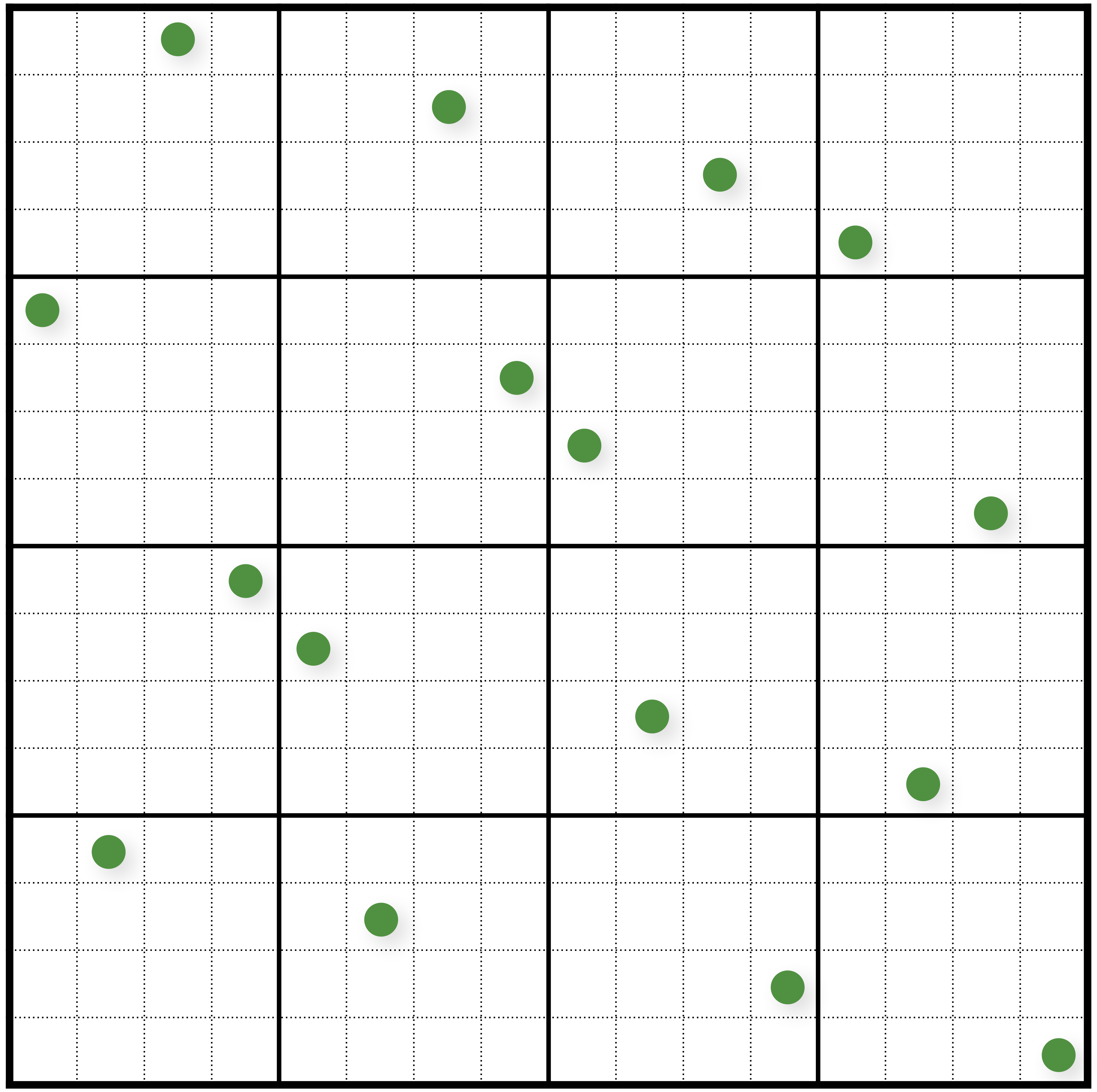
Multi-Jittered Sampling



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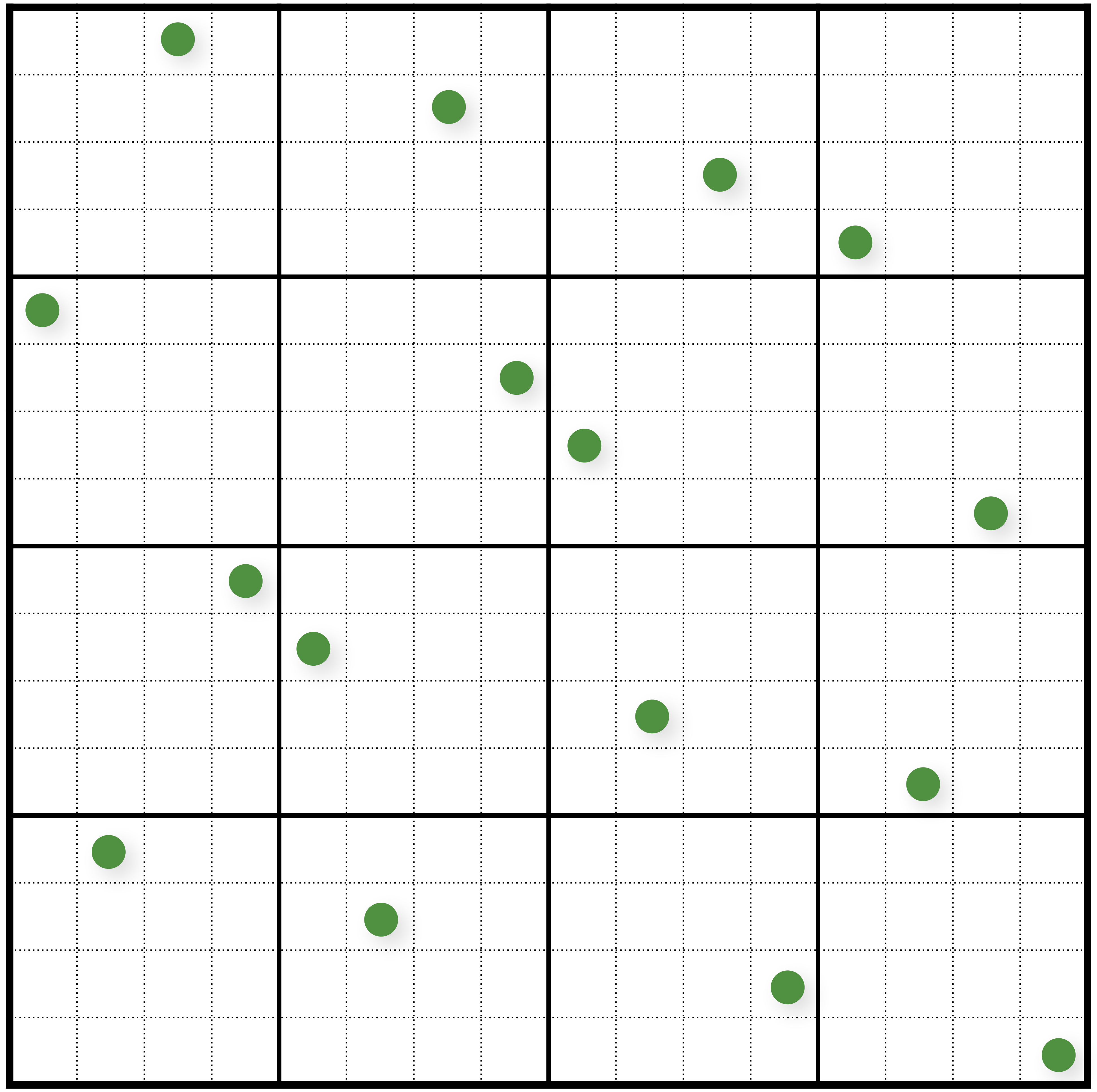
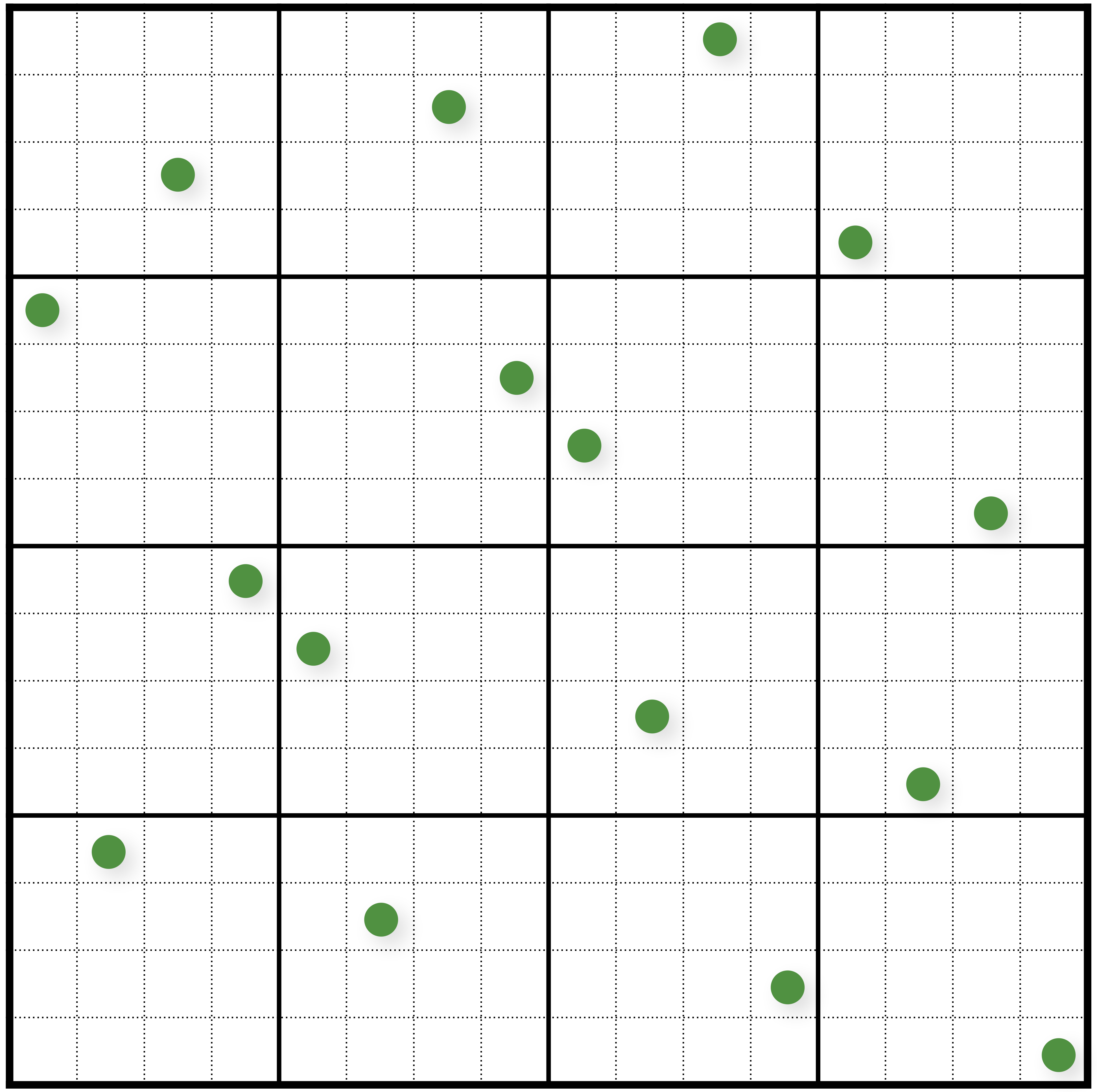


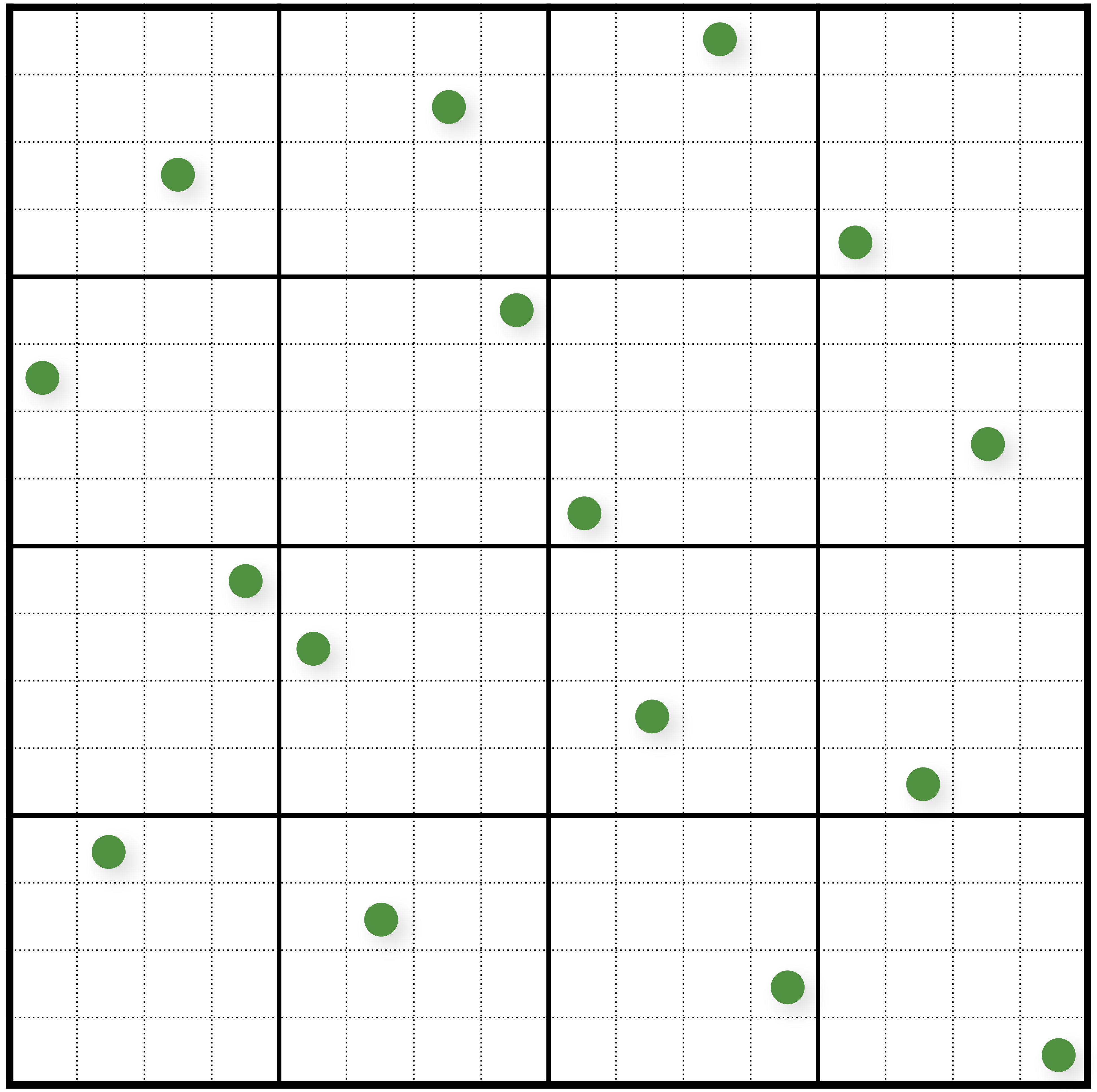
Shuffle y-coords

Multi-Jittered Sampling



Shuffle y-coords

Multi-Jittered Sampling



Shuffle y-coords

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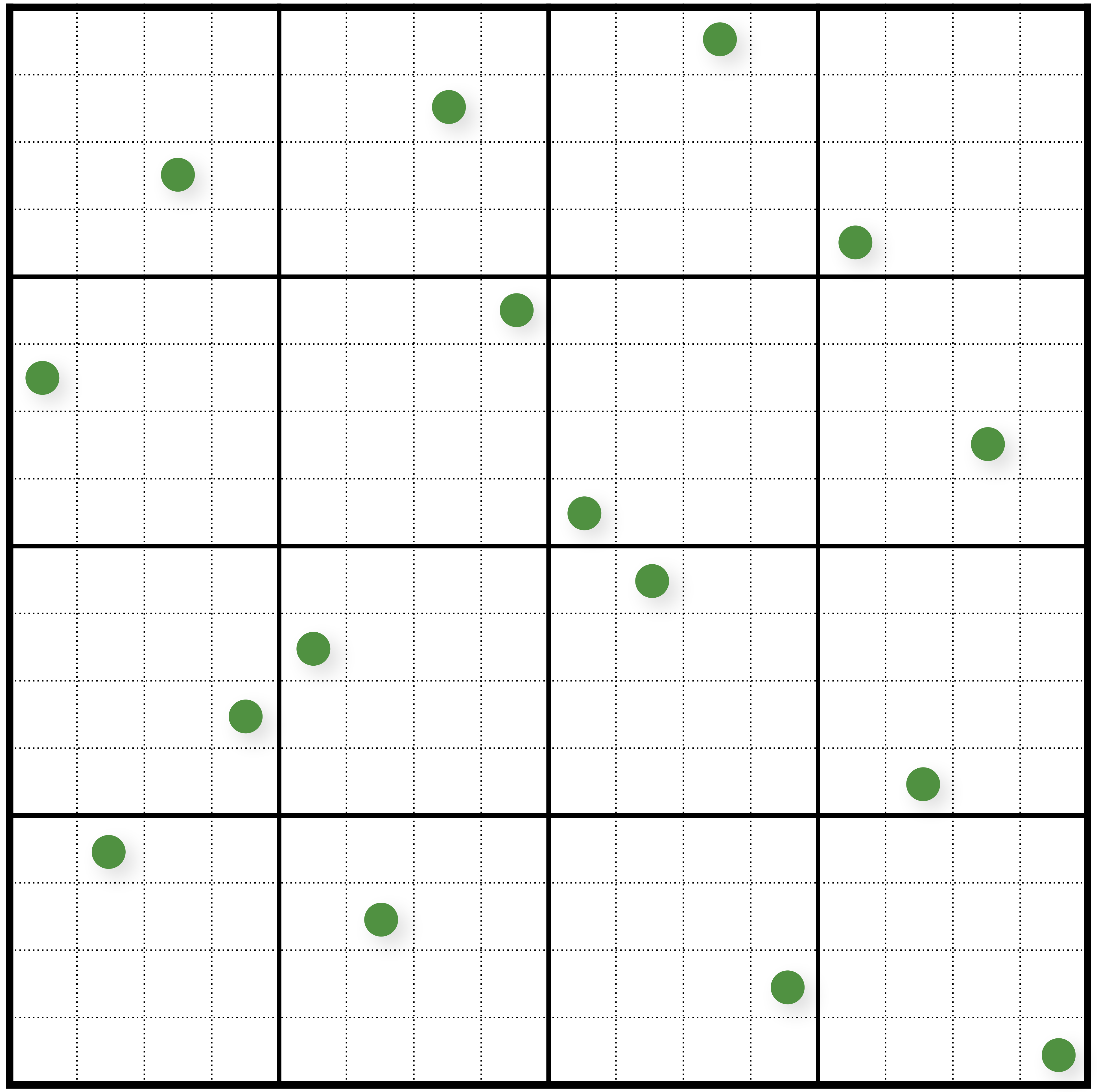

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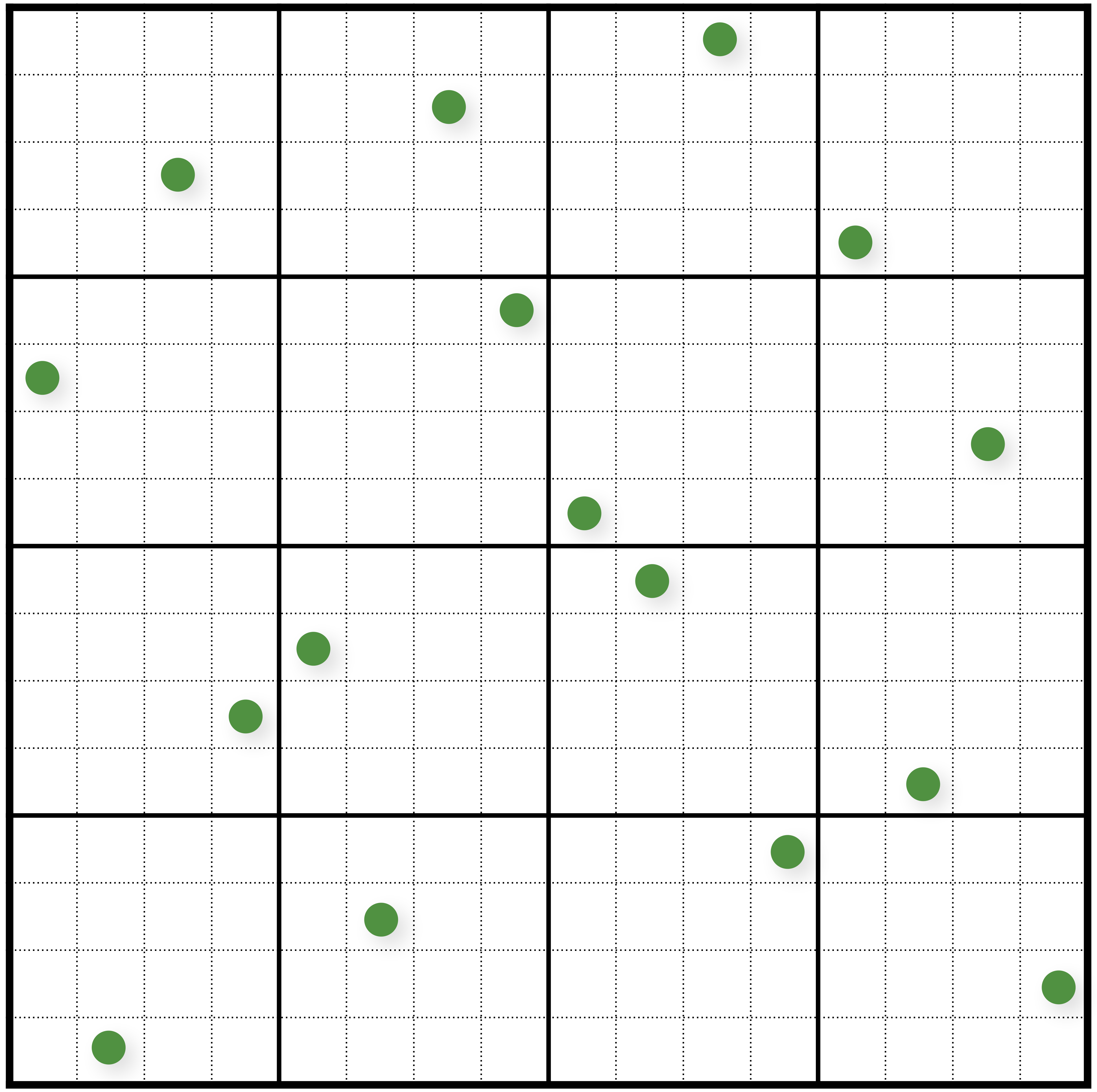
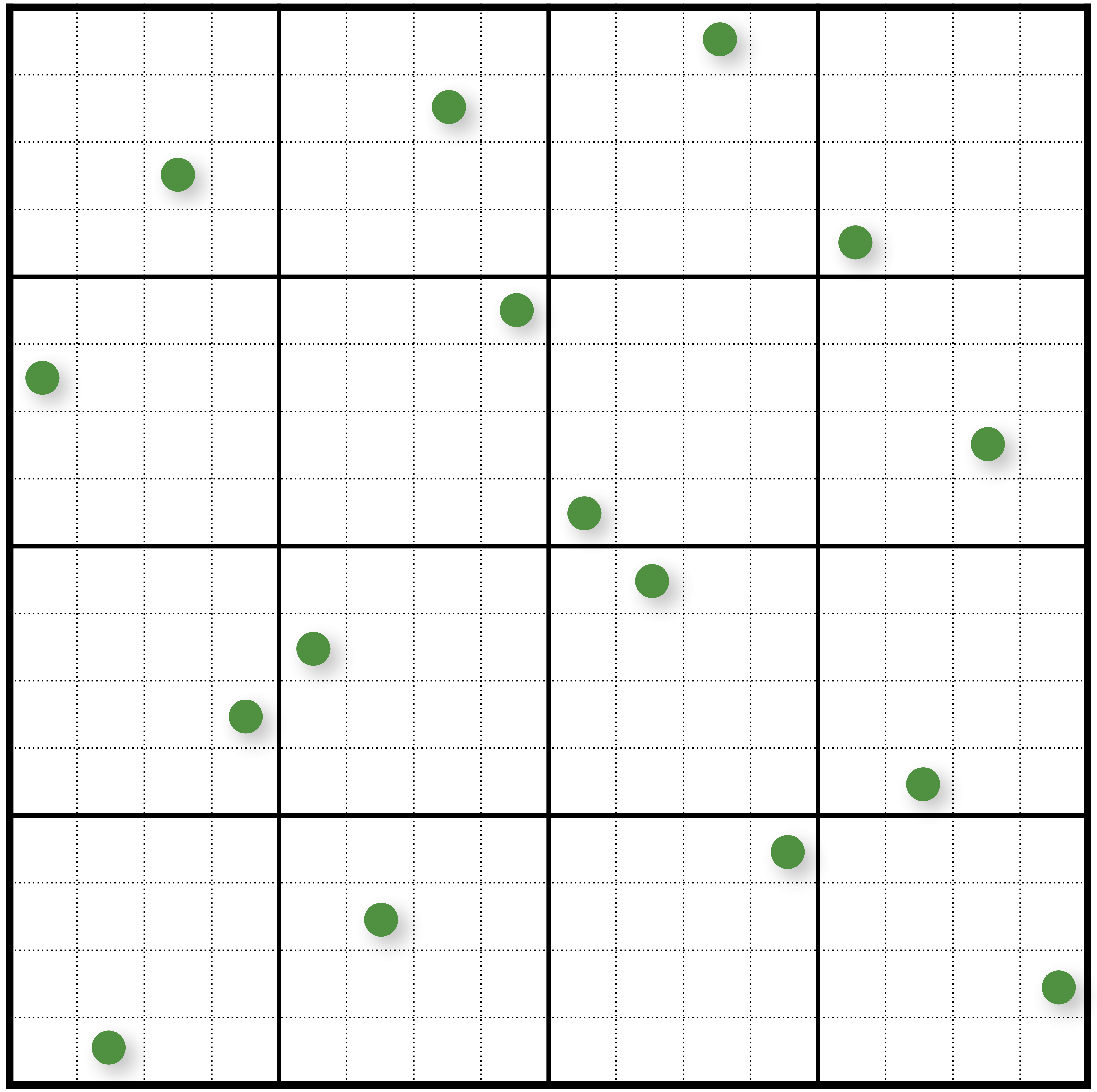
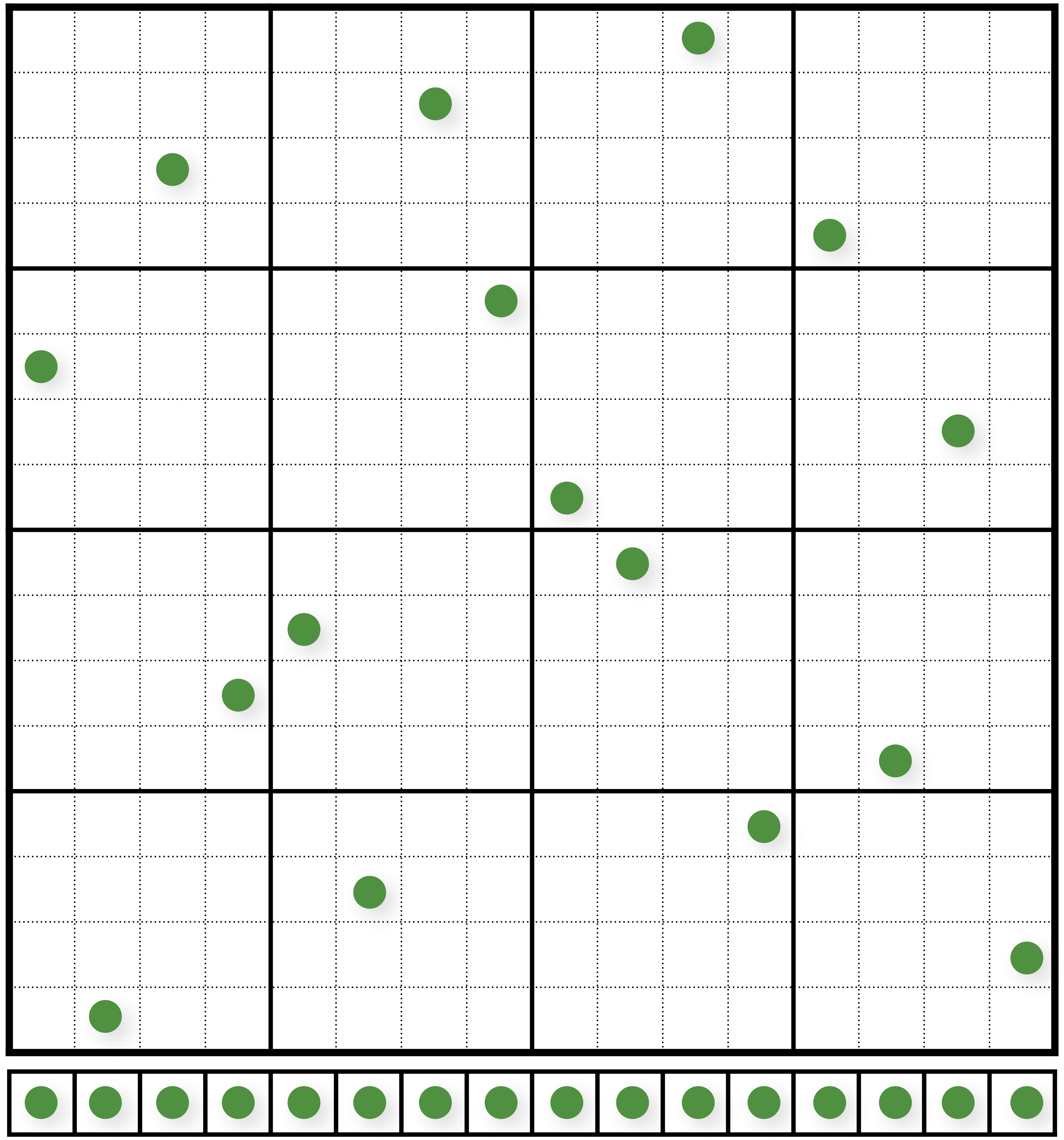


Shuffle y-coords

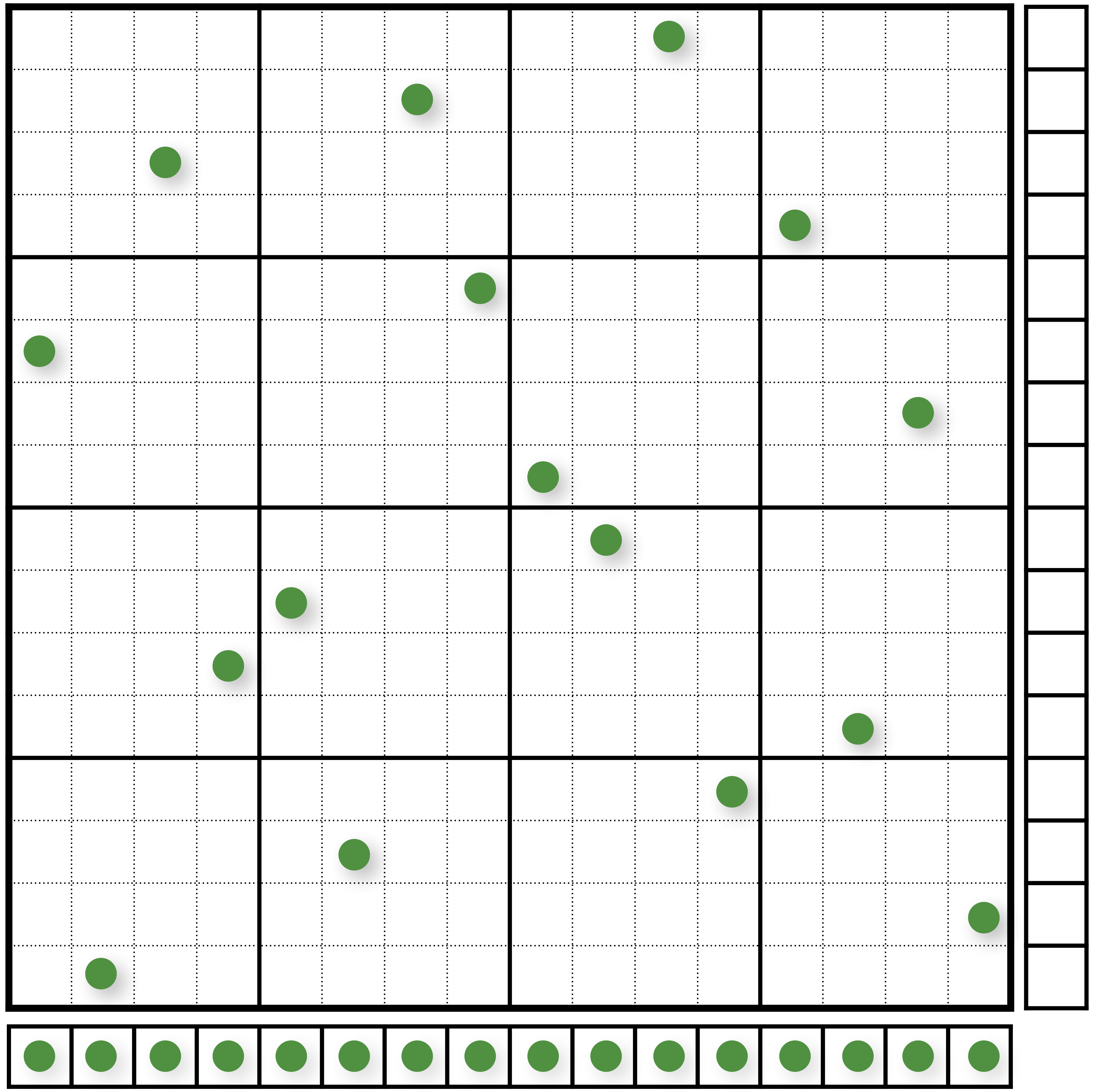
Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)



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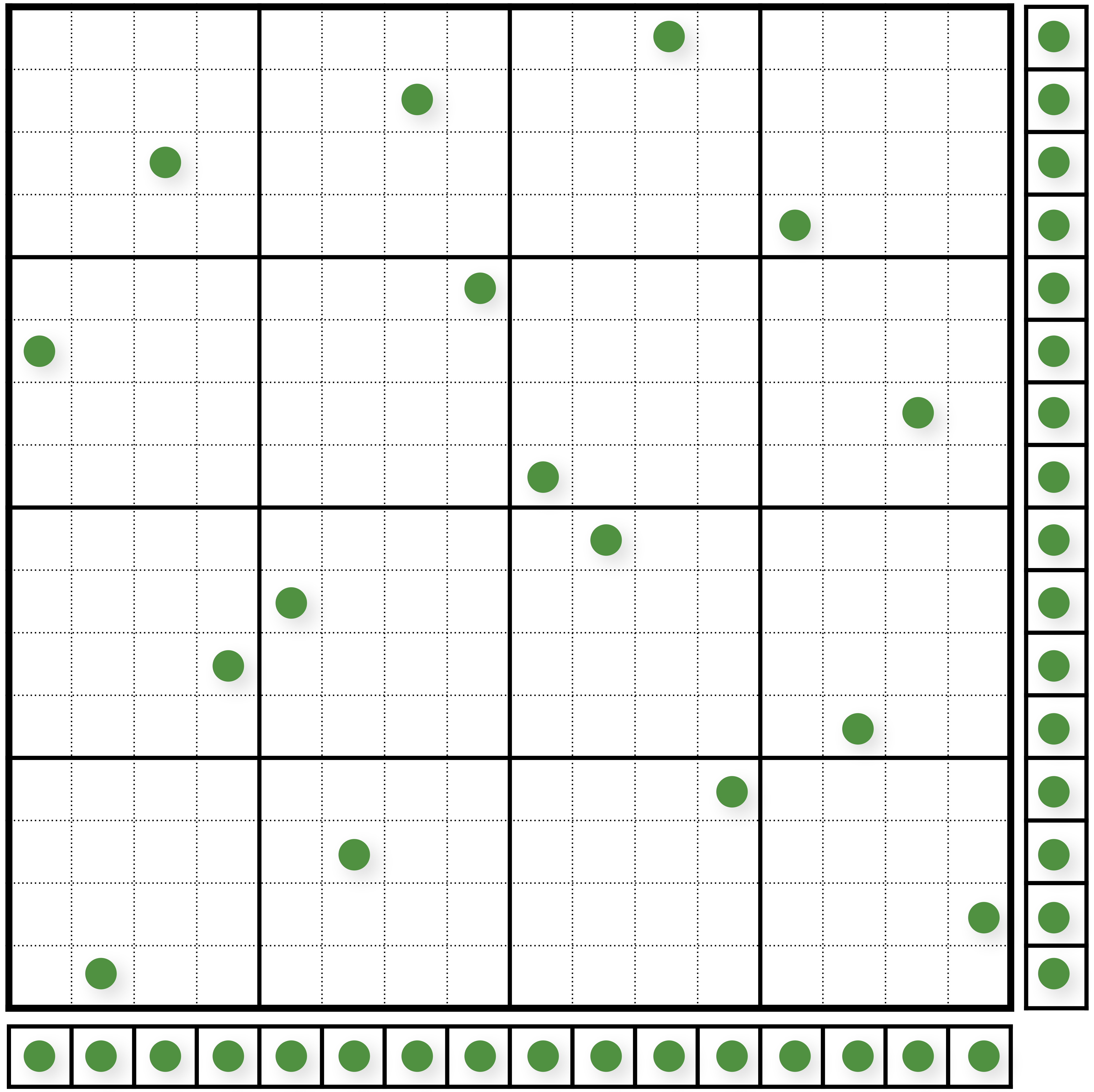
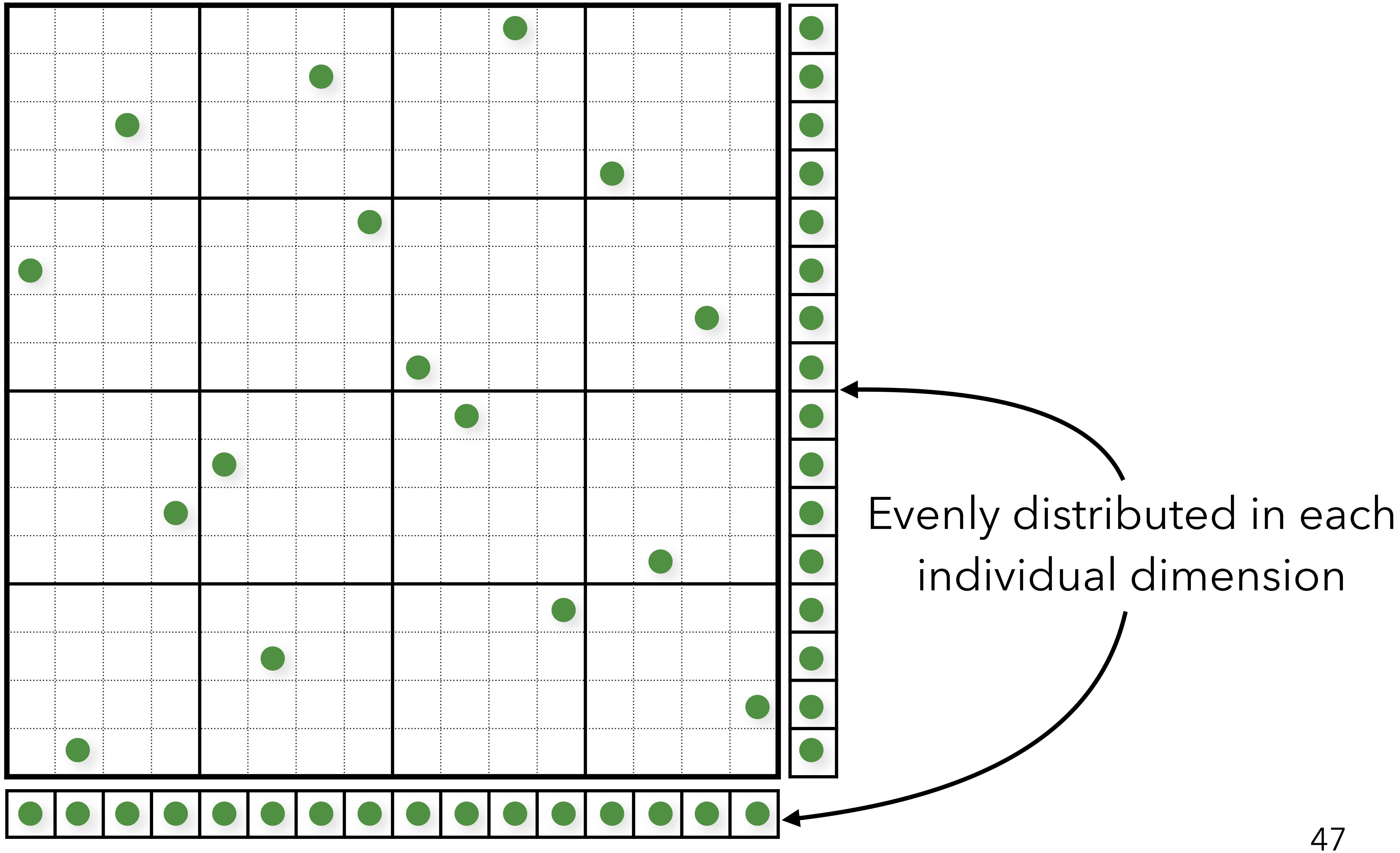
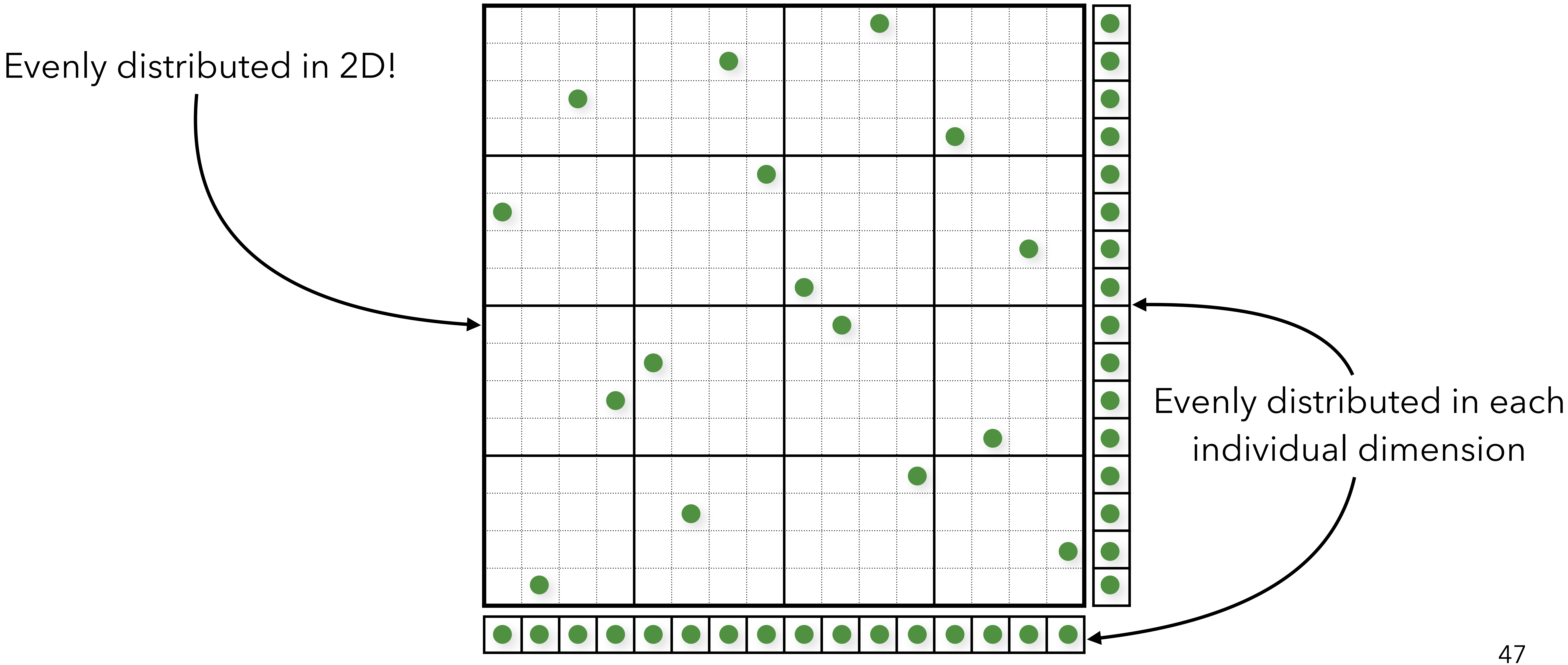
Multi-Jittered Sampling (Projections)



Multi-Jittered Sampling (Projections)

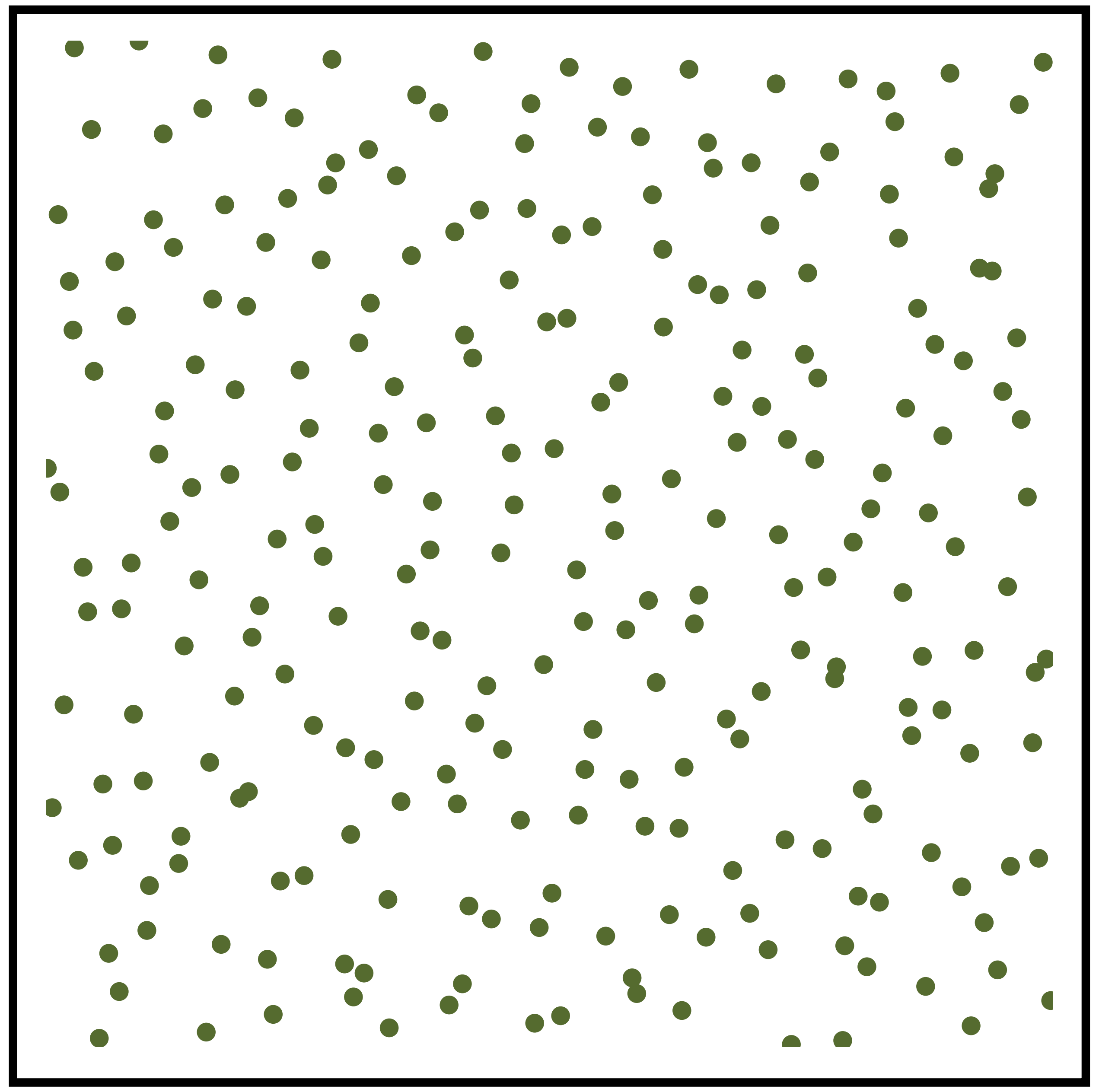


Multi-Jittered Sampling (Projections)

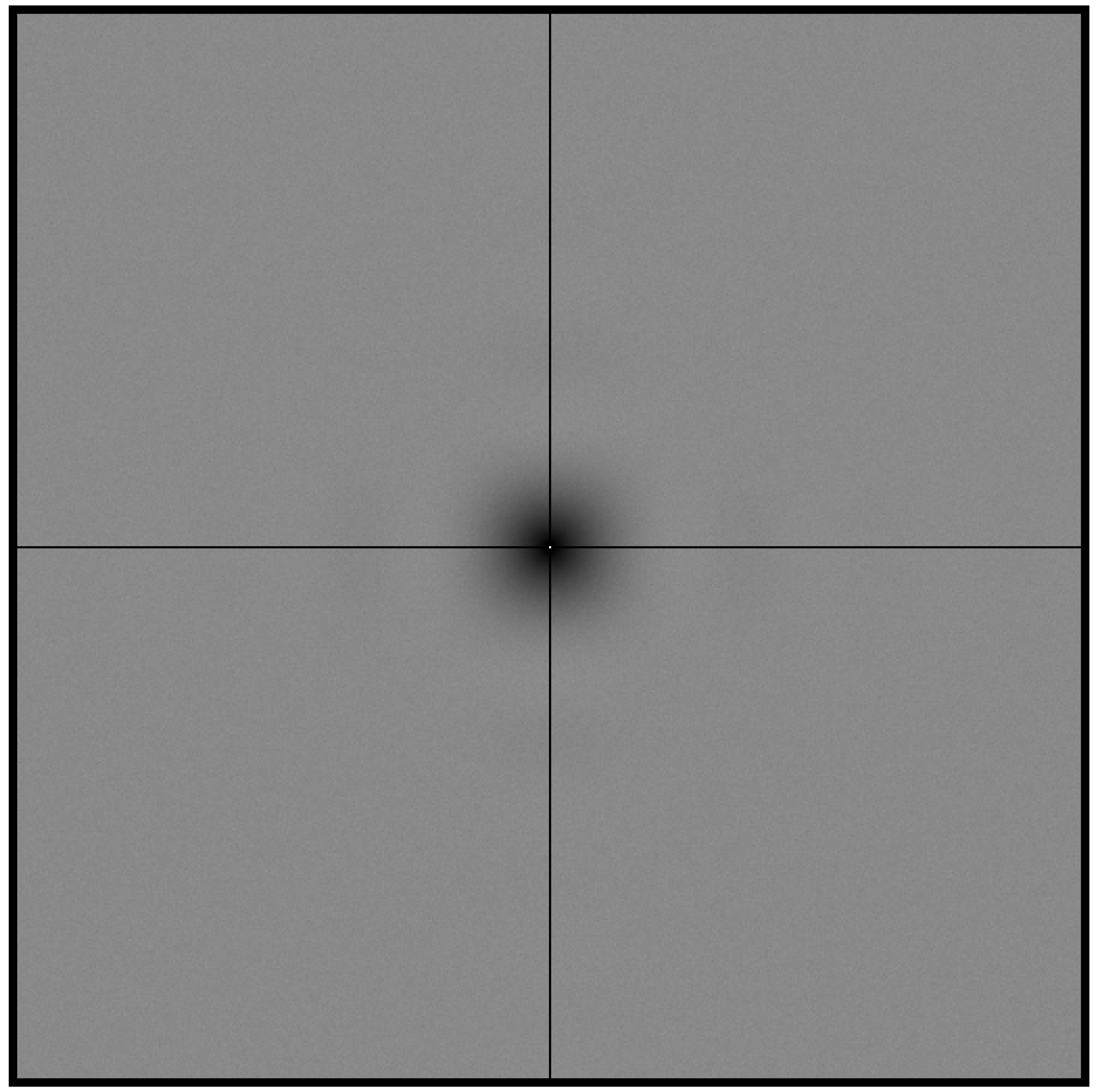


Multi-Jittered Sampling

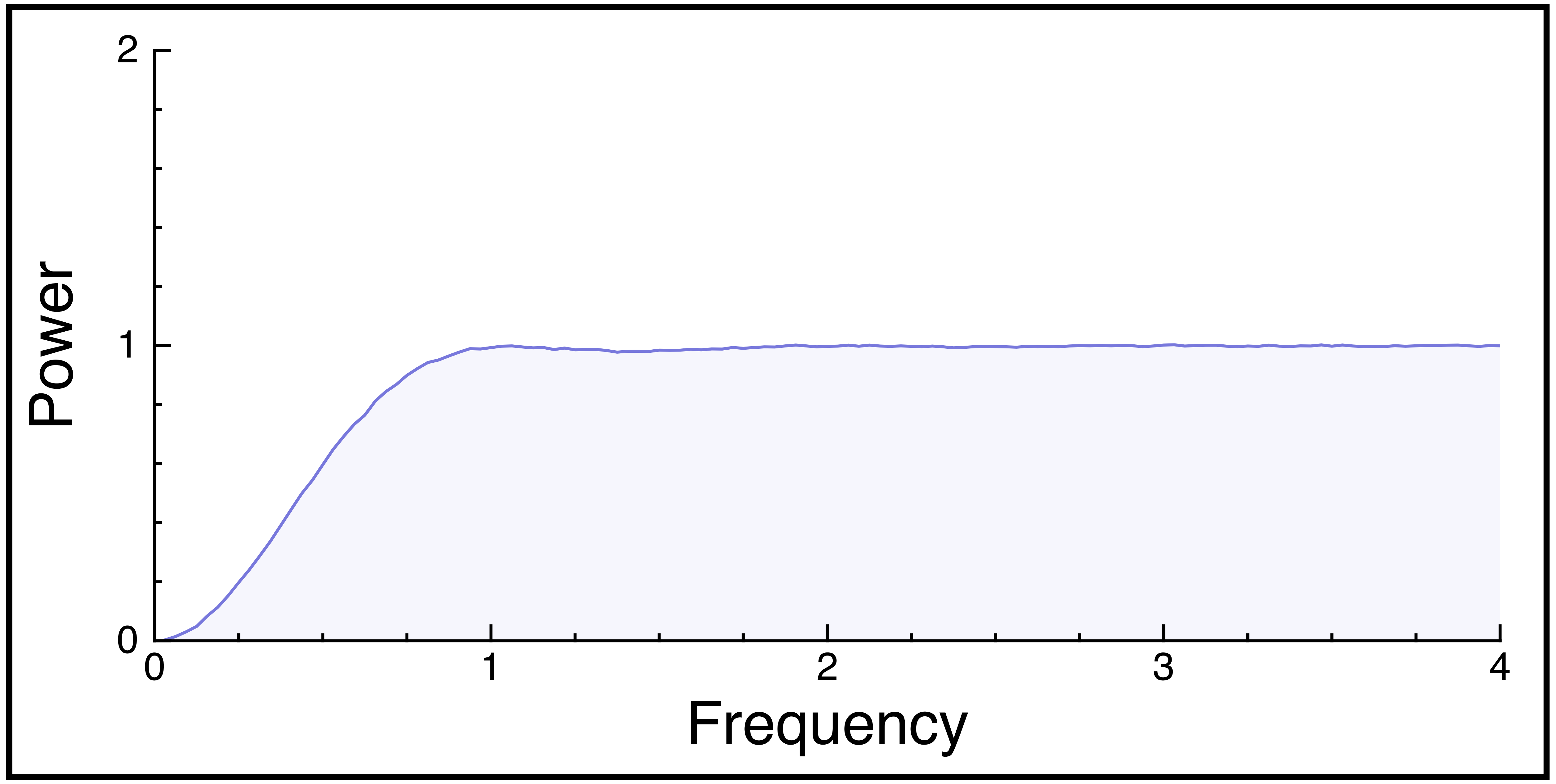
Samples



Expected power spectrum

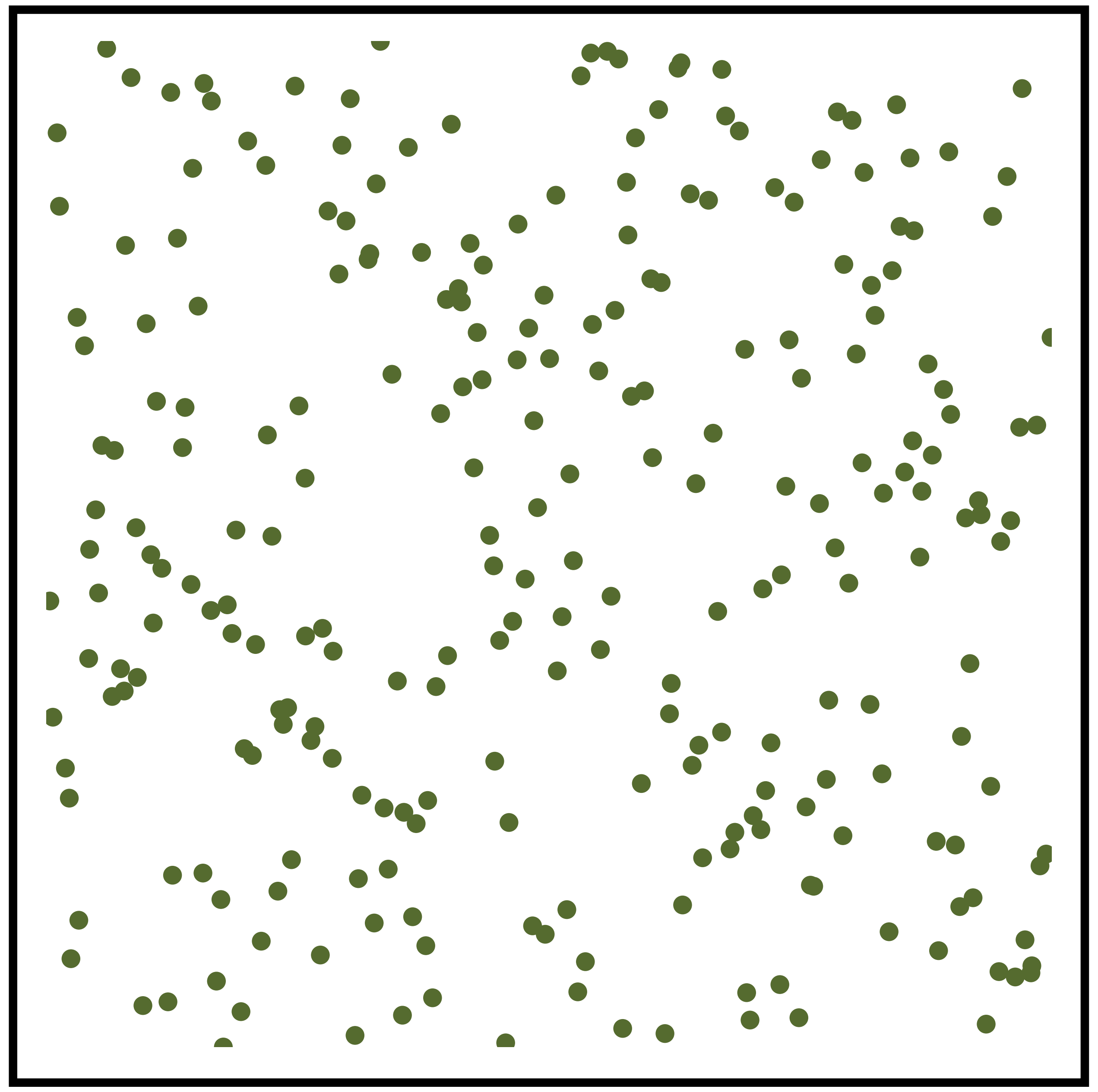


Radial mean

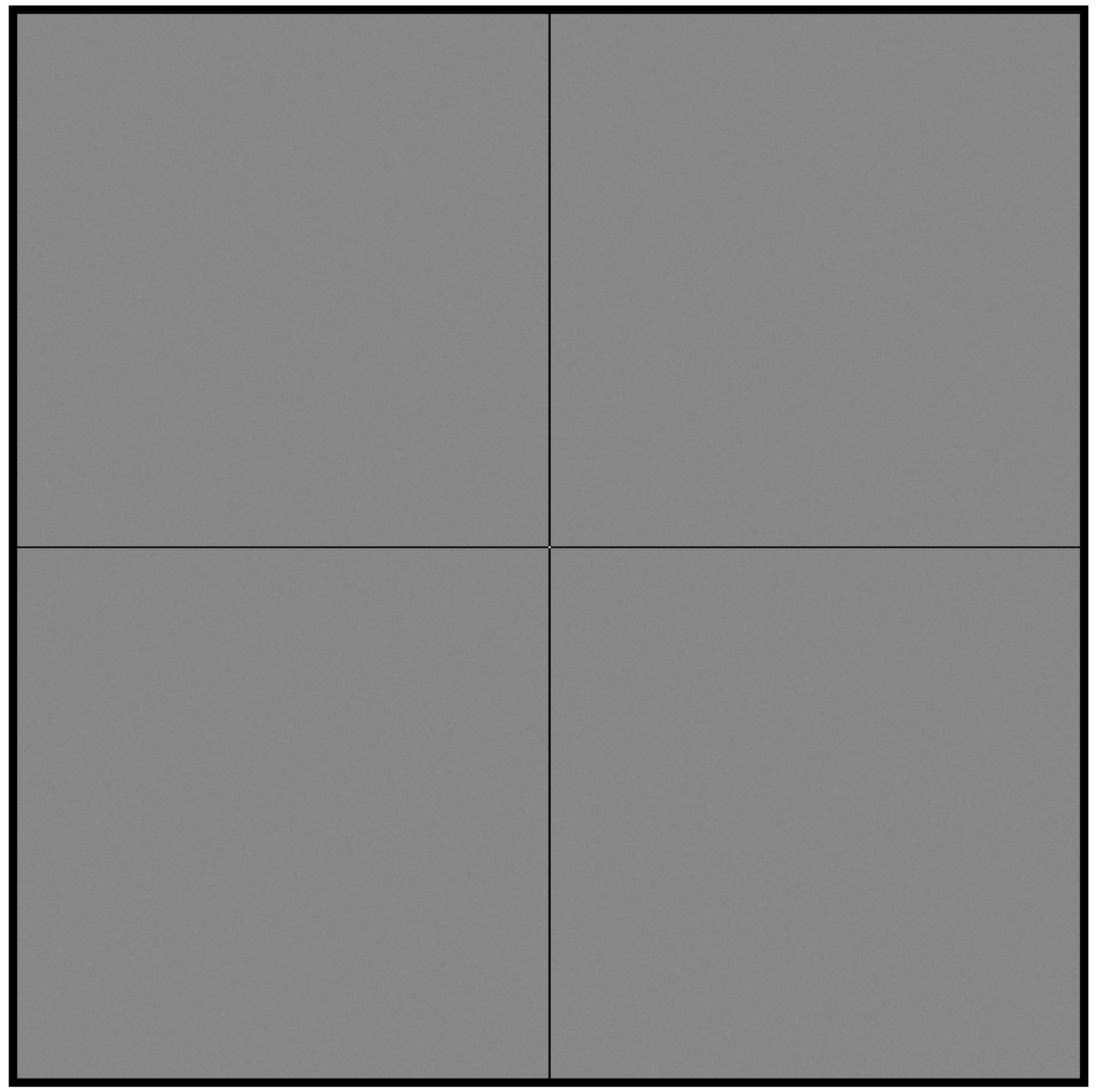


N-Rooks Sampling

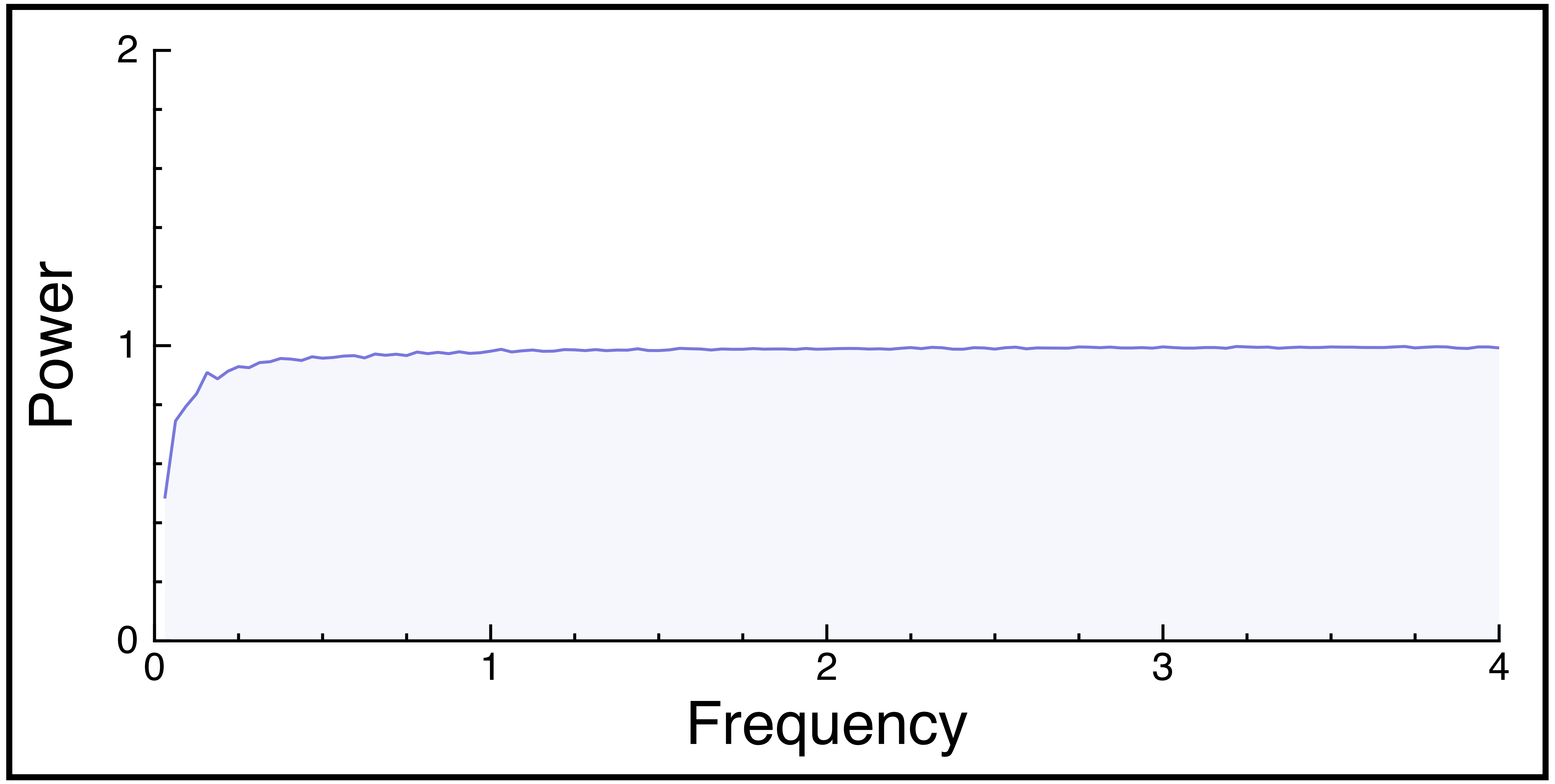
Samples



Expected power spectrum

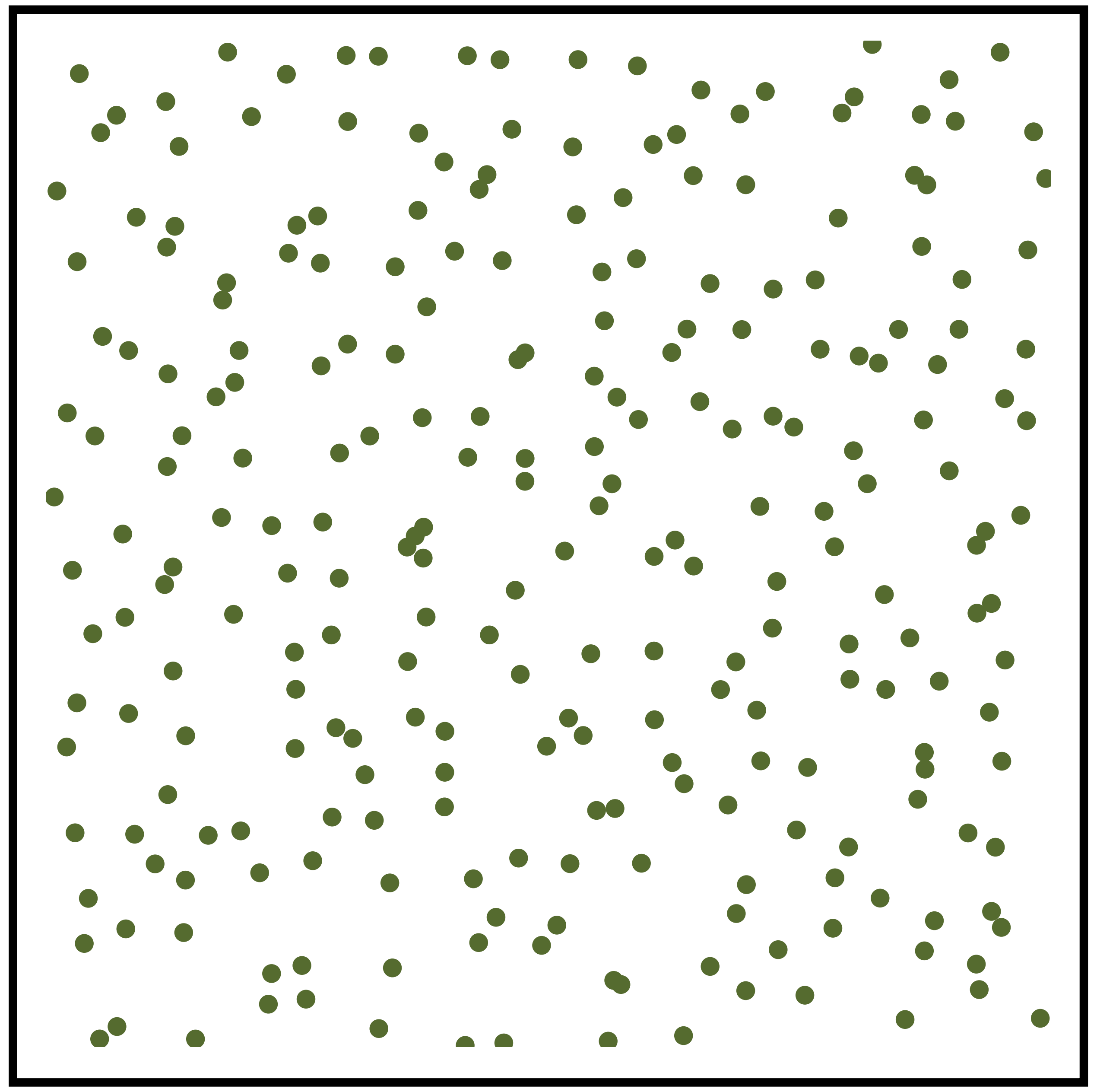


Radial mean

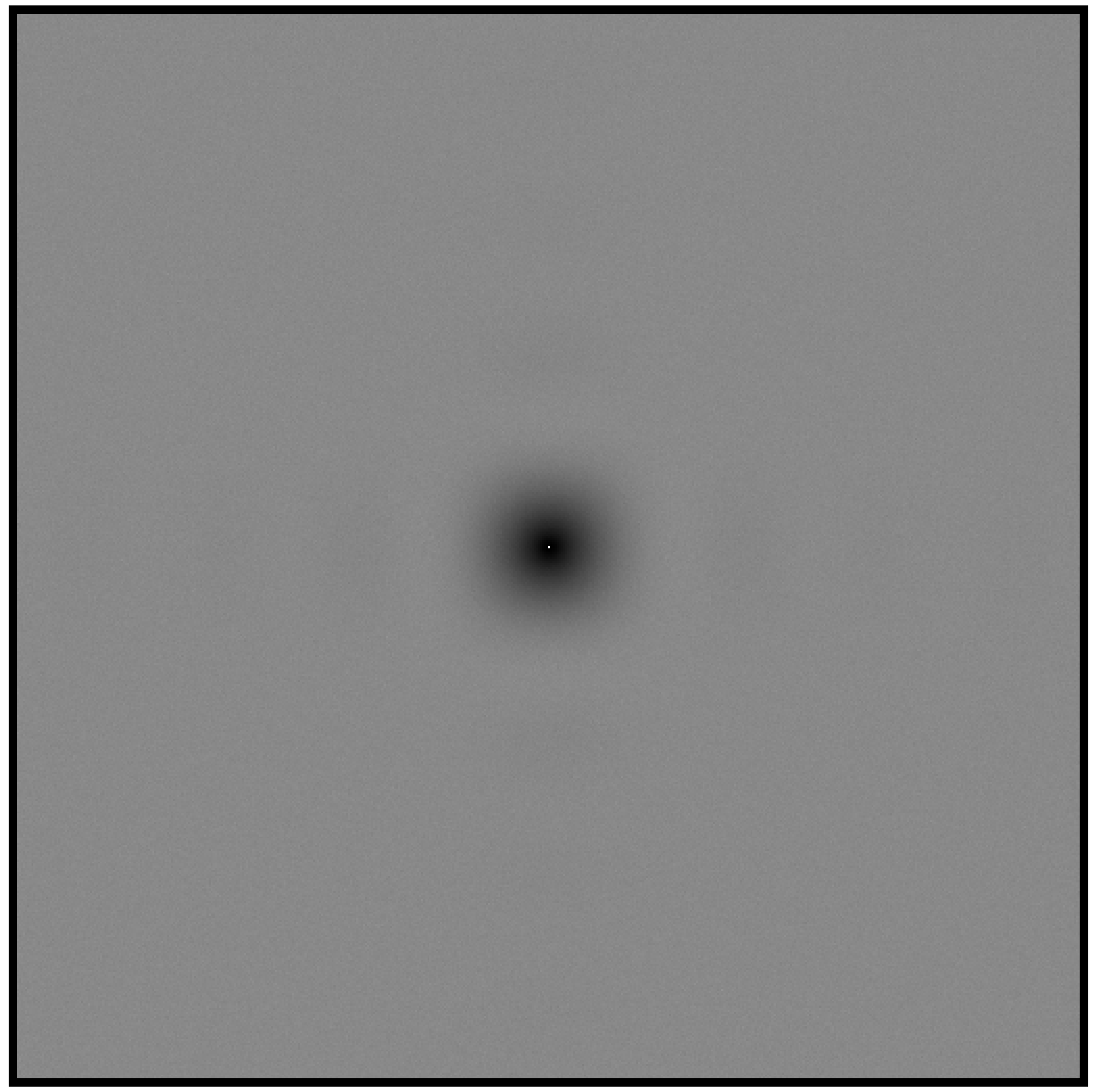


Jittered Sampling

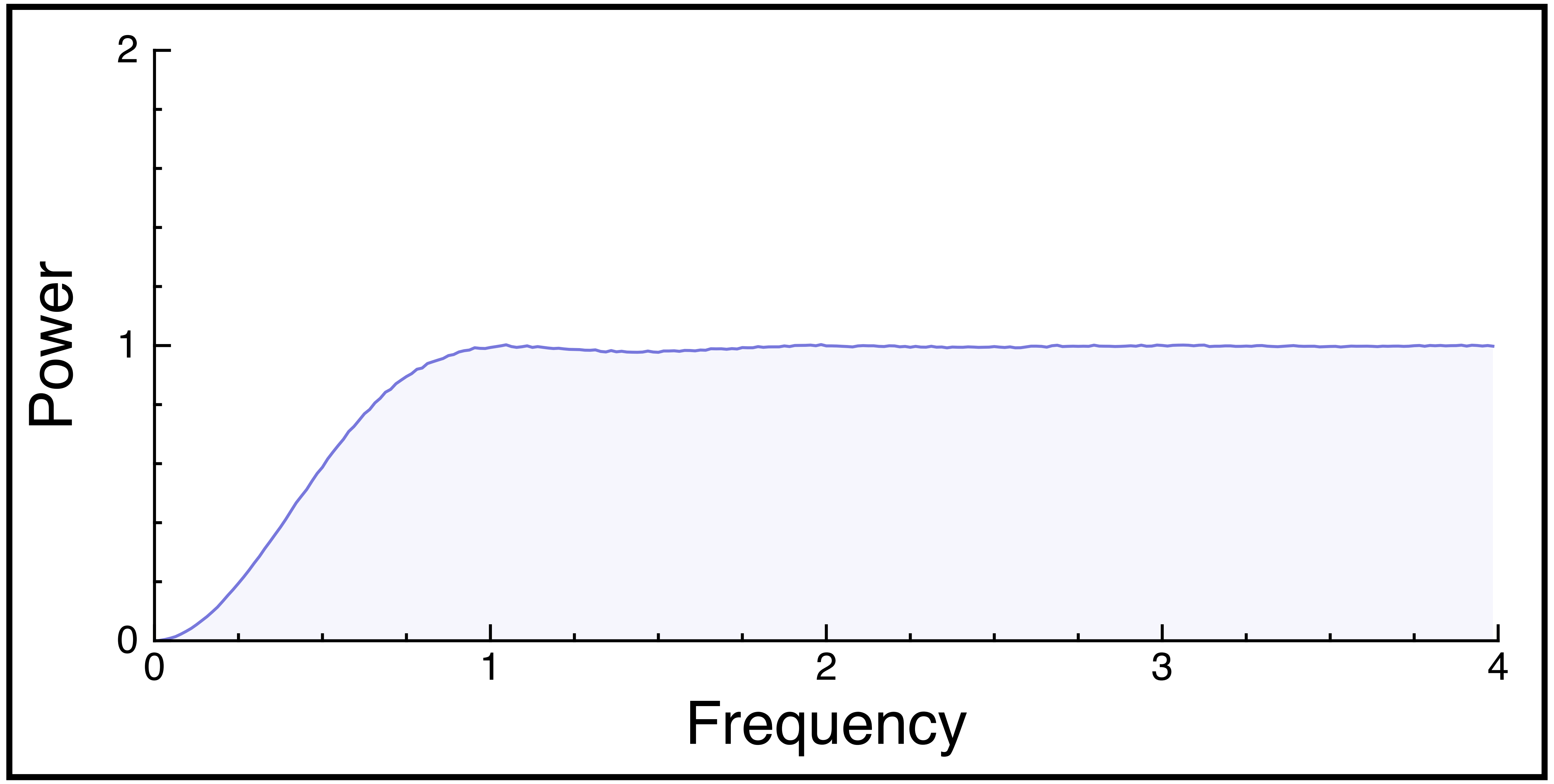
Samples



Expected power spectrum



Radial mean



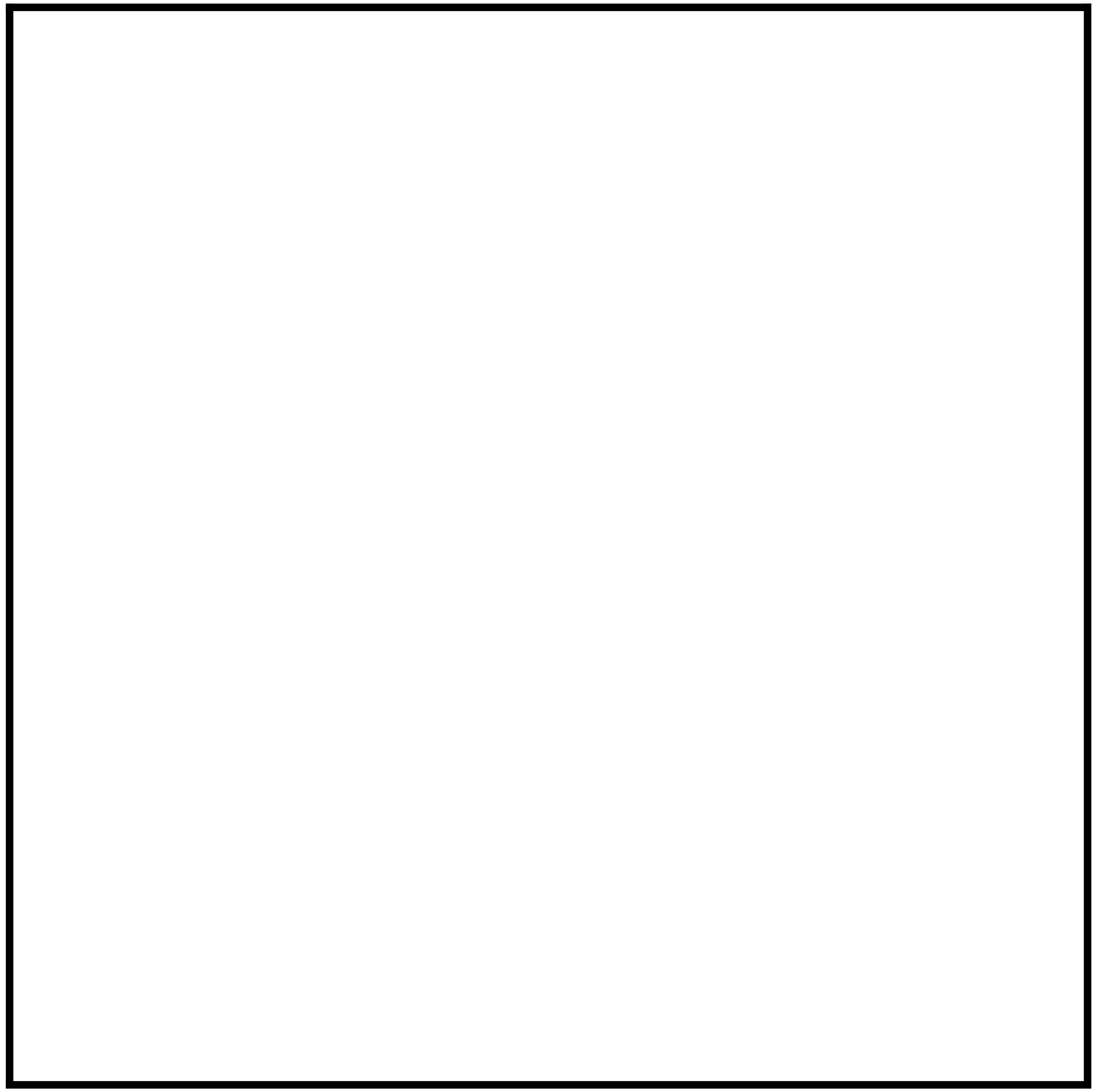
Poisson-Disk/Blue-Noise Sampling

Enforce a minimum distance between points

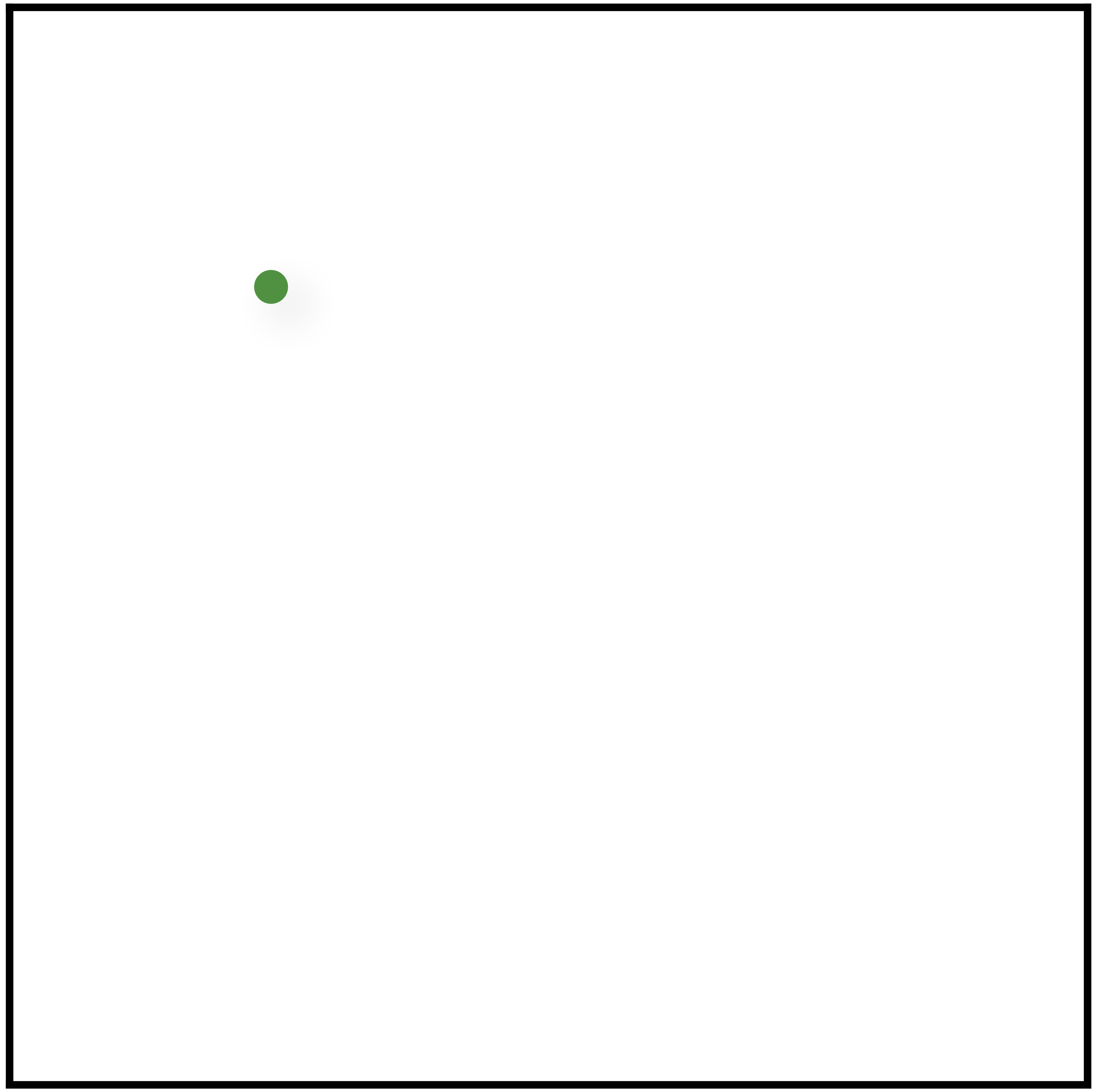
Poisson-Disk Sampling:

- Mark A. Z. Dippé and Erling Henry Wold. “Antialiasing through stochastic sampling.” *ACM SIGGRAPH,* 1985.
- Robert L. Cook. “Stochastic sampling in computer graphics.” *ACM Transactions on Graphics*, 1986.
- Ares Lagae and Philip Dutré. “A comparison of methods for generating Poisson disk distributions.” *Computer Graphics Forum*, 2008.

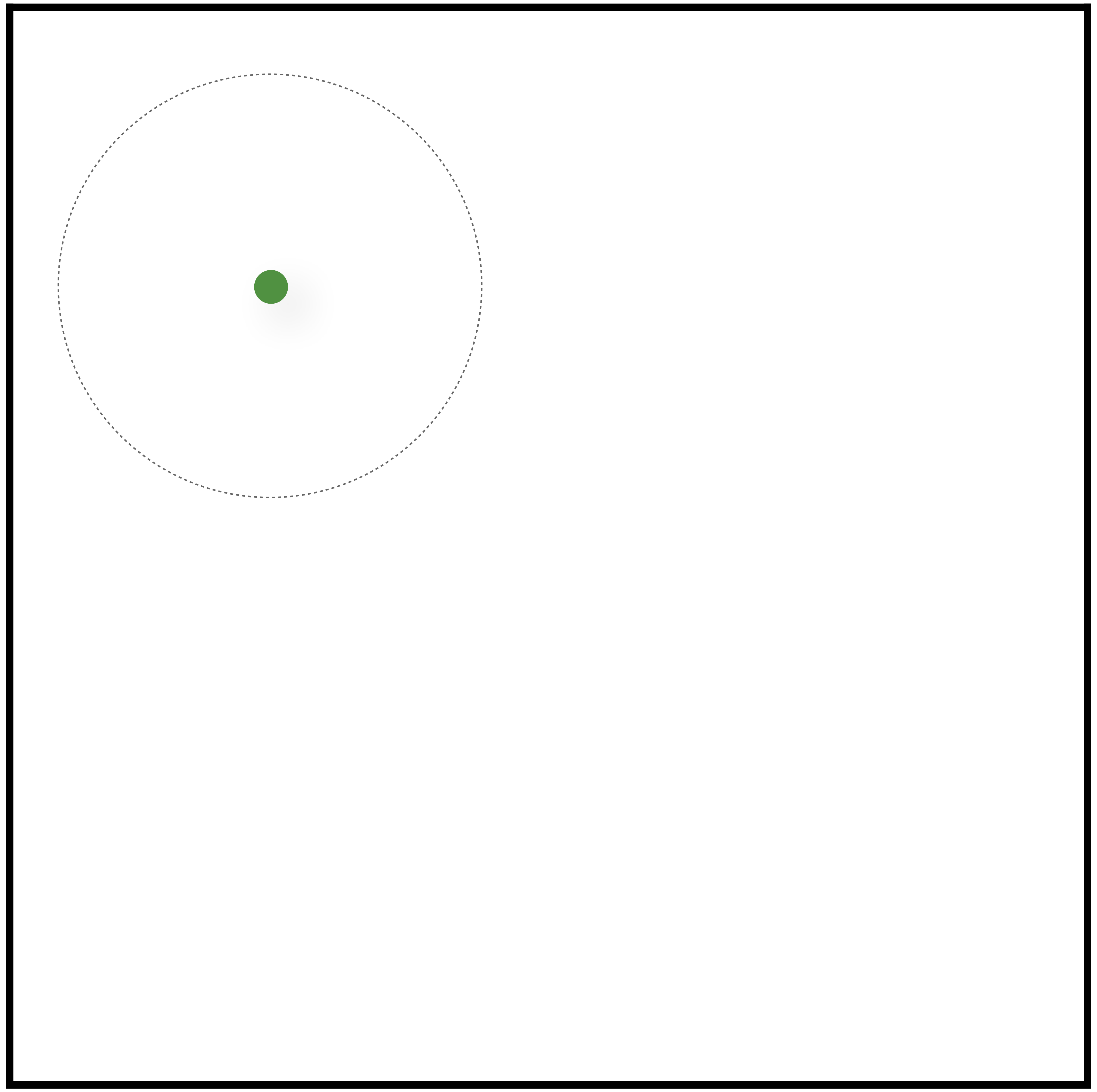
Random Dart Throwing



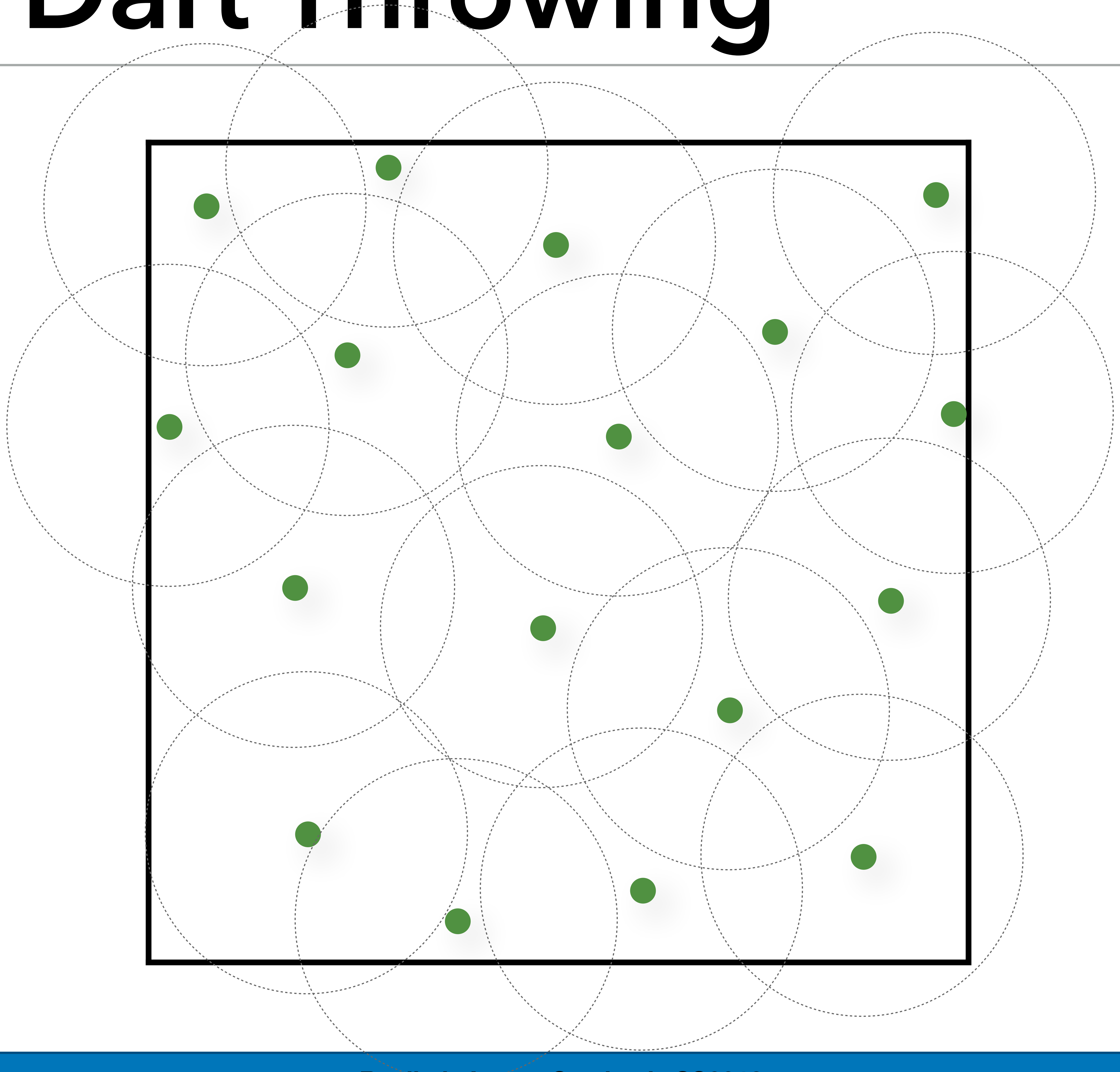
Random Dart Throwing



Random Dart Throwing



Random Dart Throwing



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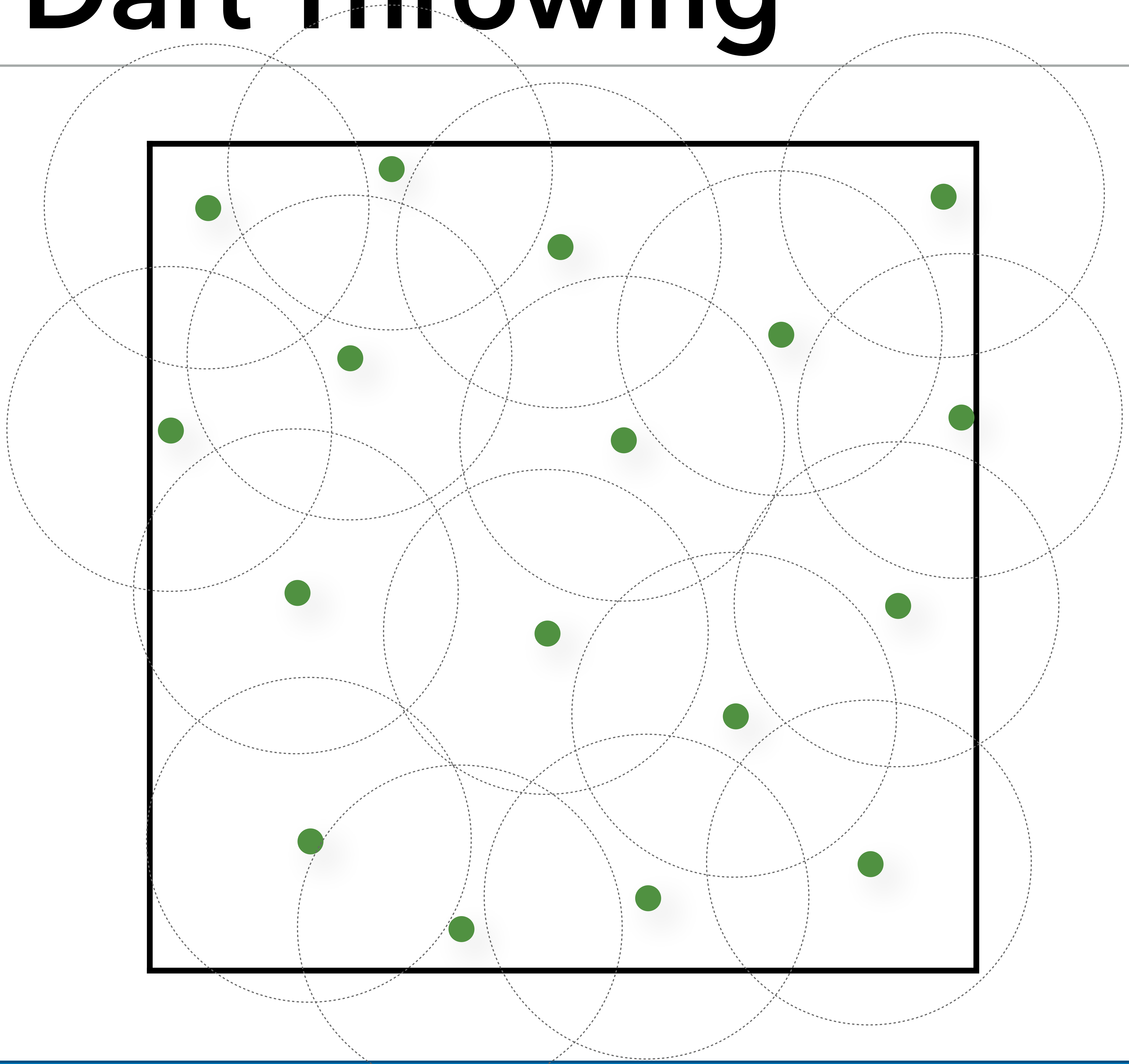
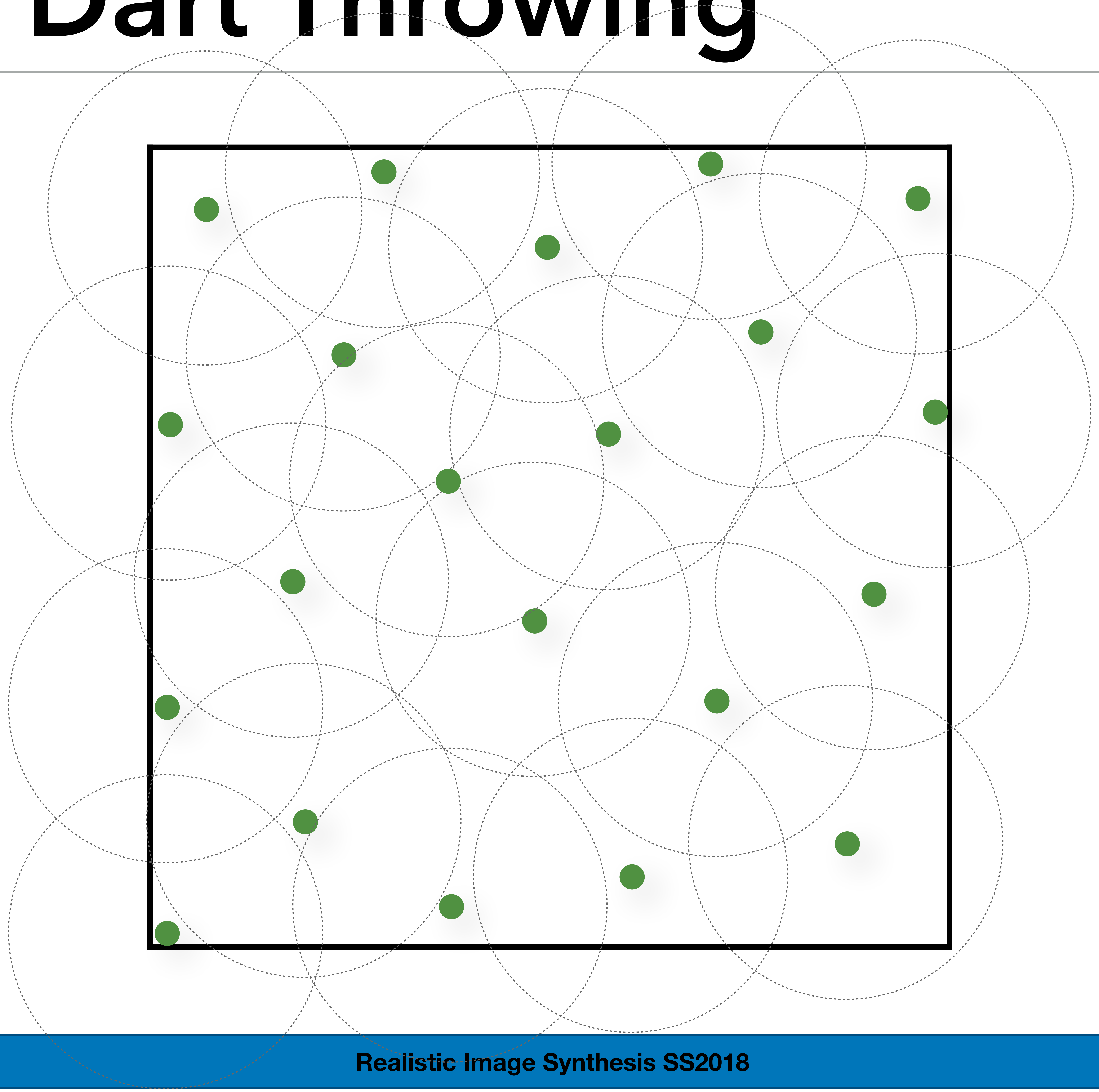
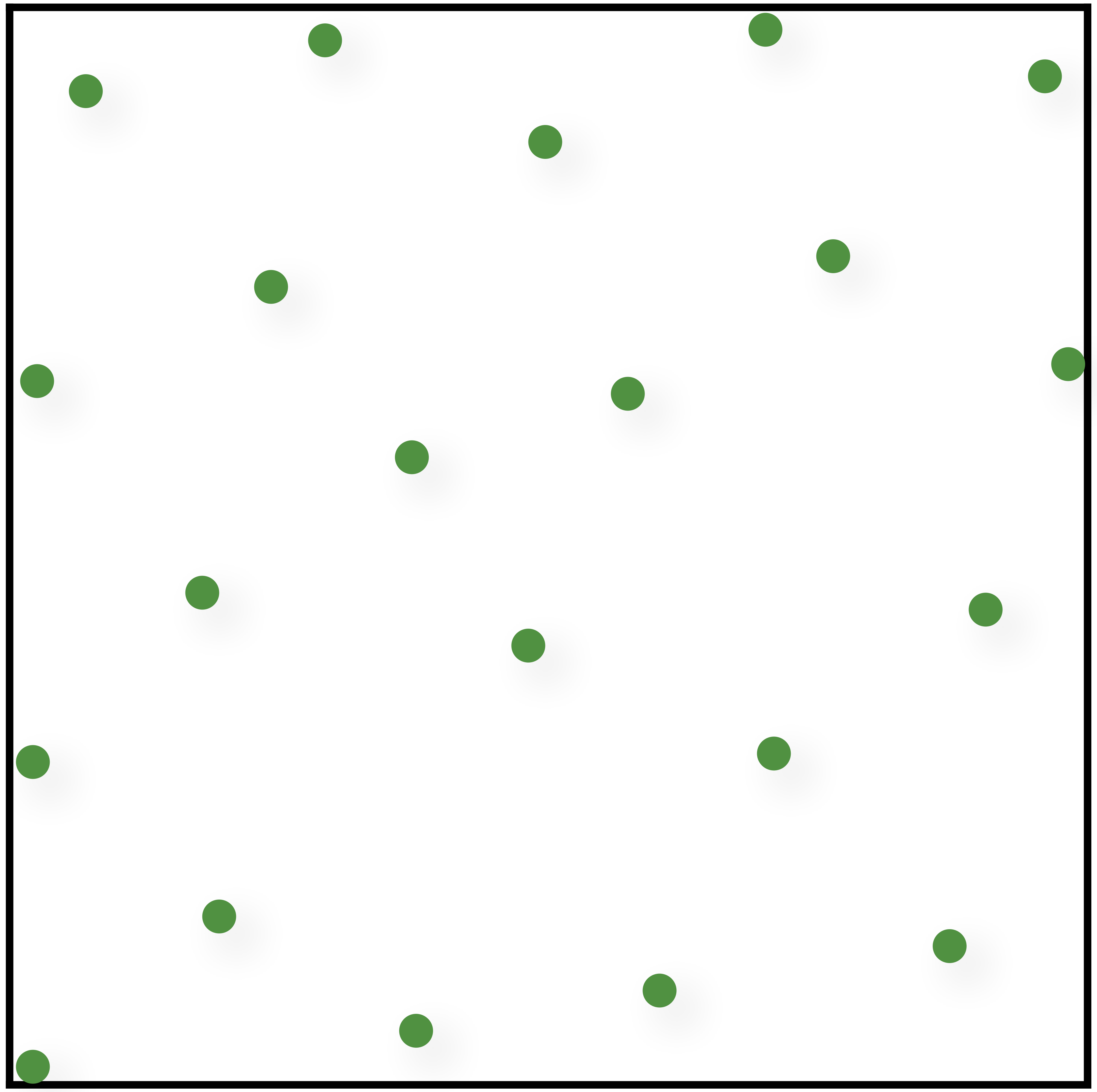
Random Dart Throwing



Random Dart Throwing

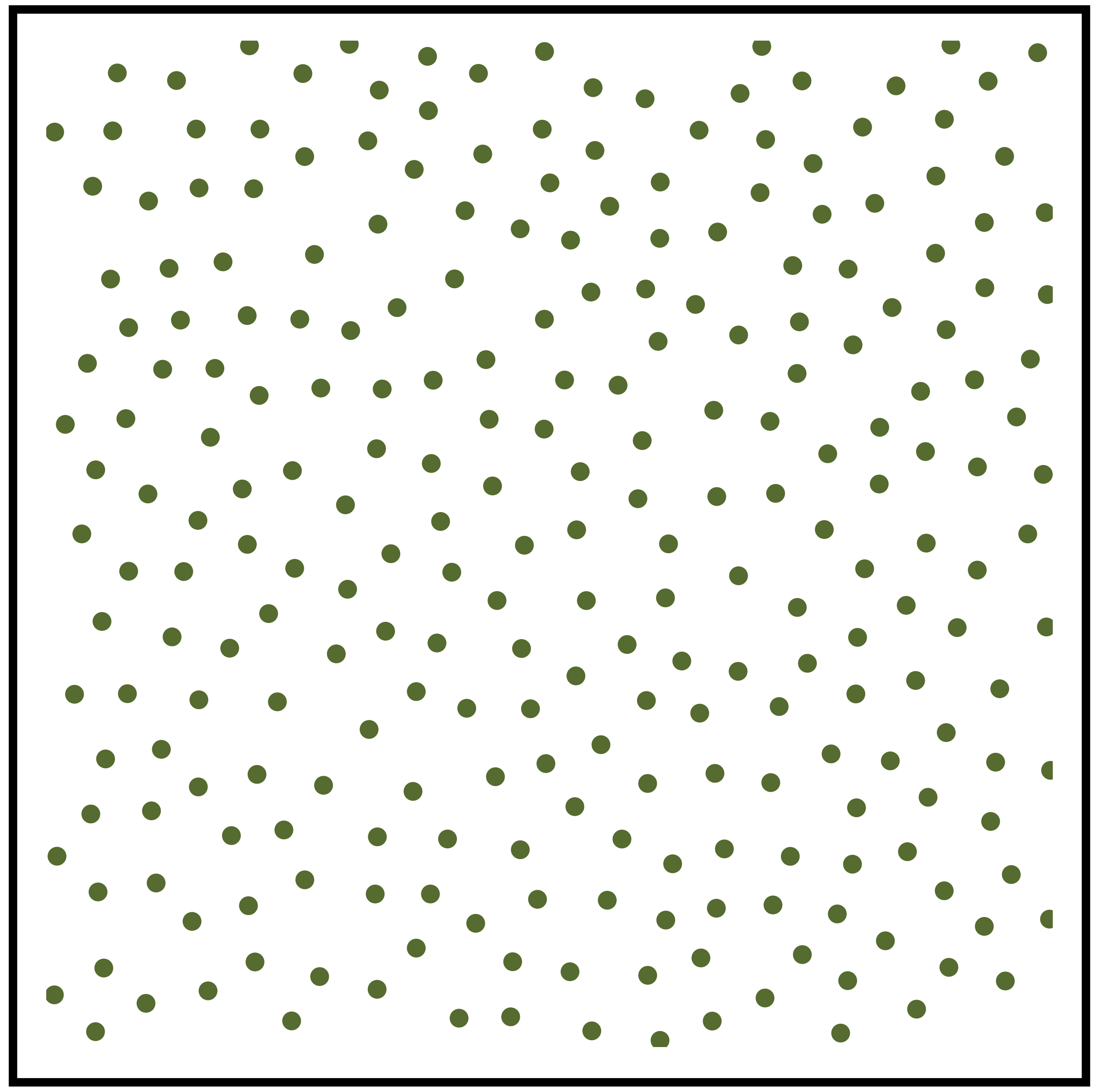


Random Dart Throwing

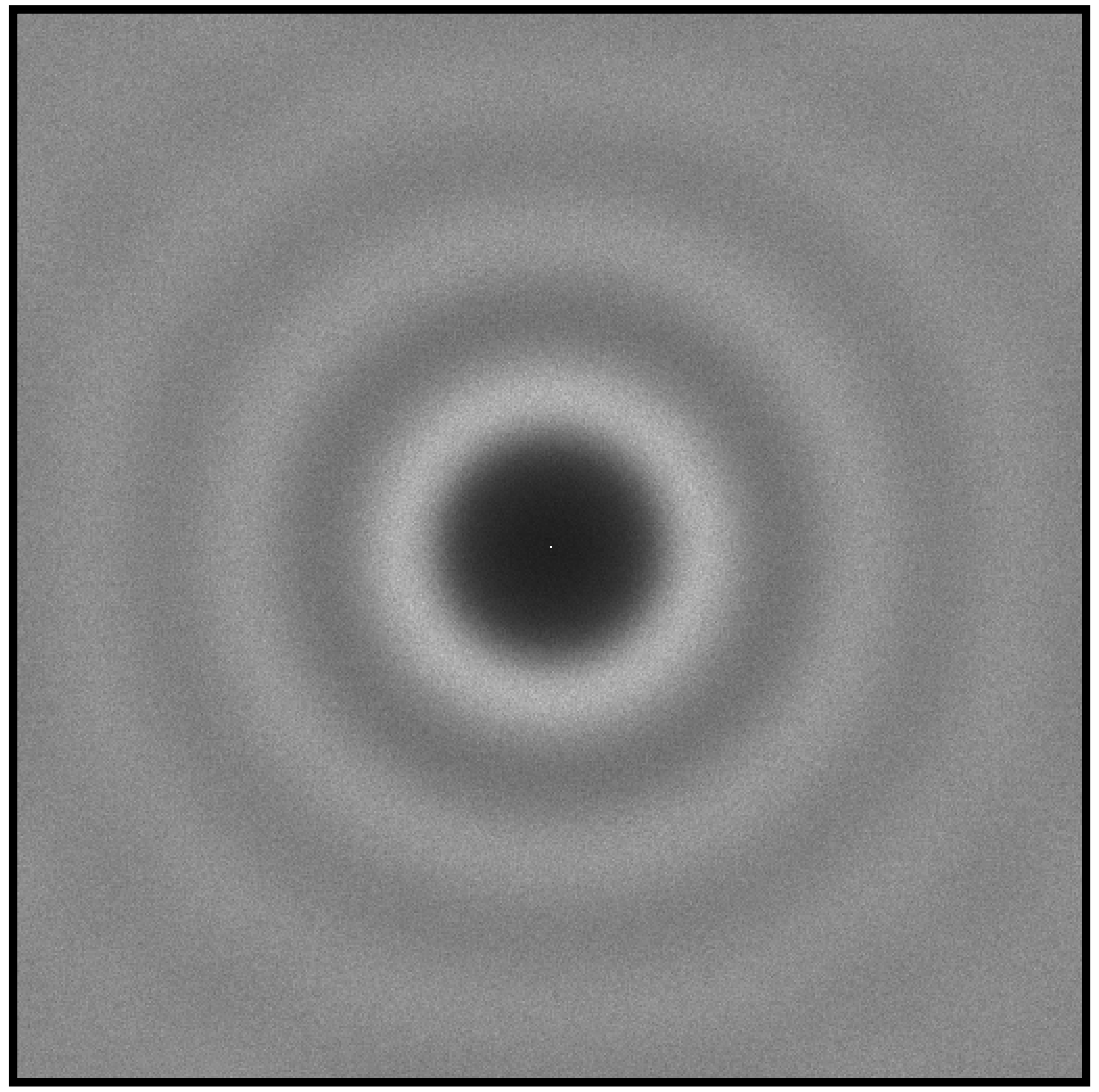


Poisson Disk Sampling

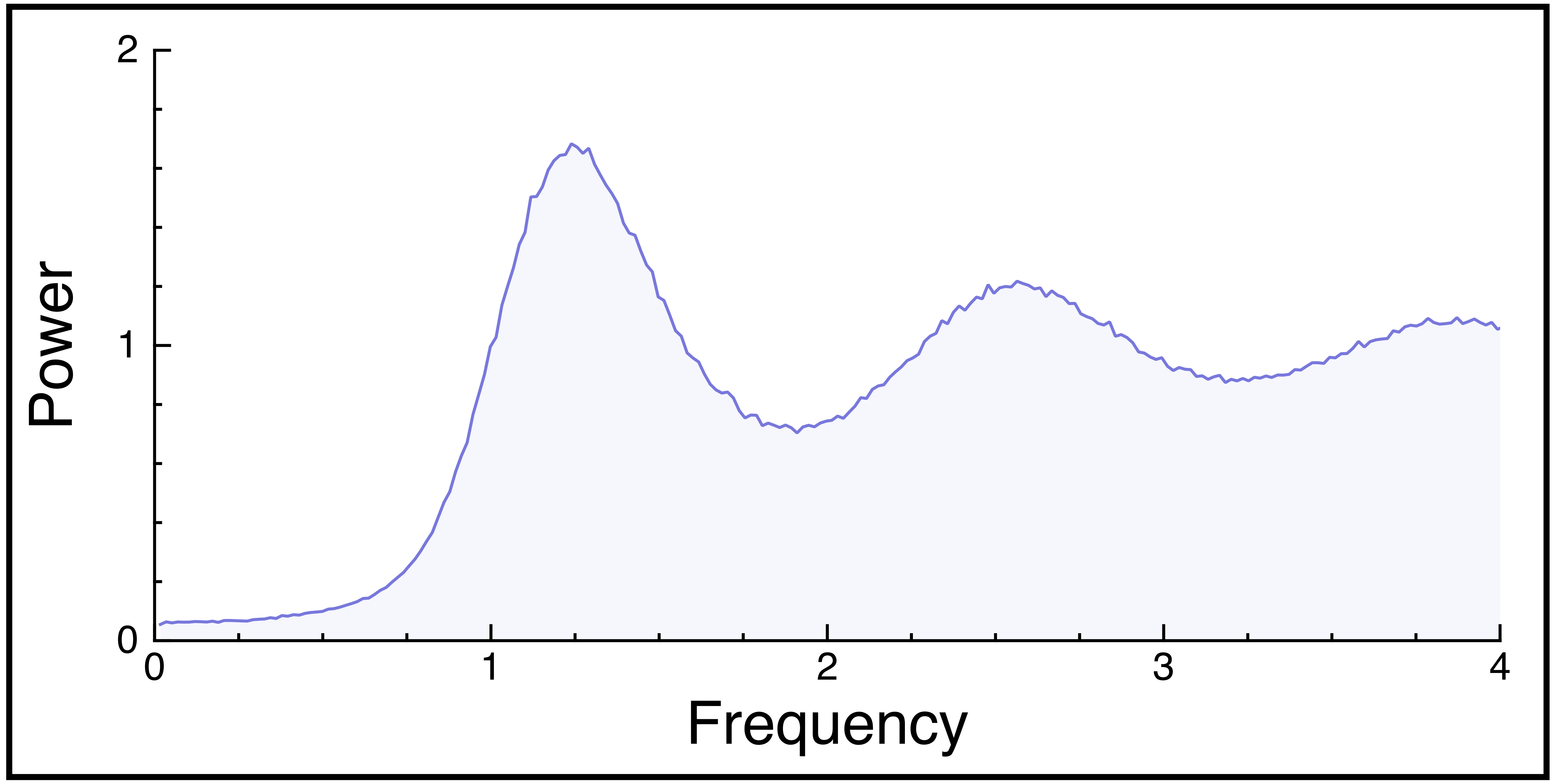
Samples



Expected power spectrum



Radial mean



Blue-Noise Sampling (Relaxation-based)

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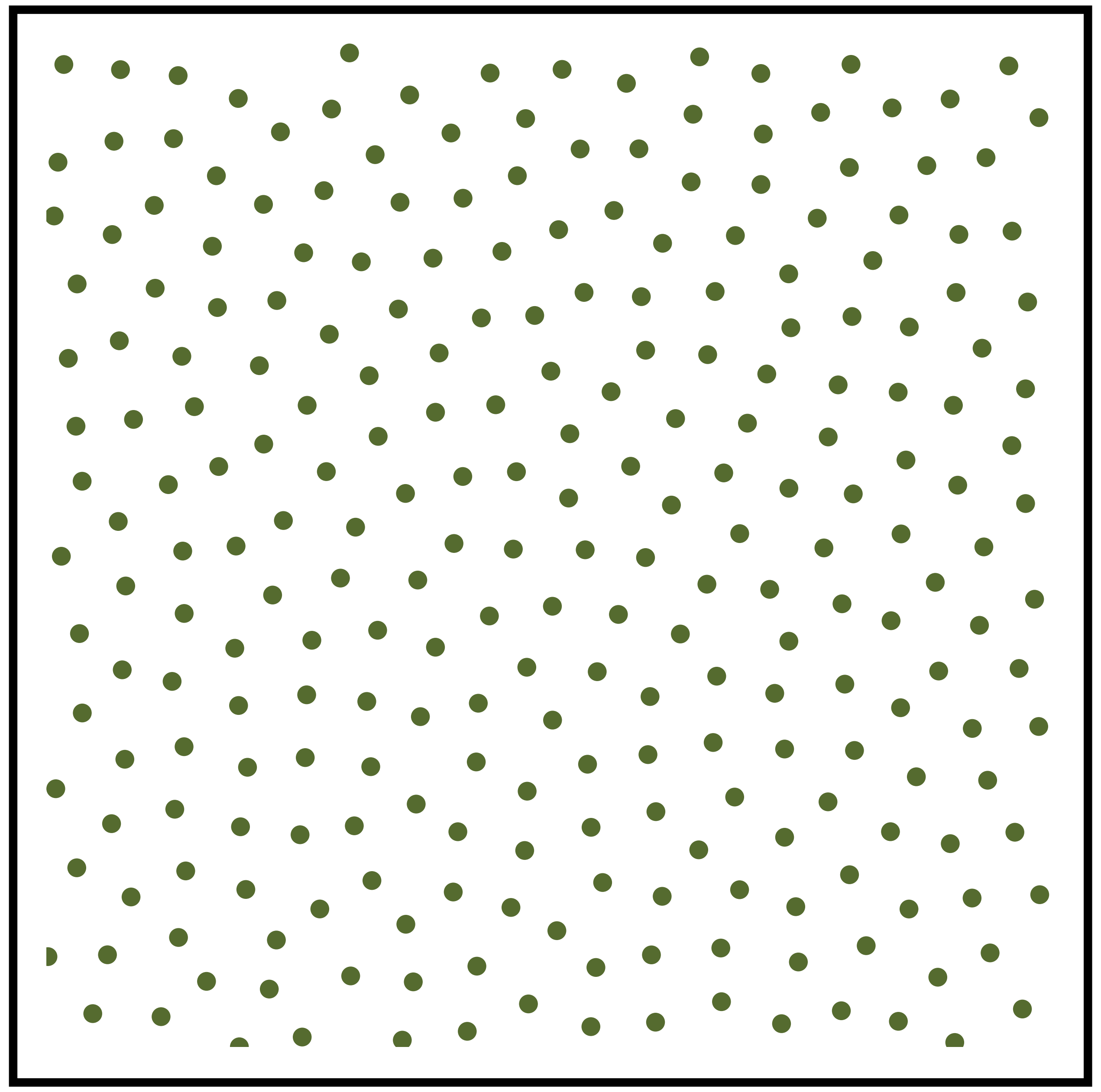
1. Initialize sample positions (e.g. random)

Blue-Noise Sampling (Relaxation-based)

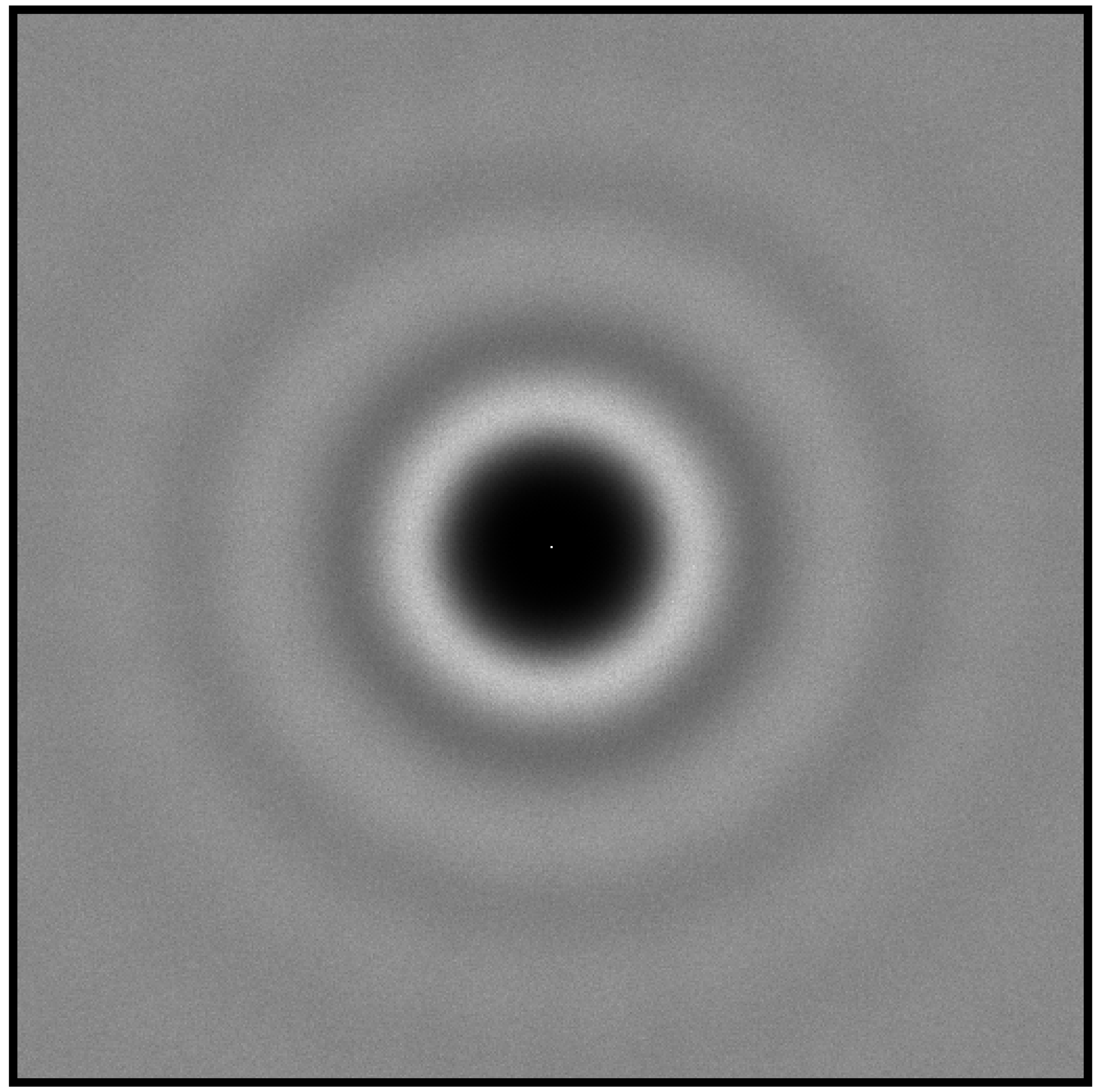
1. Initialize sample positions (e.g. random)
2. Use an iterative relaxation to move samples away from each other.

CCVT Sampling [Balzer et al. 2009]

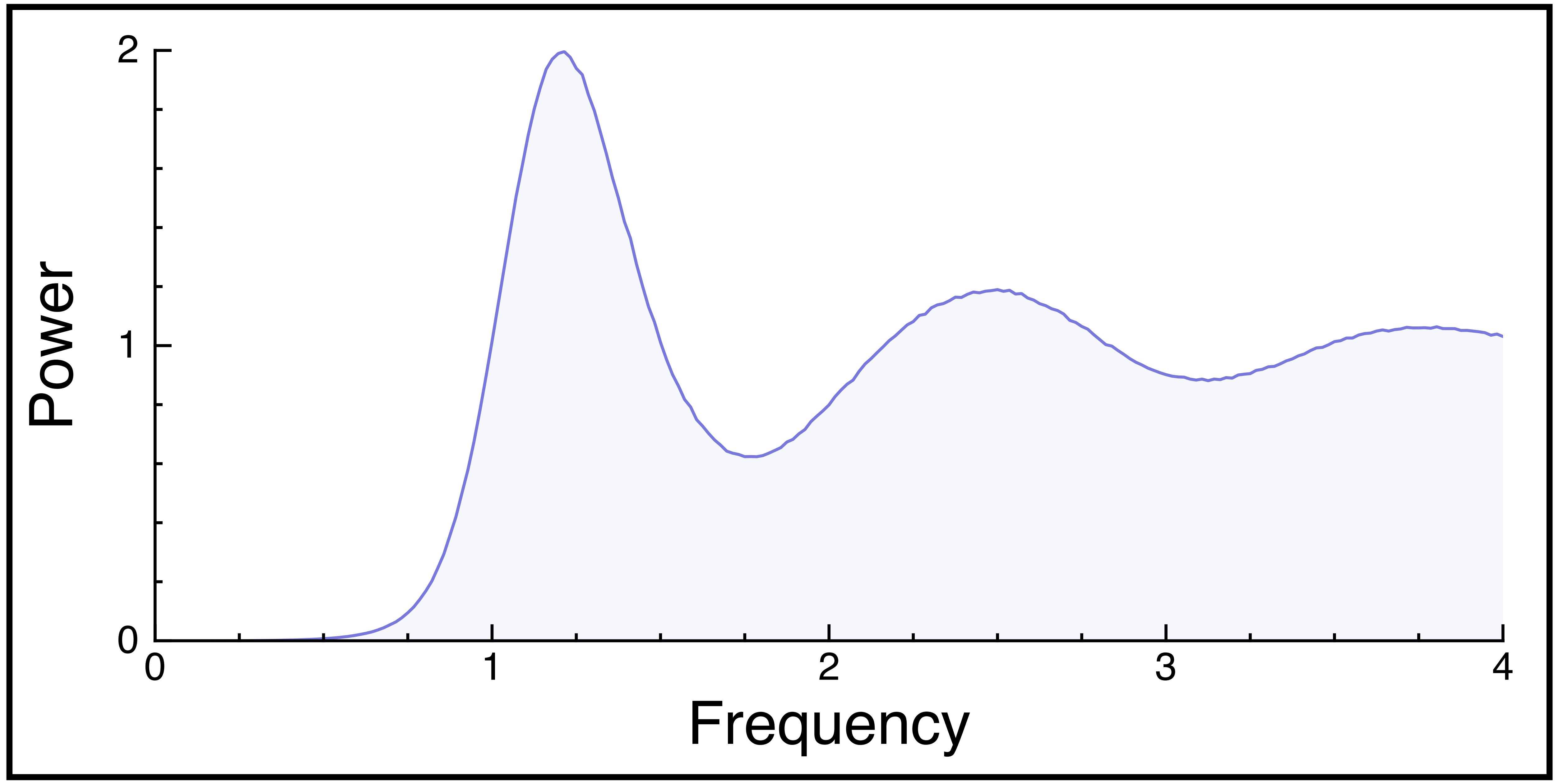
Samples



Expected power spectrum

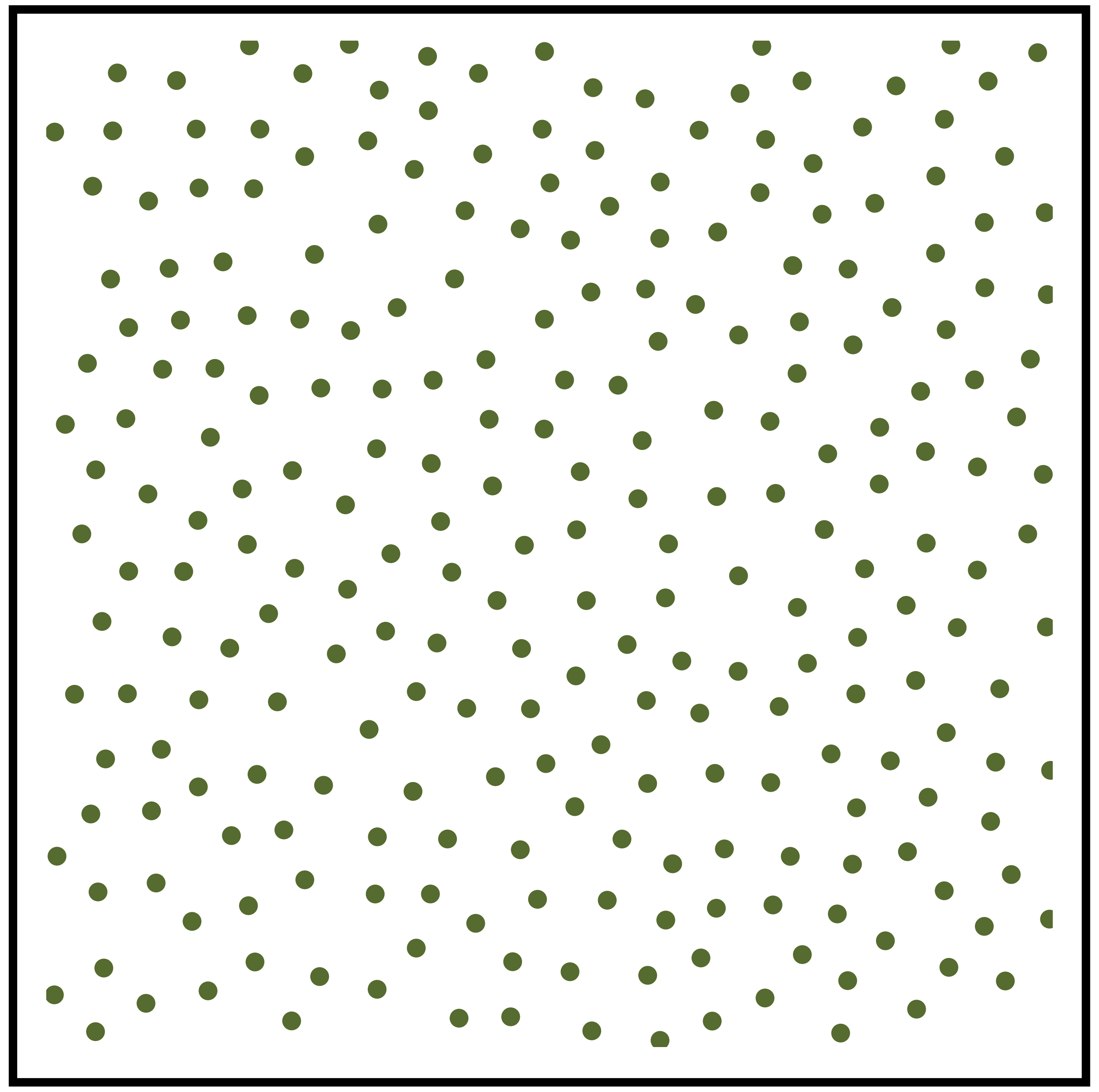


Radial mean

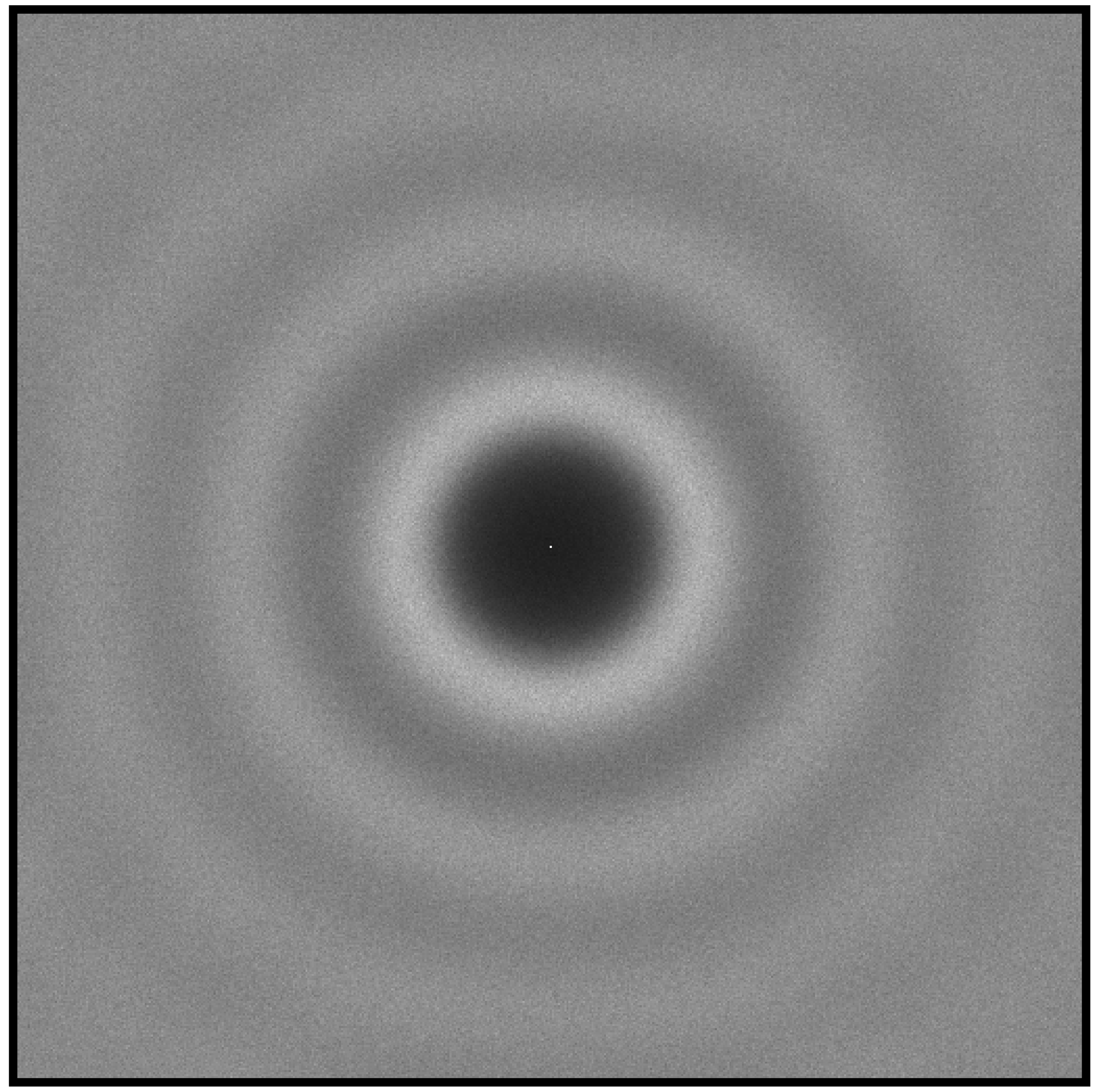


Poisson Disk Sampling

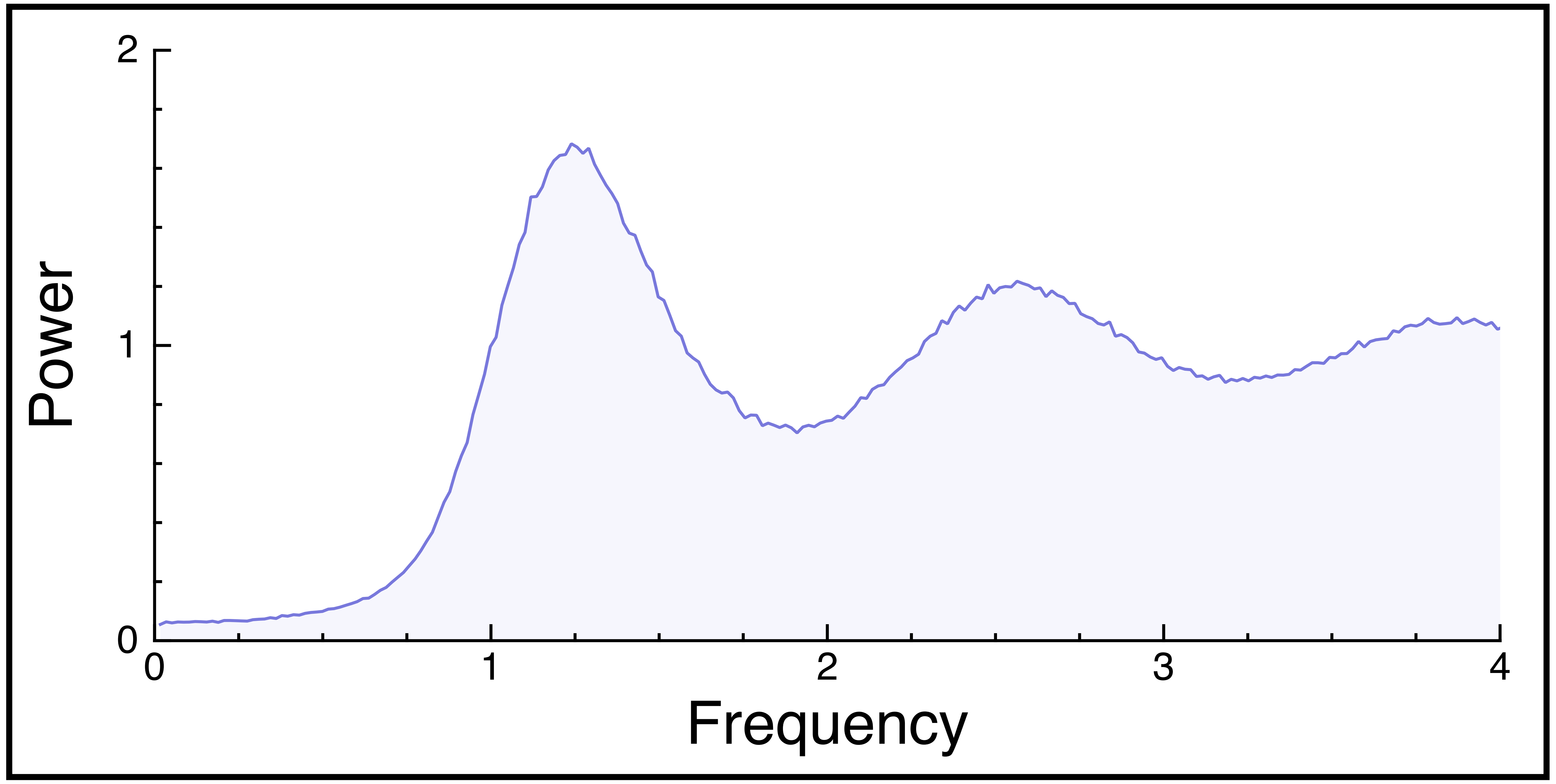
Samples



Expected power spectrum



Radial mean



Low-Discrepancy Sampling

Deterministic sets of points specially crafted to be evenly distributed (have low discrepancy).

Entire field of study called Quasi-Monte Carlo (QMC)

The Van der Corput Sequence

Radical Inverse Φ_b in base 2

Subsequent points “fall into biggest holes”

k	Base 2	Φ_b

The Van der Corput Sequence

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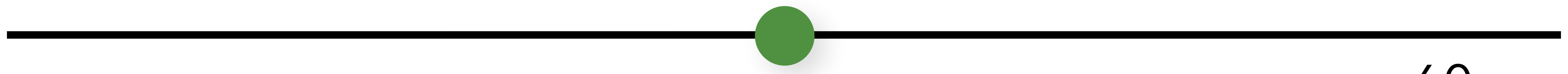
k	Base 2	Φ_b
1	1	.1 = 1/2



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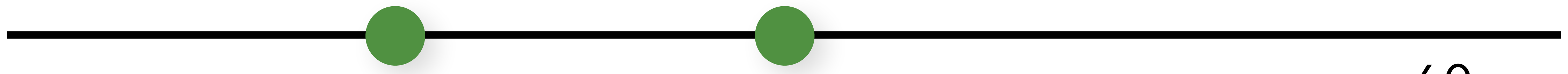
Subsequent points “fall into biggest holes”

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4



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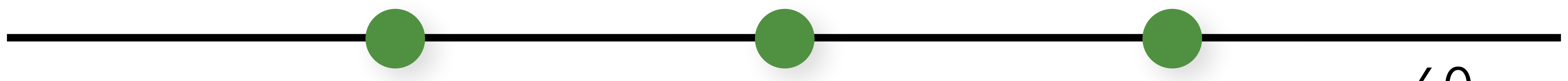

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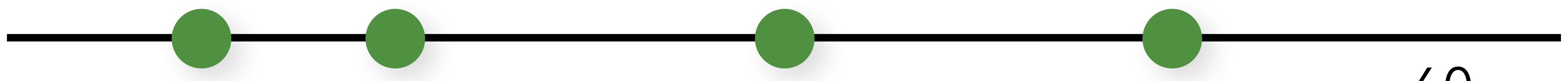
3	11	.11 = 3/4
4	100	.001 = 1/8



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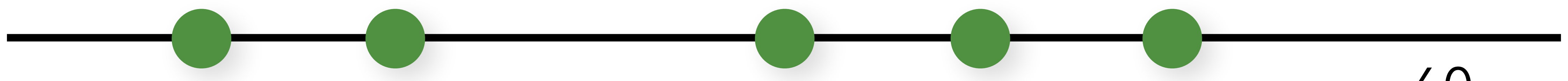

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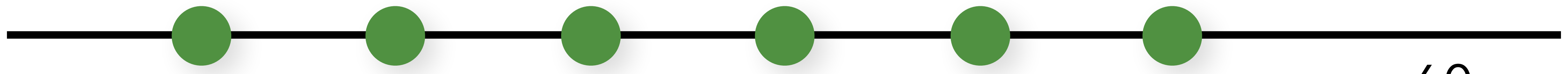

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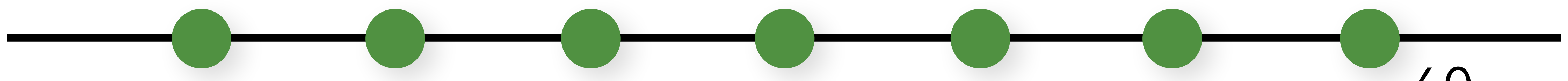
4	100	.001 = 1/8
5	101	.101 = 5/8
6	110	.011 = 3/8
7	111	.111 = 7/8



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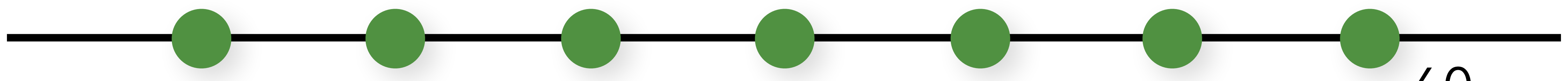
6	110	.011 = 3/8
7	111	.111 = 7/8
...		



Halton and Hammersley Points

Halton: Radical inverse with different base for each dimension:

$$\vec{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \dots, \Phi_{p_n}(k))$$

Halton and Hammersley Points

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- The bases should all be relatively prime.

Halton and Hammersley Points

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Hammersley: Same as Halton, but first dimension is k/N:

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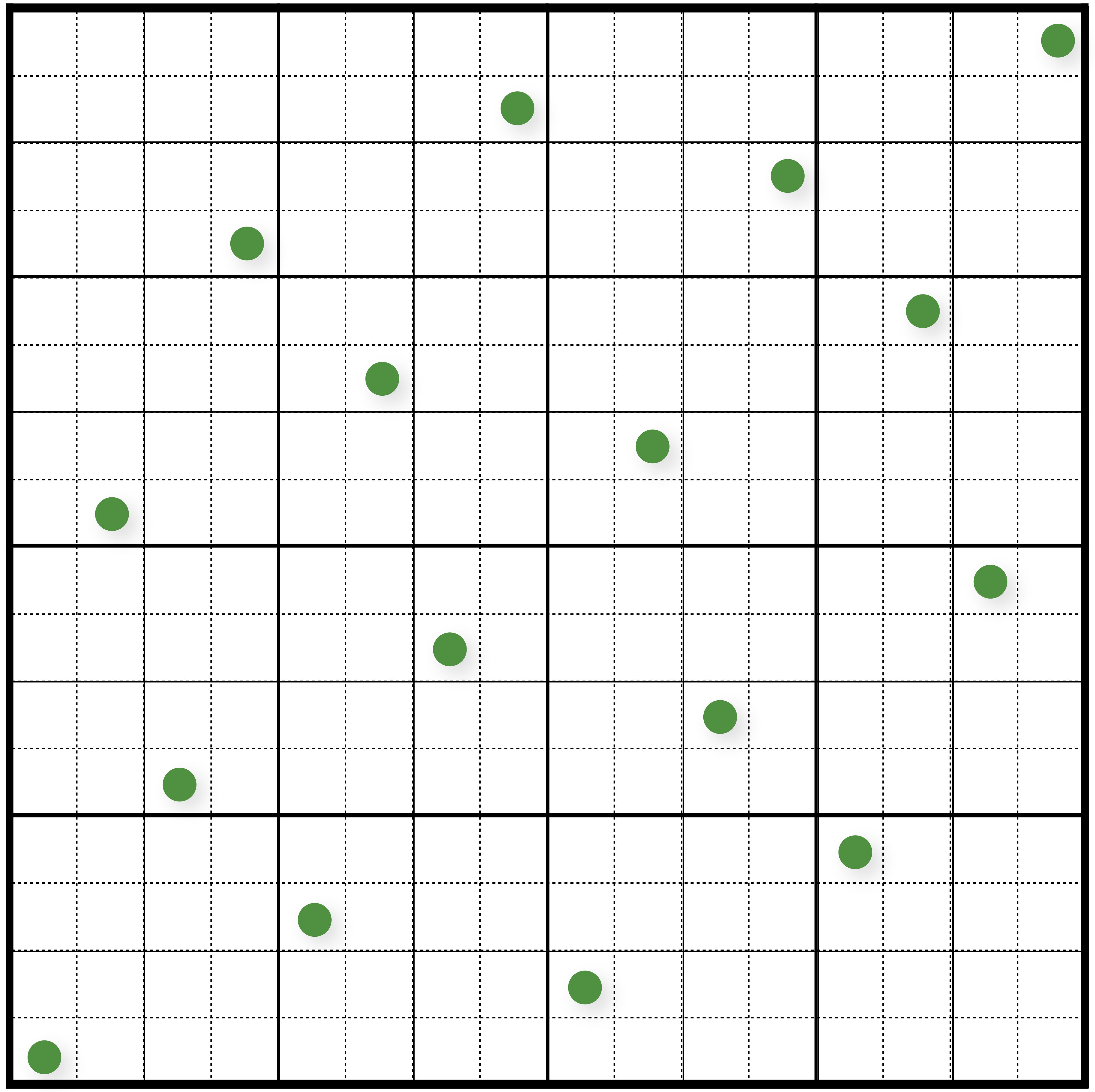
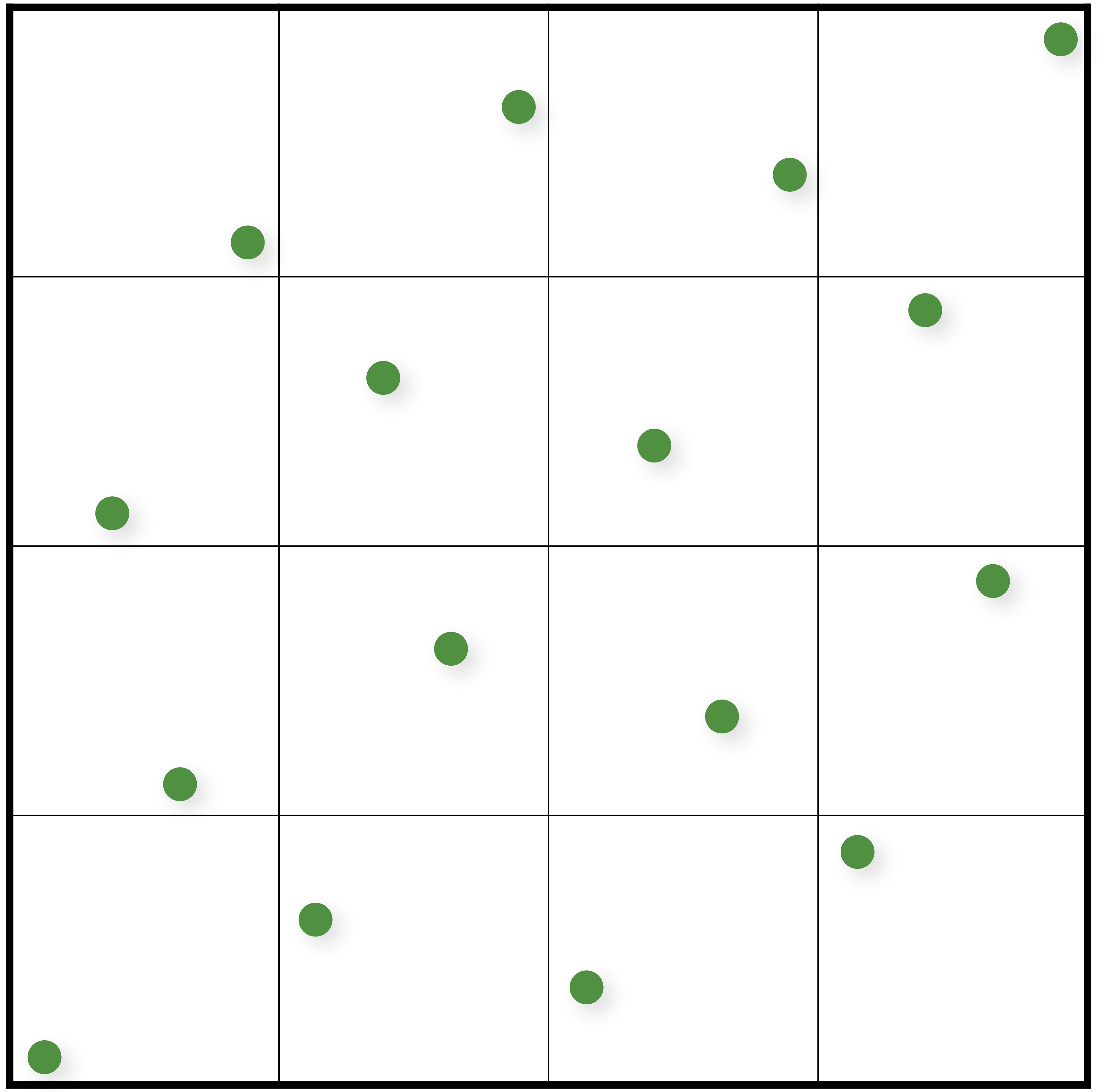
- Not incremental, need to know sample count, N, in advance

The Hammersley Sequence



1 sample in each “elementary interval”

The Hammersley Sequence



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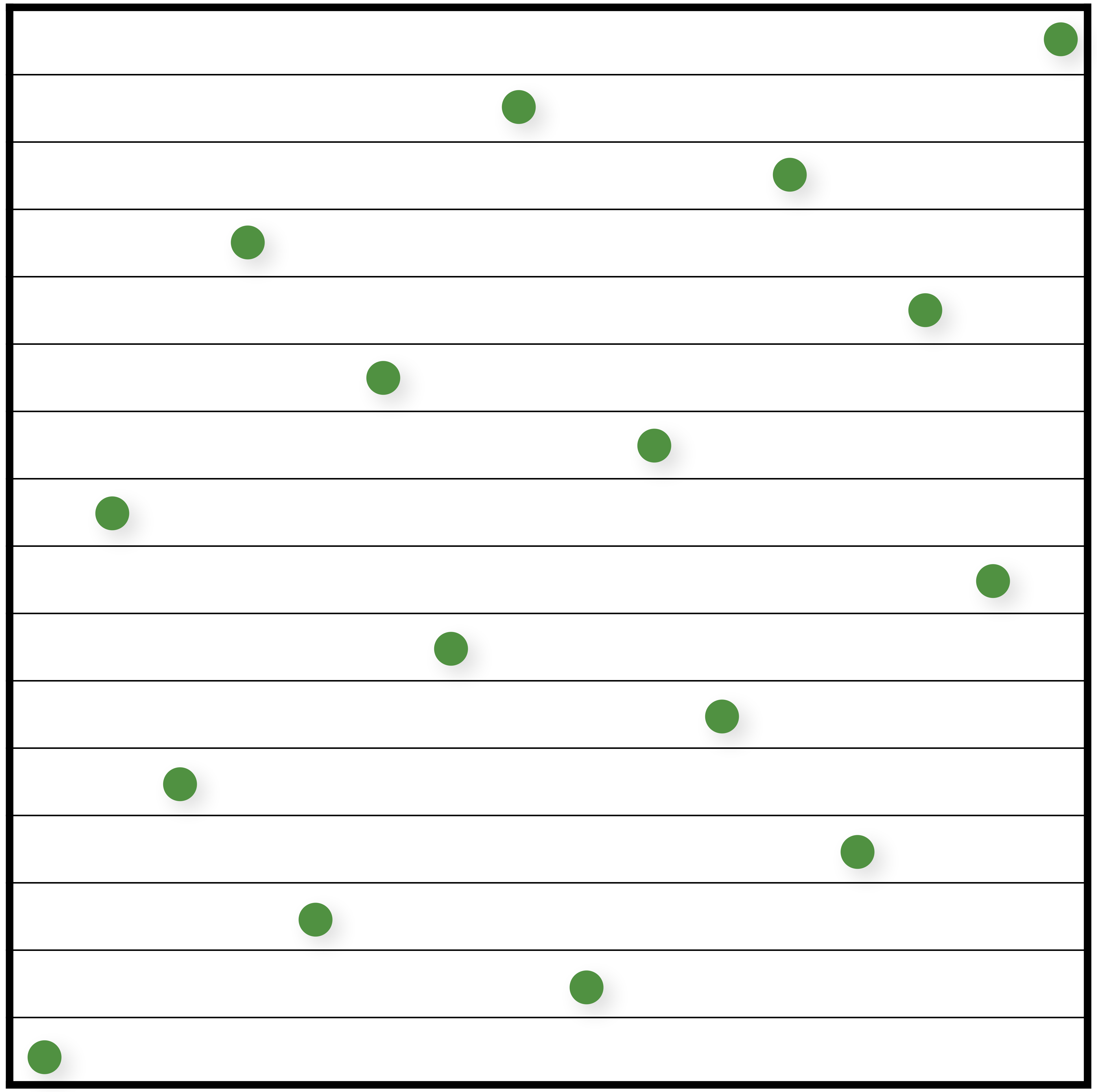

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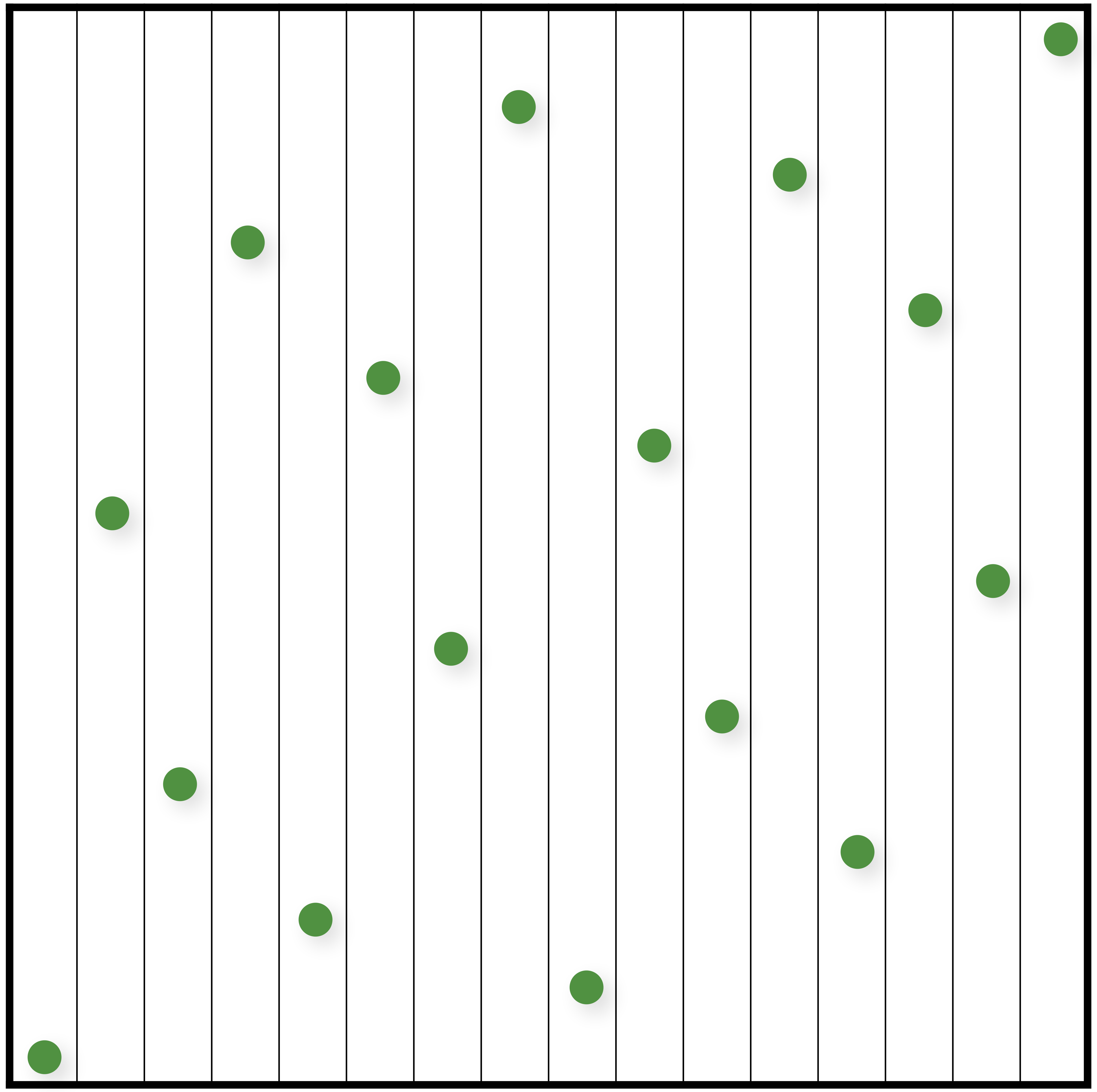

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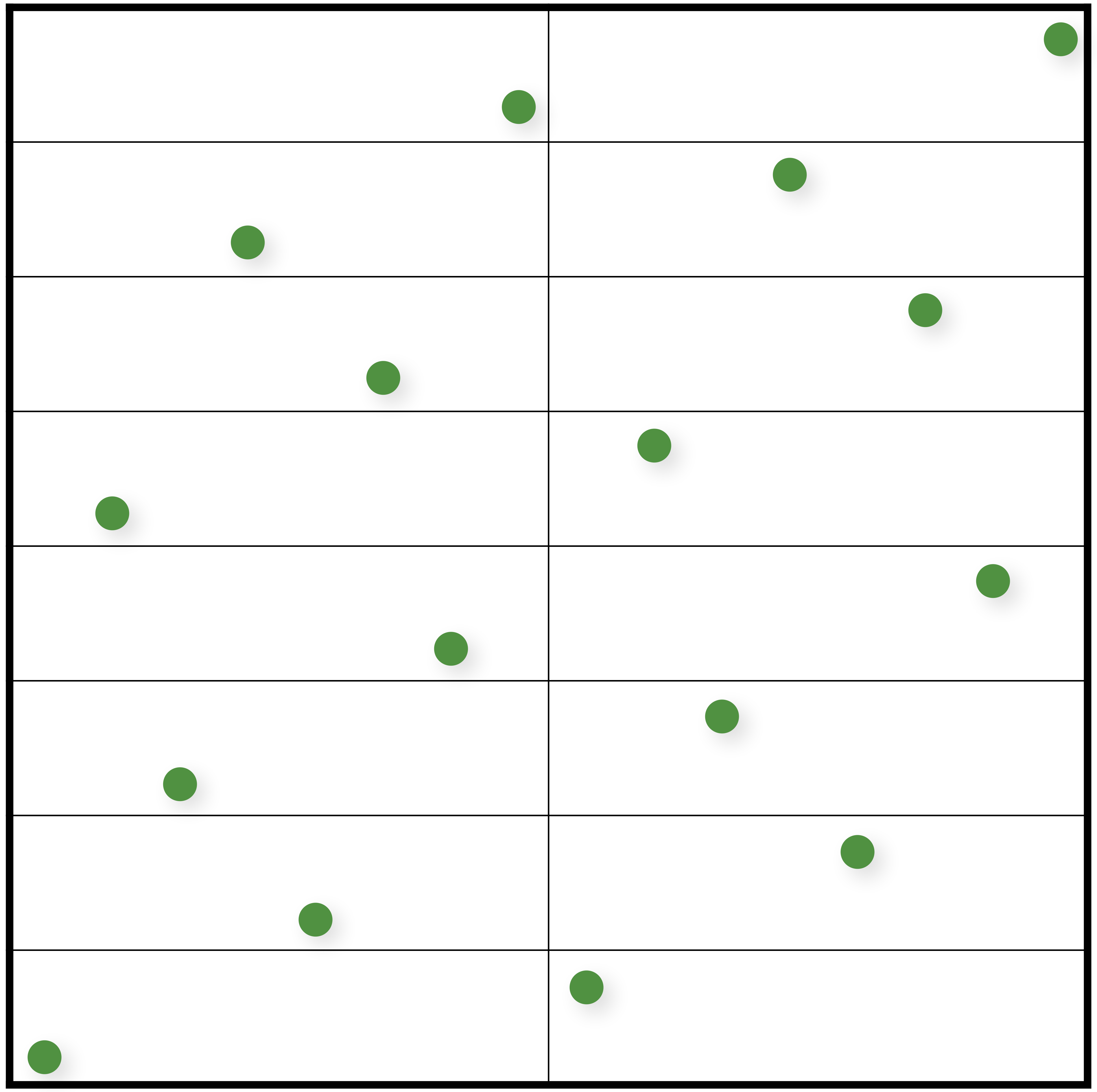

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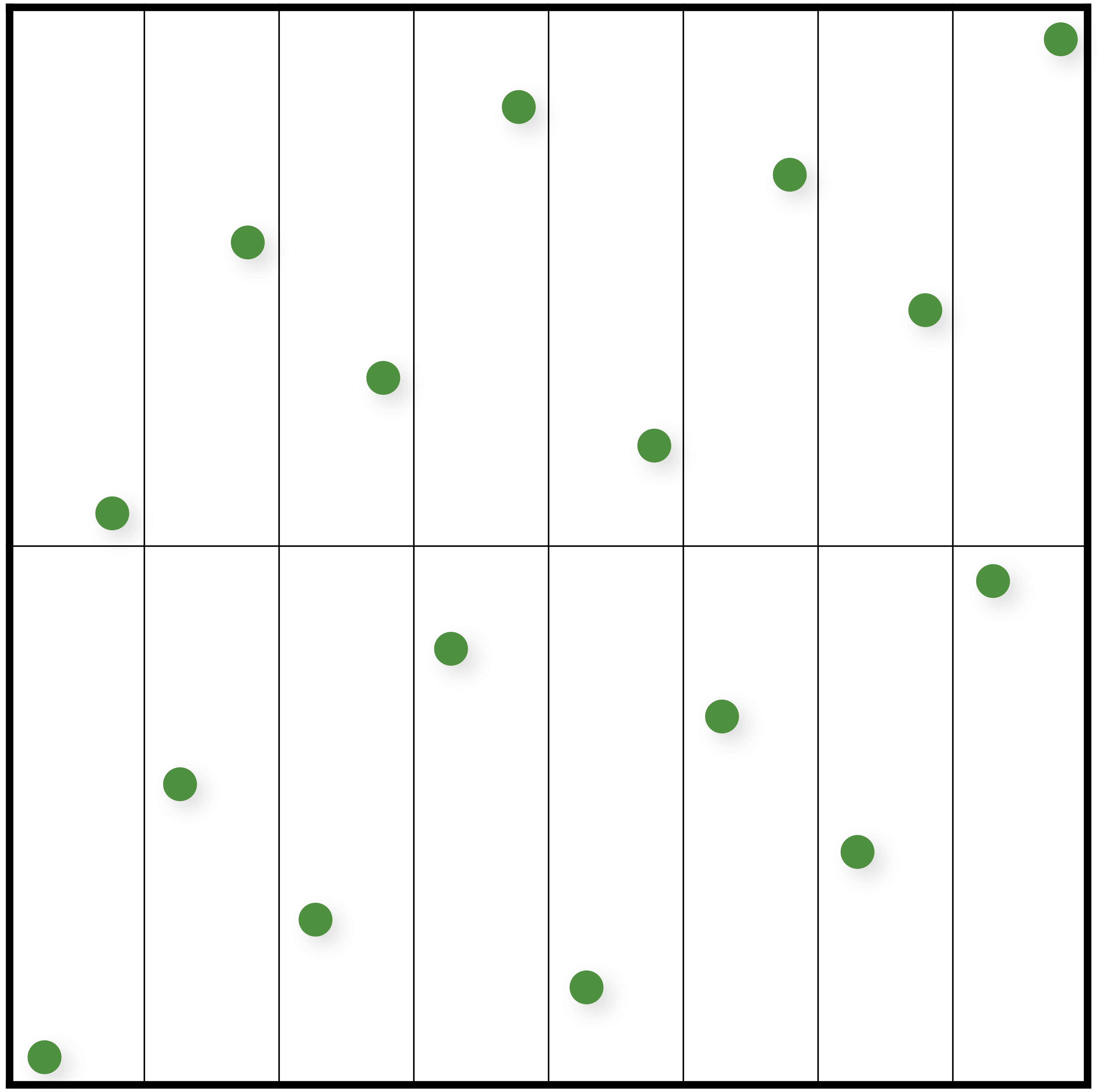

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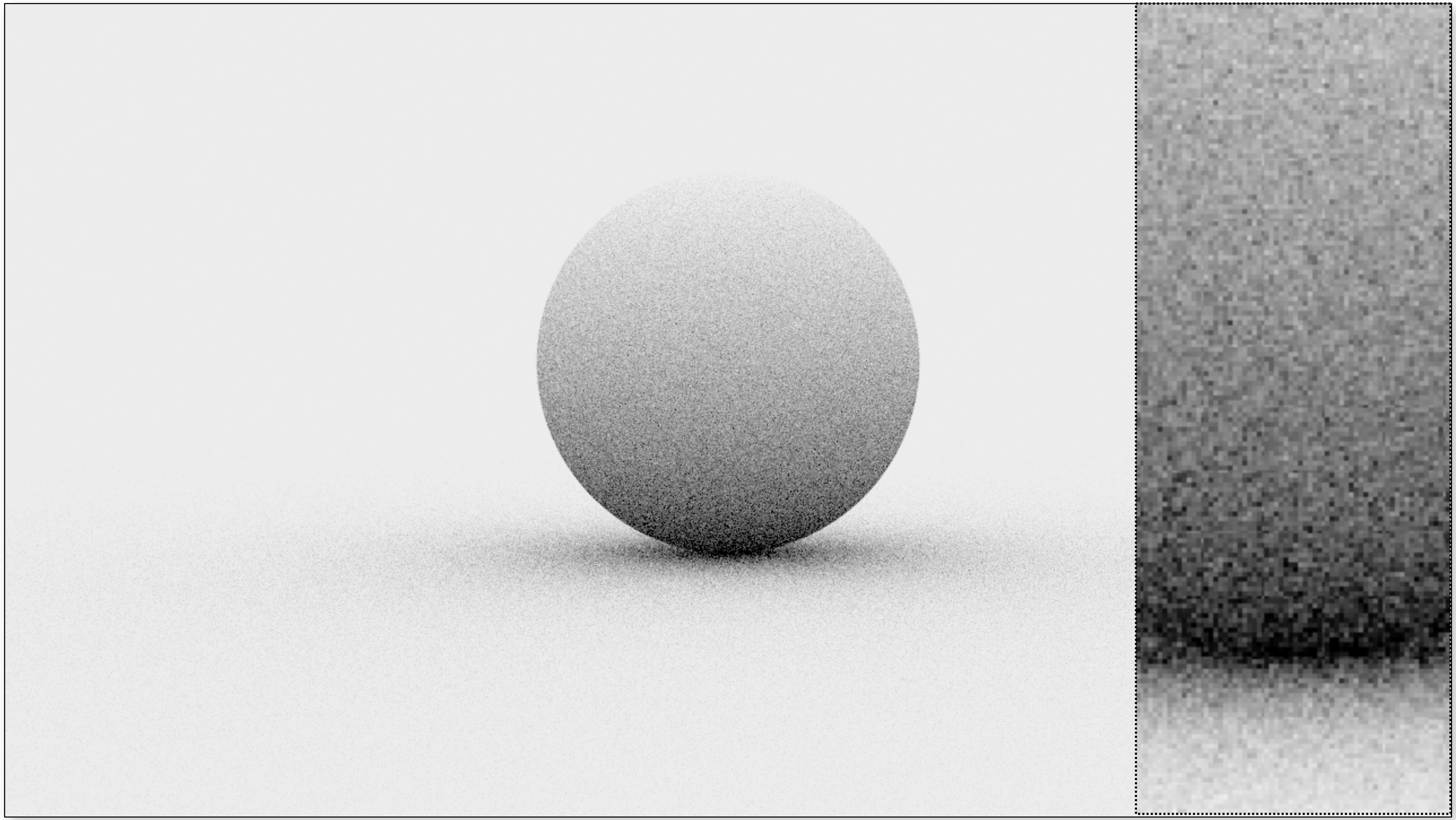
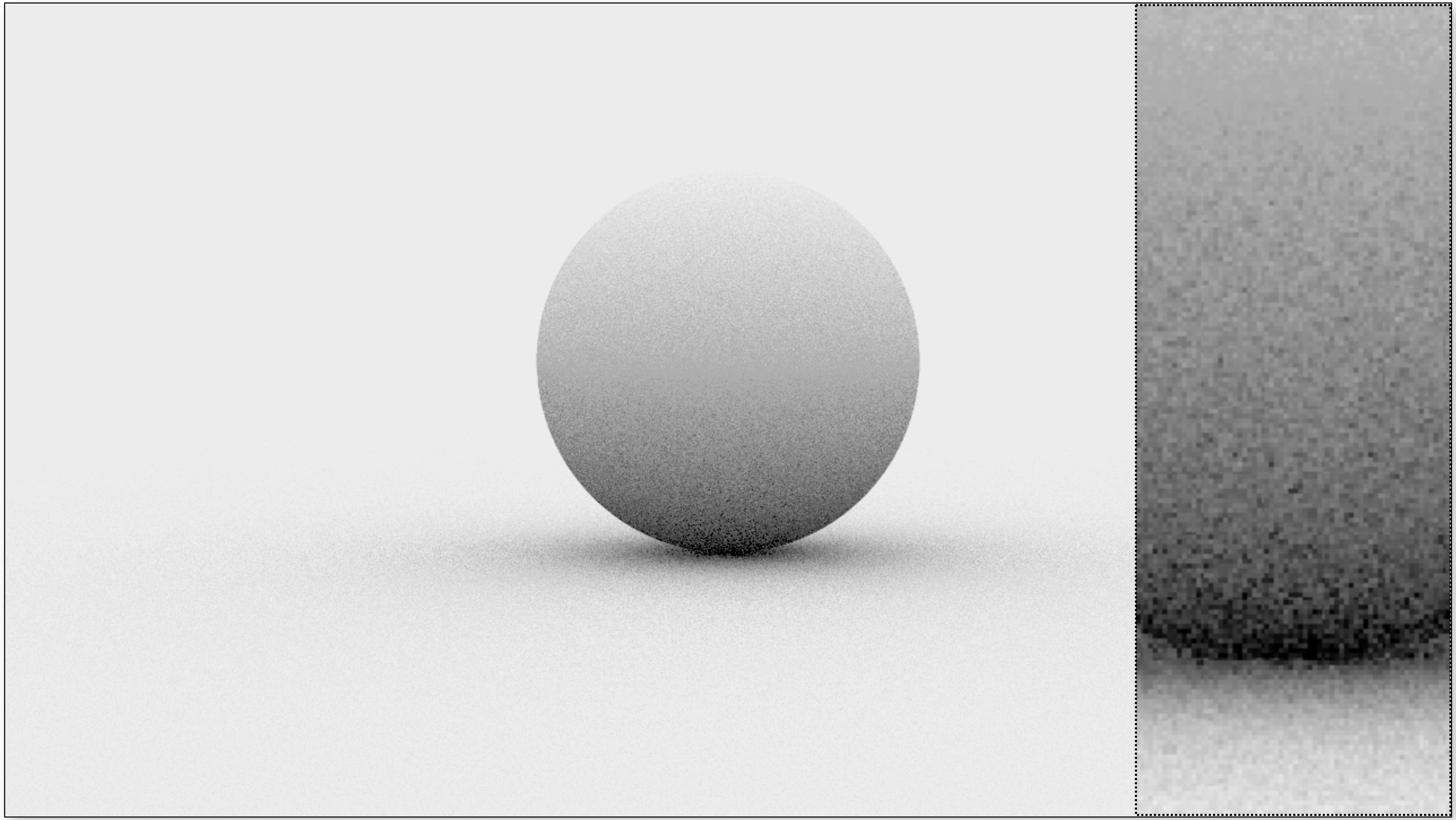
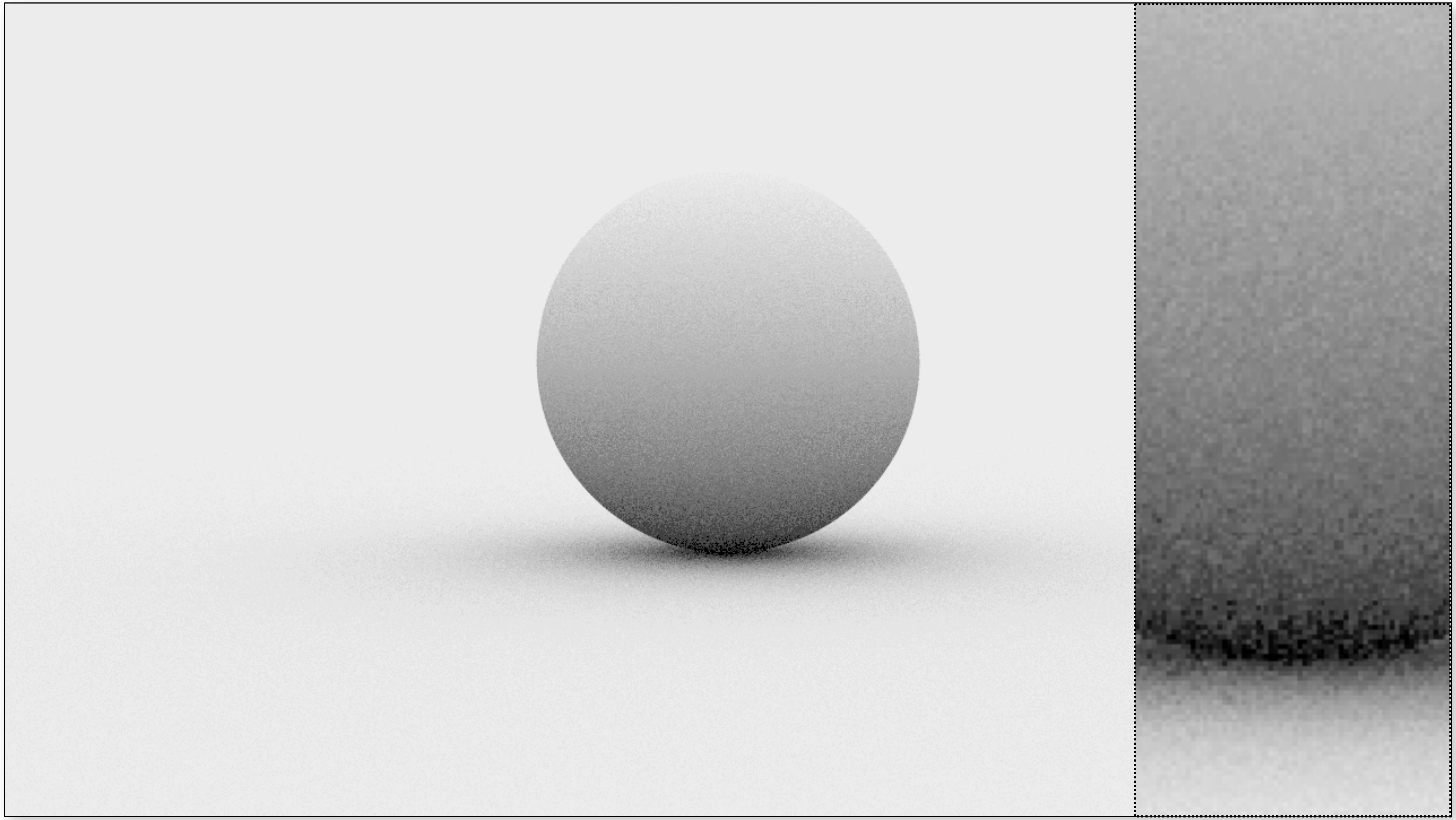
Monte Carlo (16 random samples)



Monte Carlo (16 jittered samples)



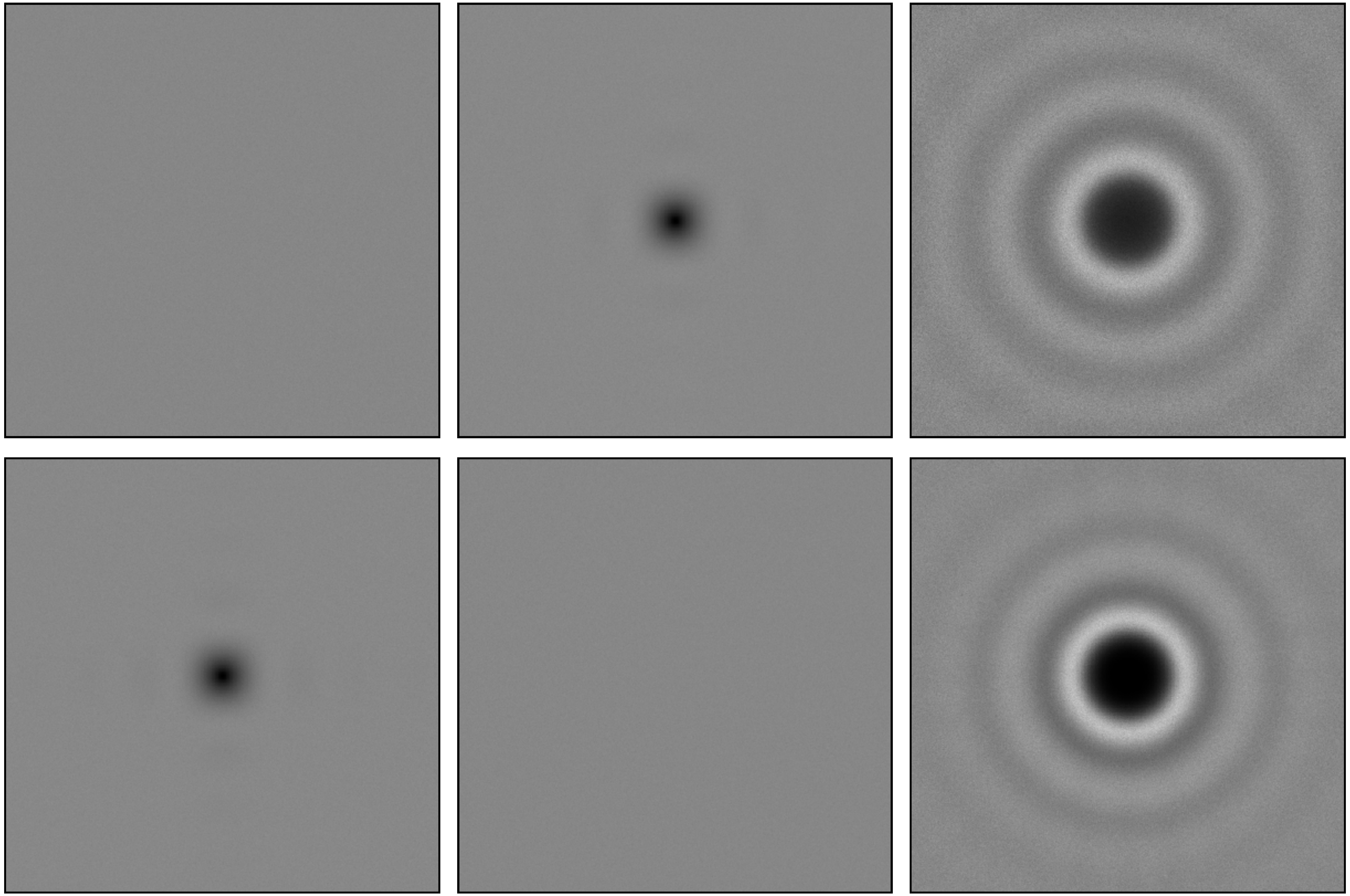
Scrambled Low-Discrepancy Sampling



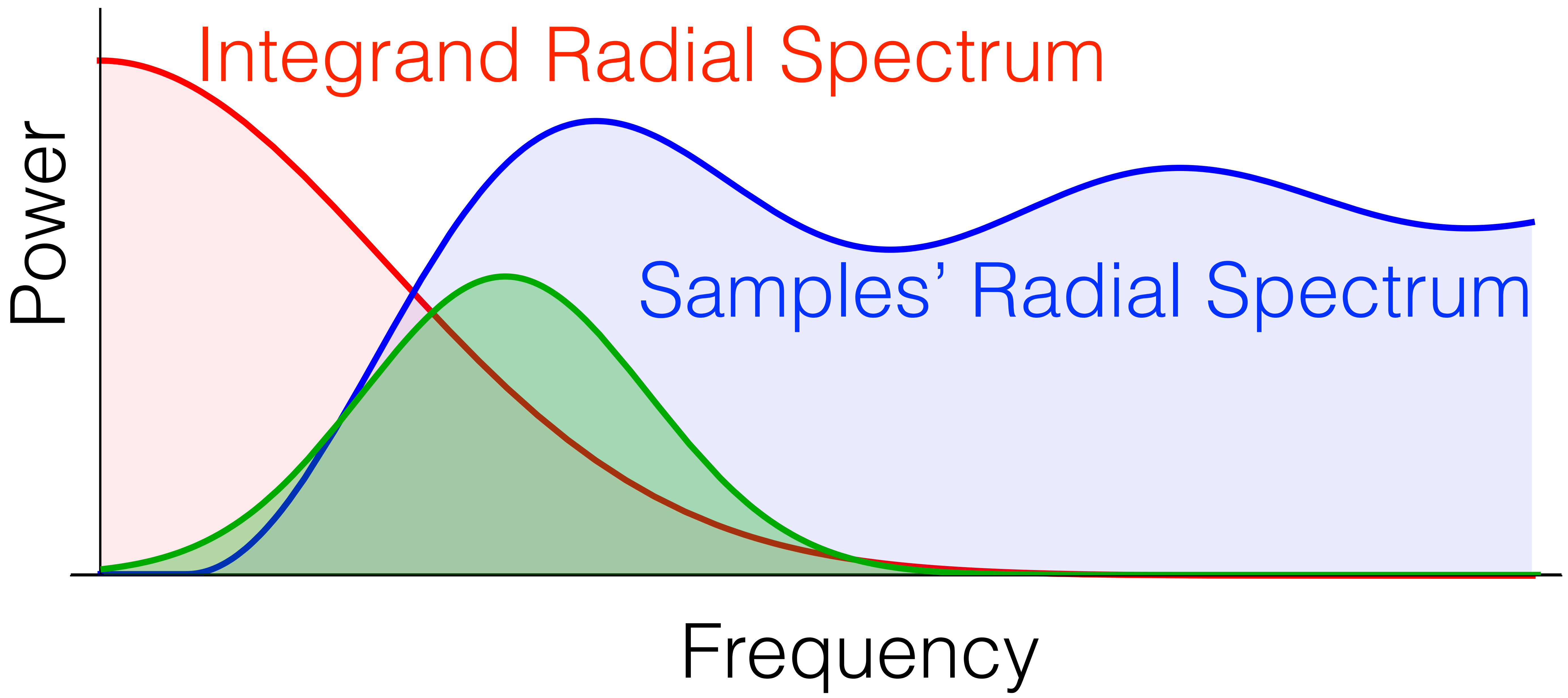
More info on QMC in Rendering

S. Premoze, A. Keller, and M. Raab.
Advanced (Quasi-) Monte Carlo Methods for Image Synthesis.
In SIGGRAPH 2012 courses.

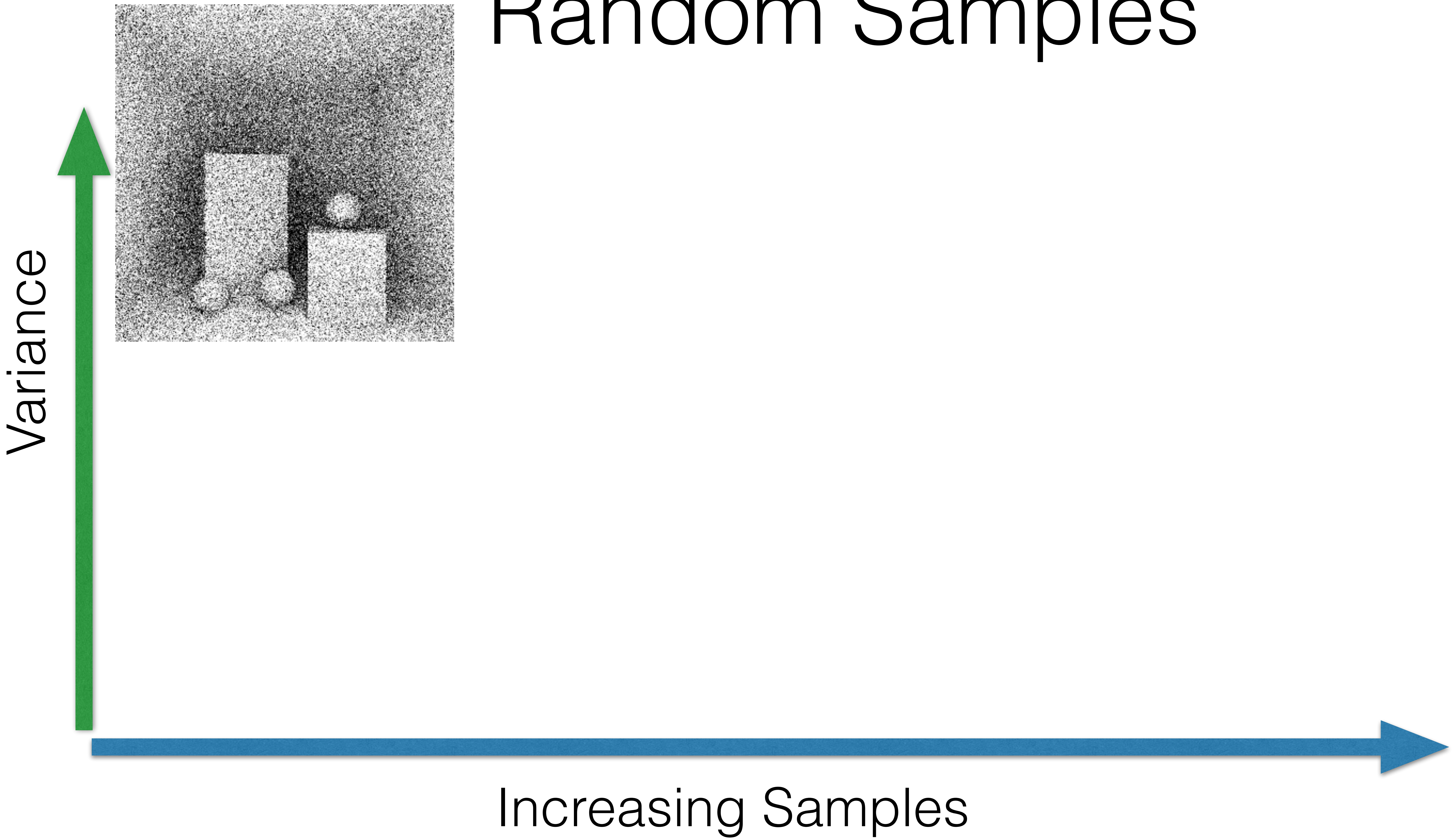
How can we predict error from these?



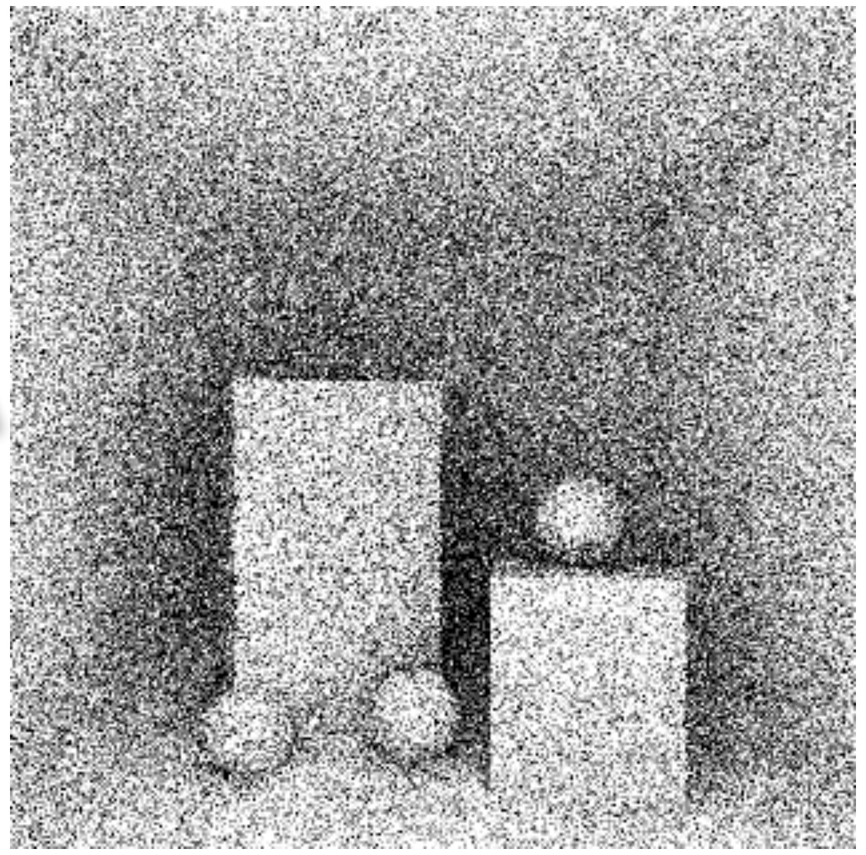
Part 2: Formal Treatment of MSE, Bias and Variance

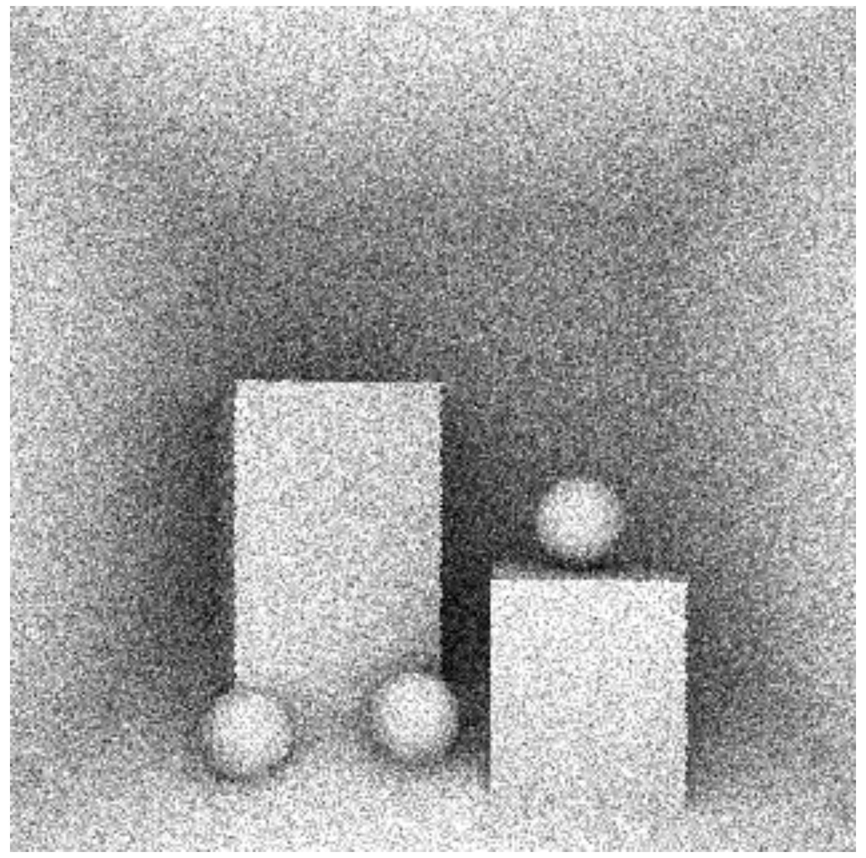


Convergence rate for Random Samples



Convergence rate for Random Samples

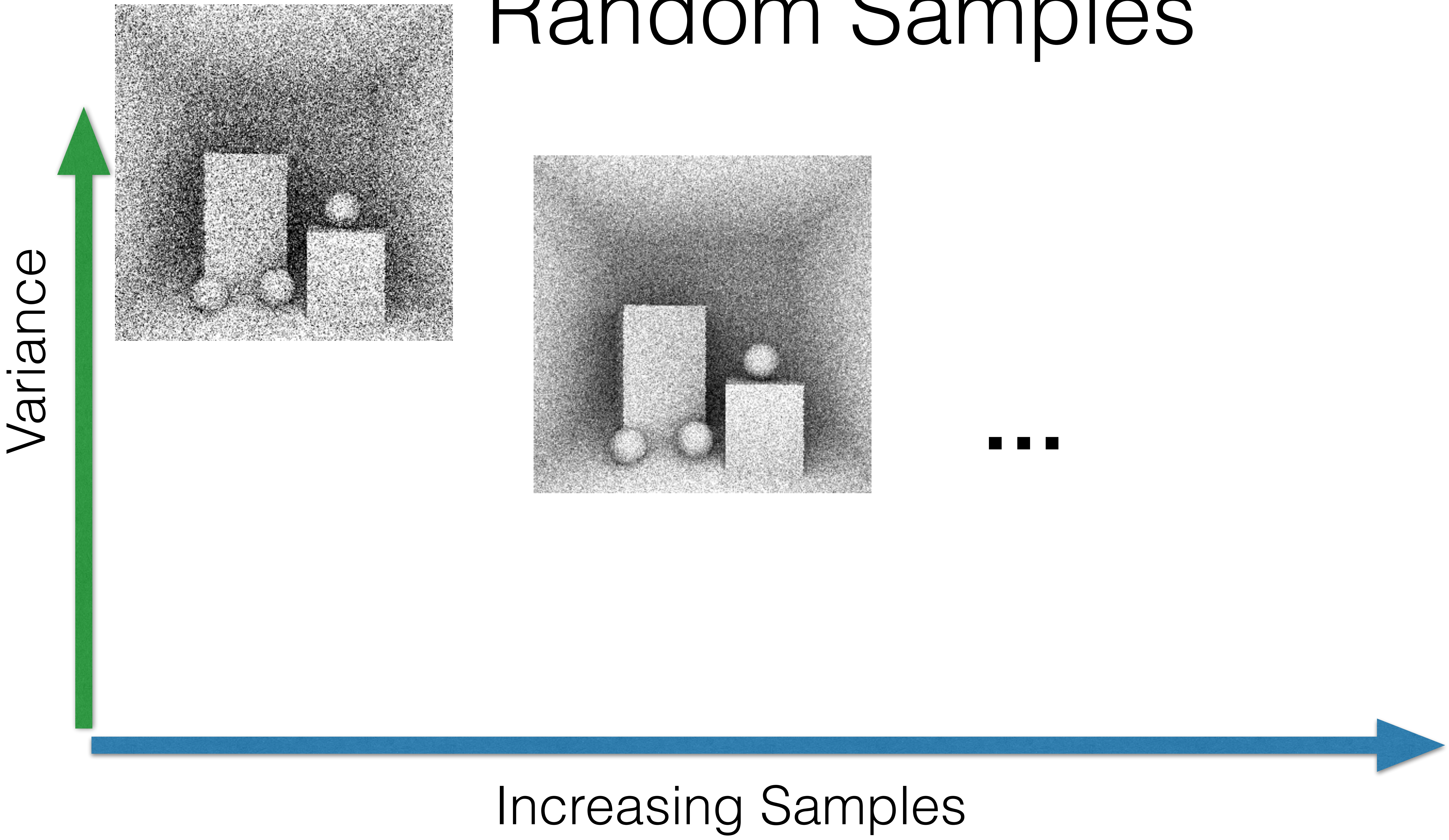




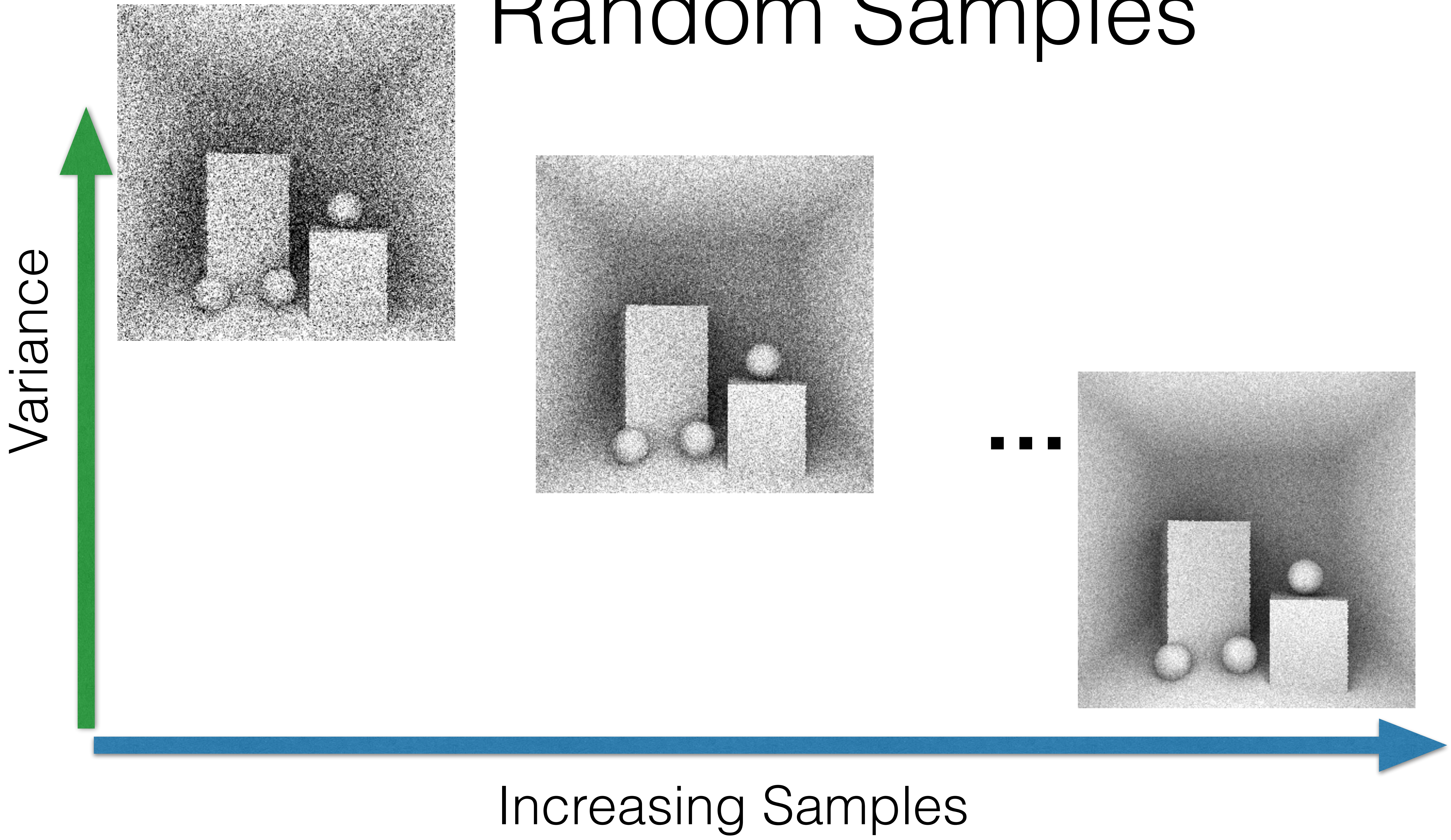
Variance

Increasing Samples

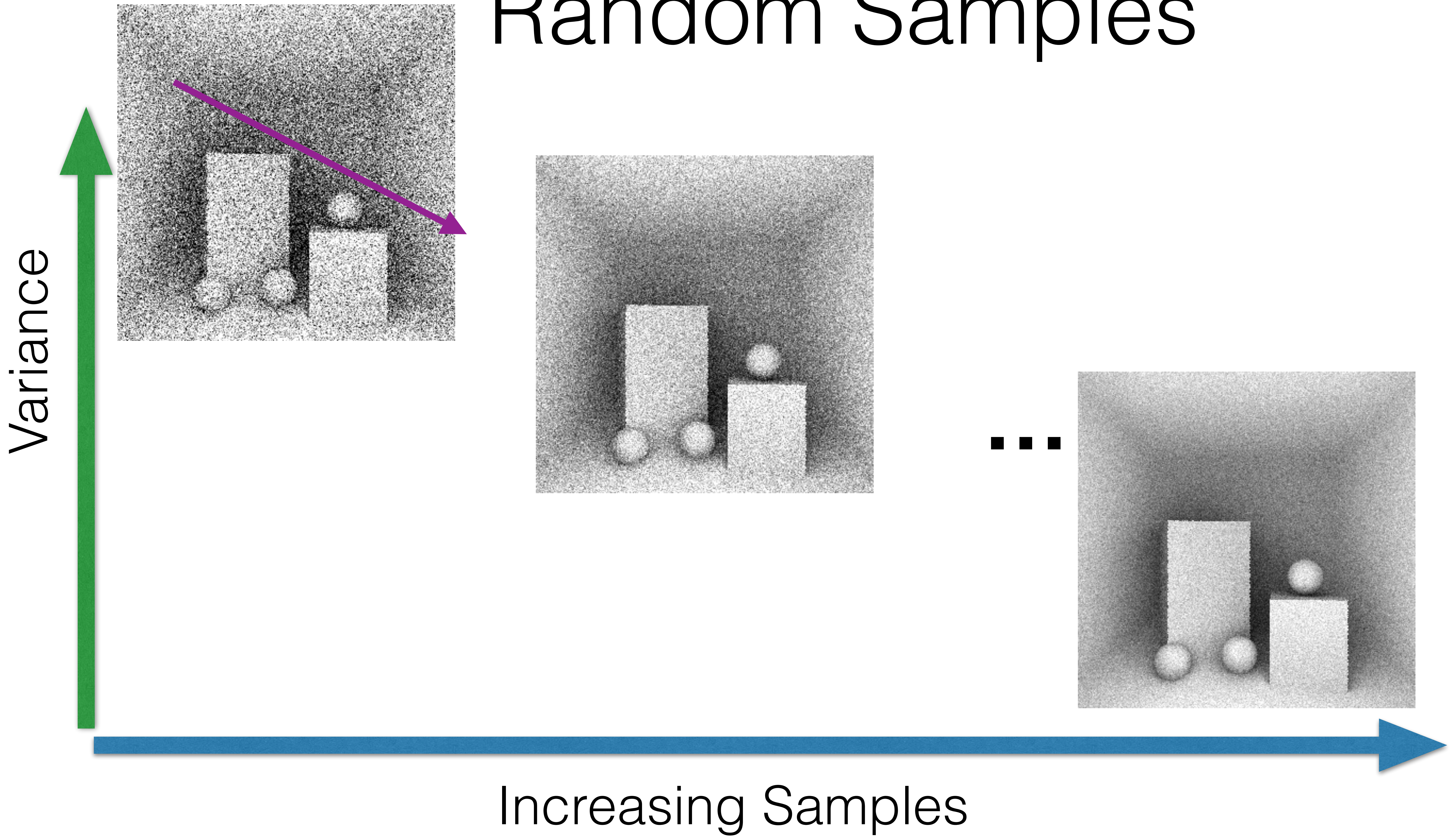
Convergence rate for Random Samples



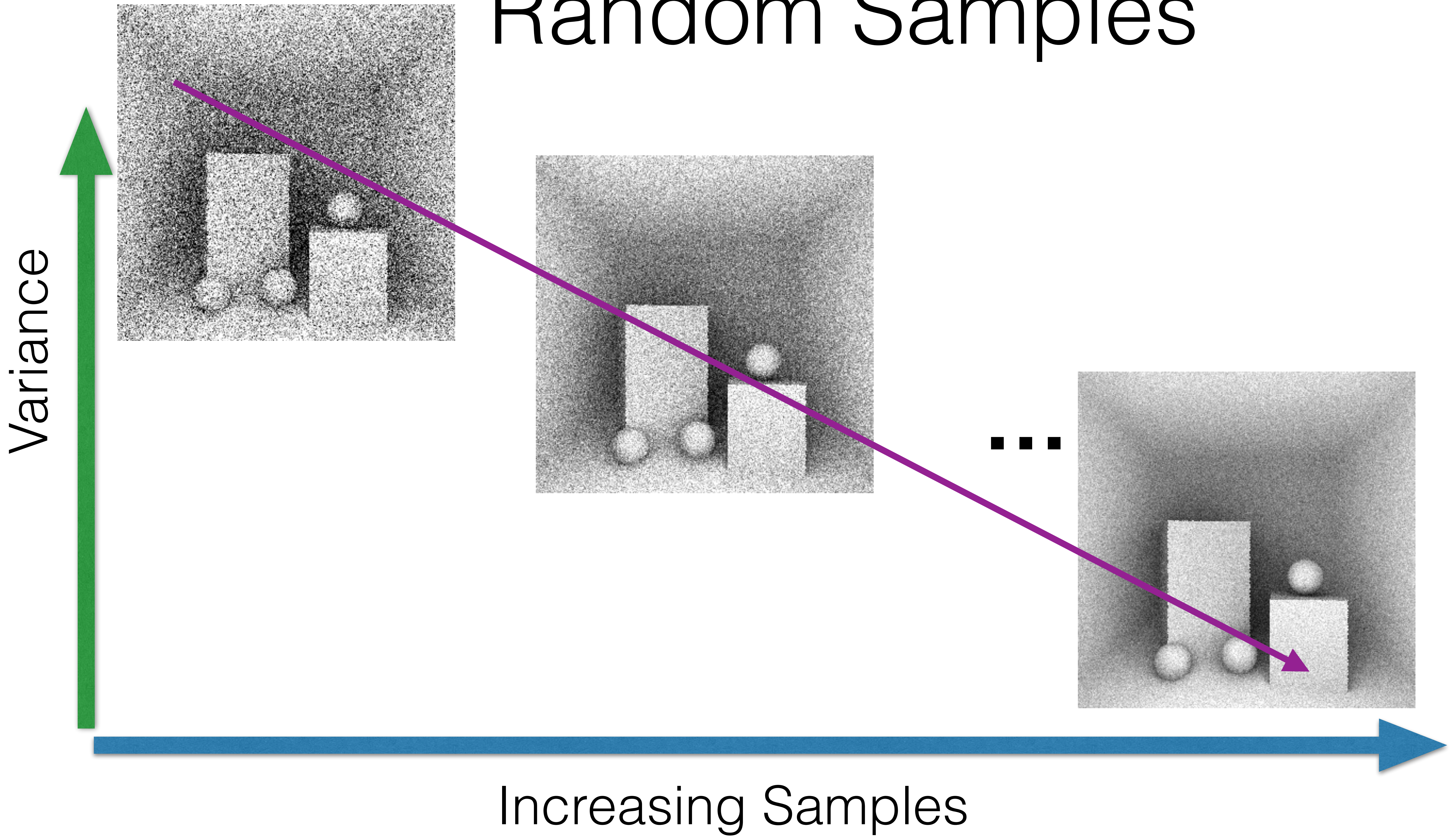
Convergence rate for Random Samples



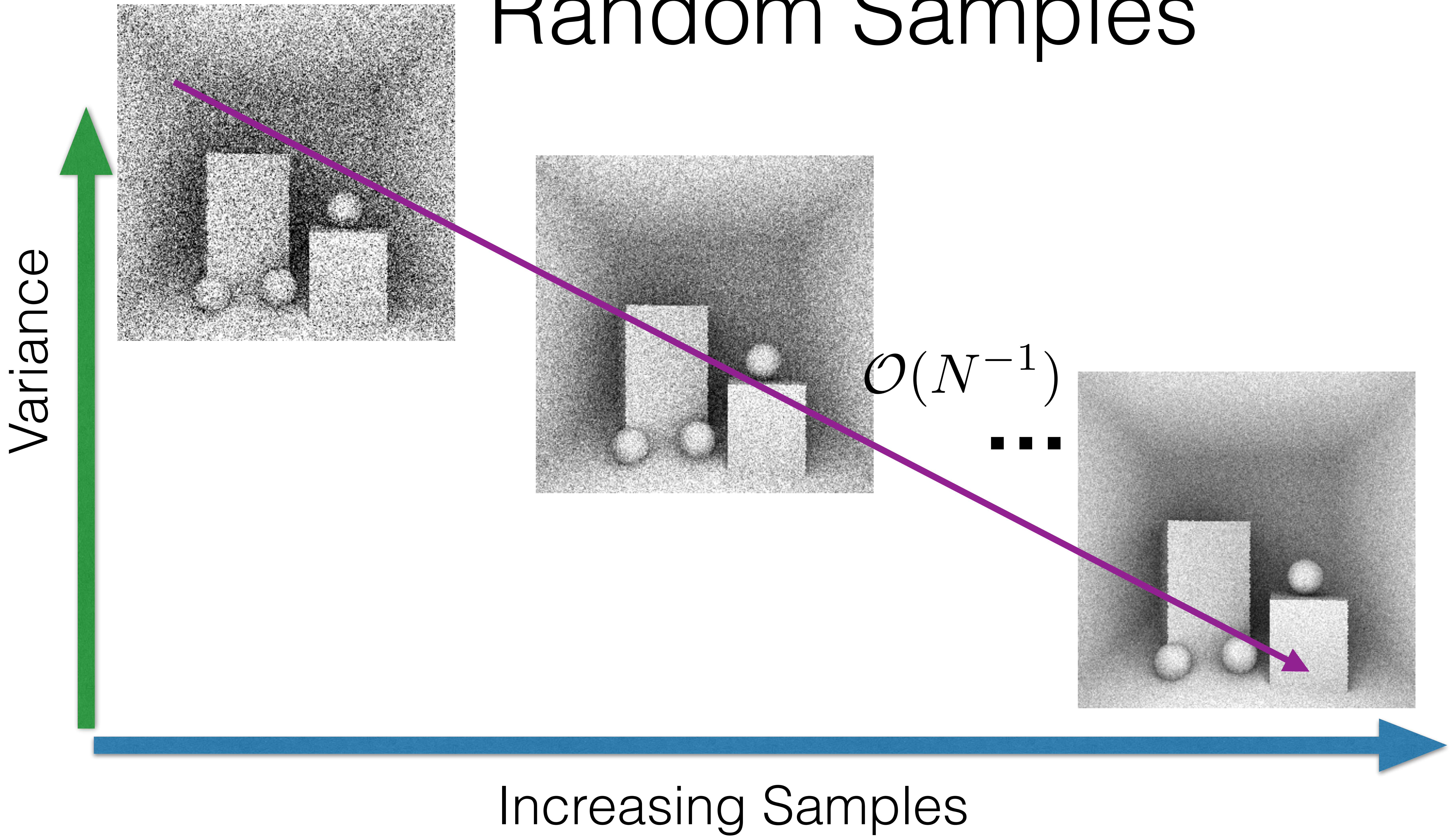
Convergence rate for Random Samples



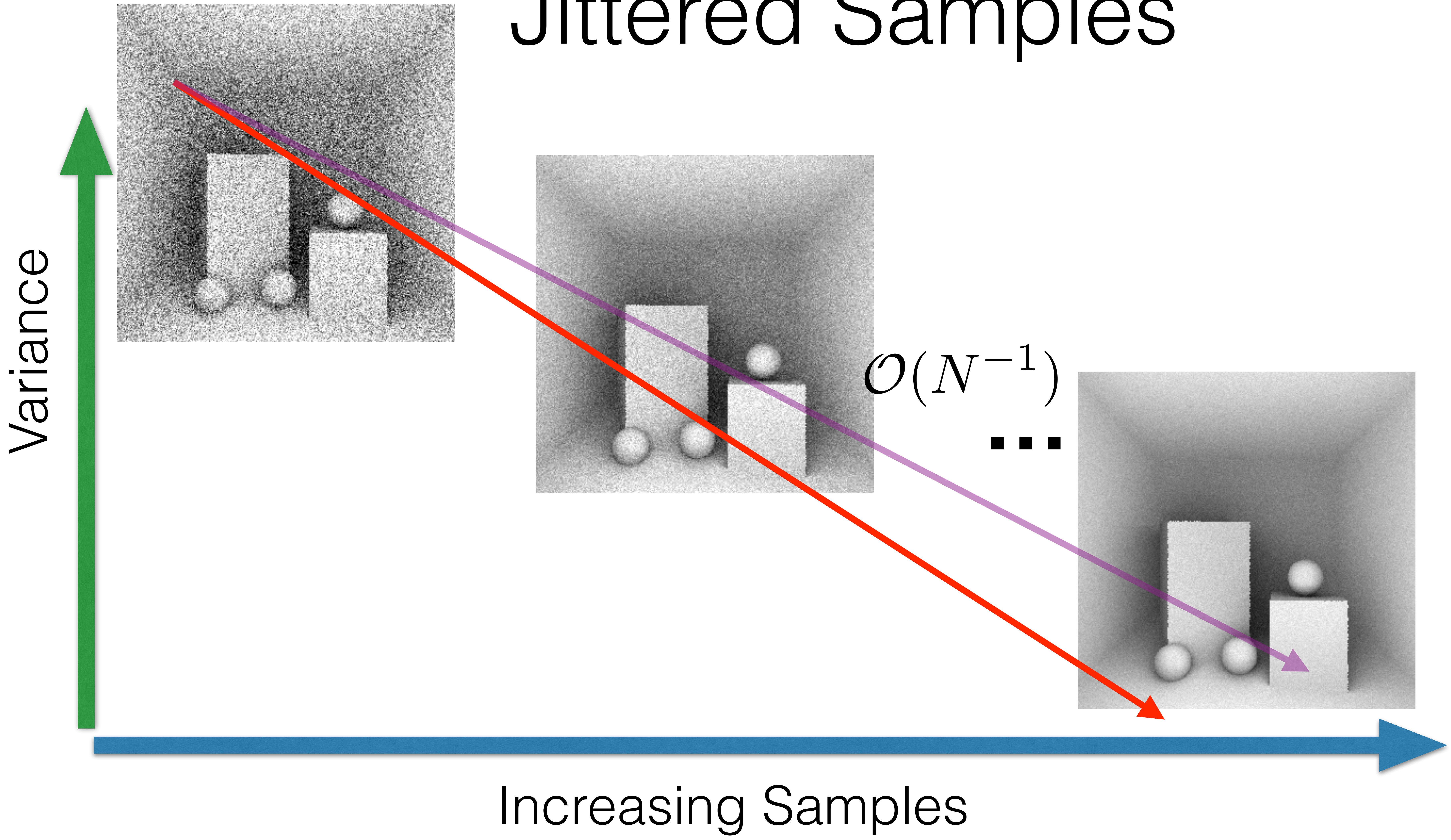
Convergence rate for Random Samples



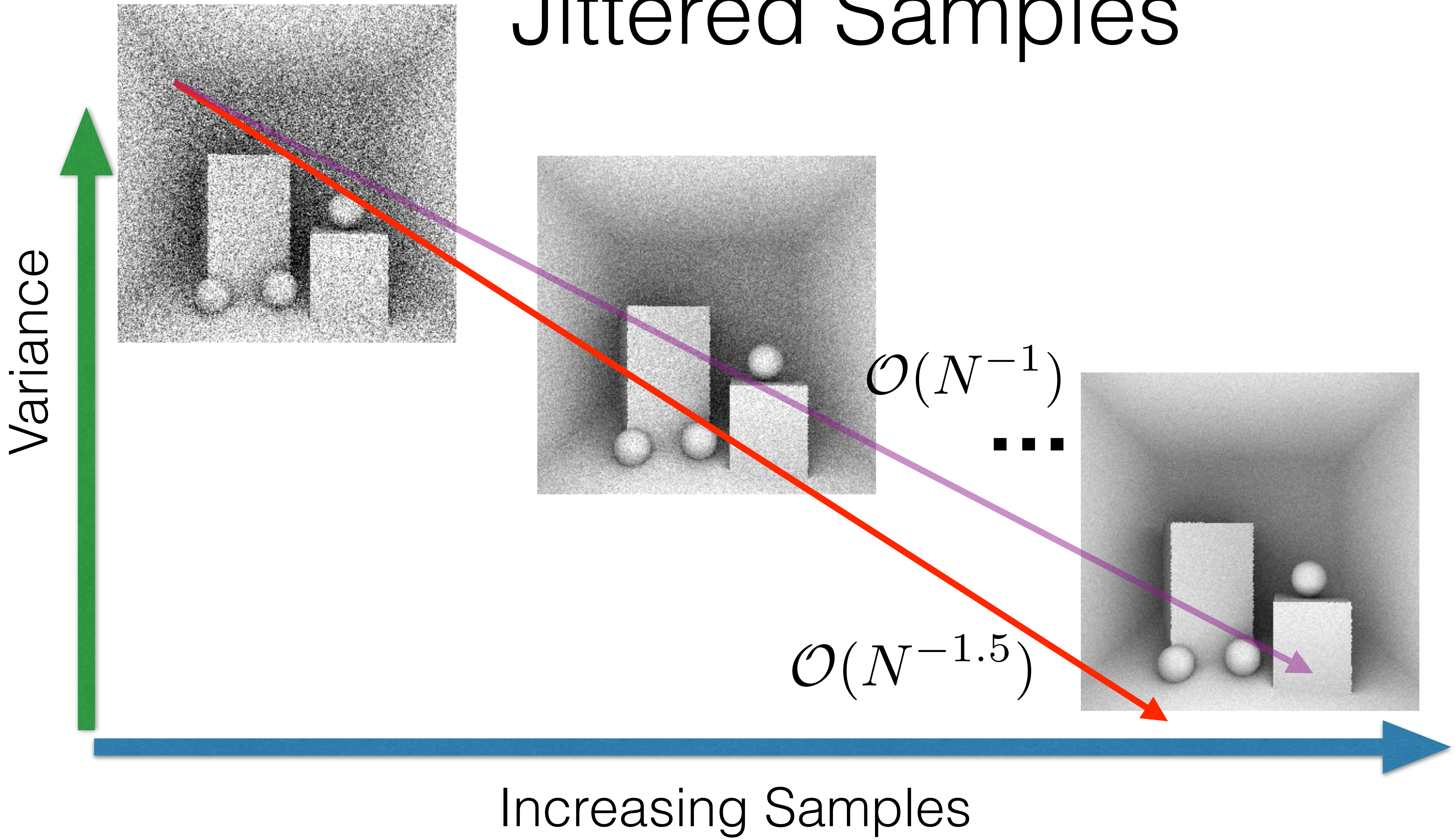
Convergence rate for Random Samples



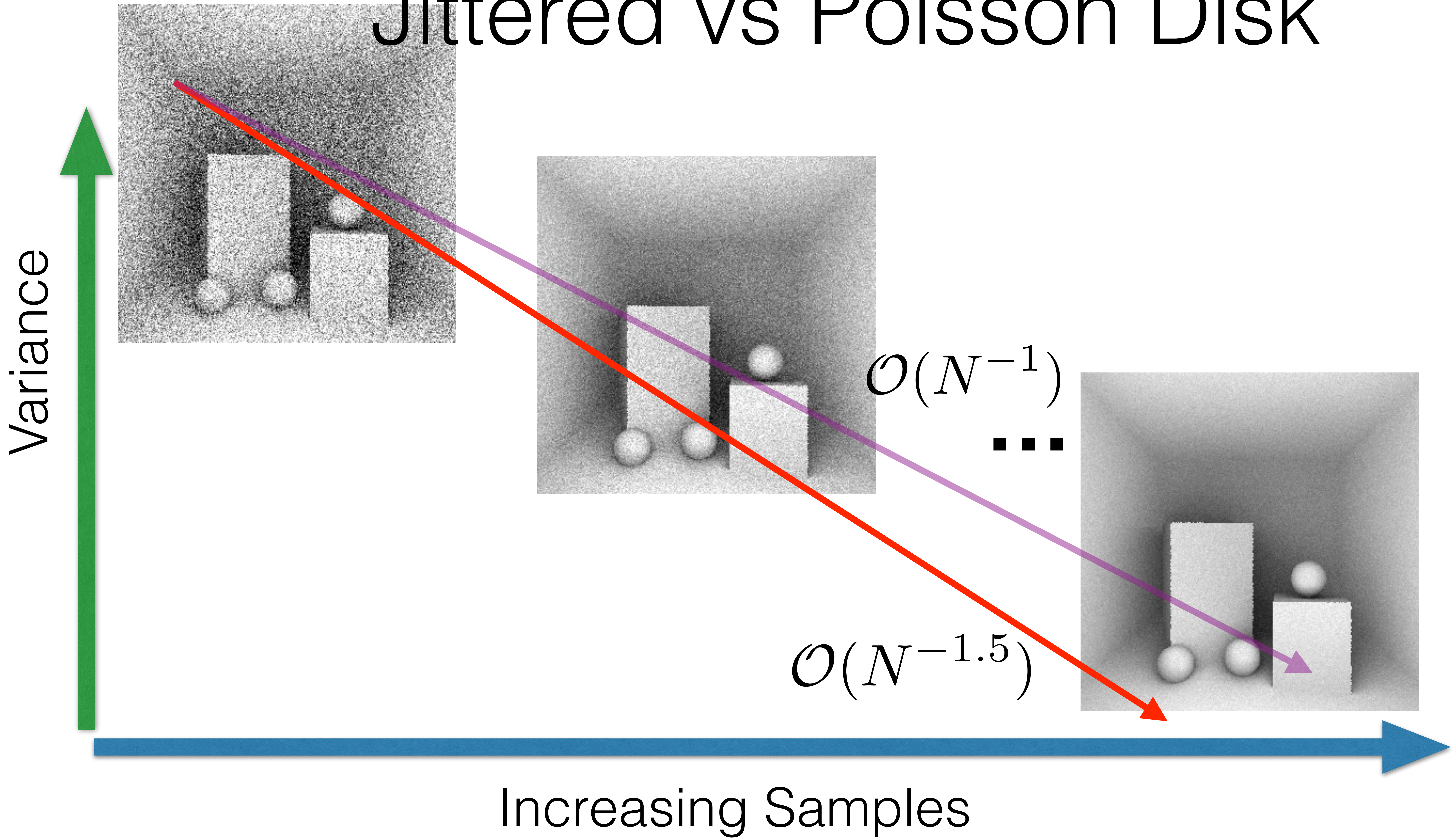
Convergence rate for Jittered Samples



Convergence rate for Jittered Samples



Convergence rate
Jittered vs Poisson Disk



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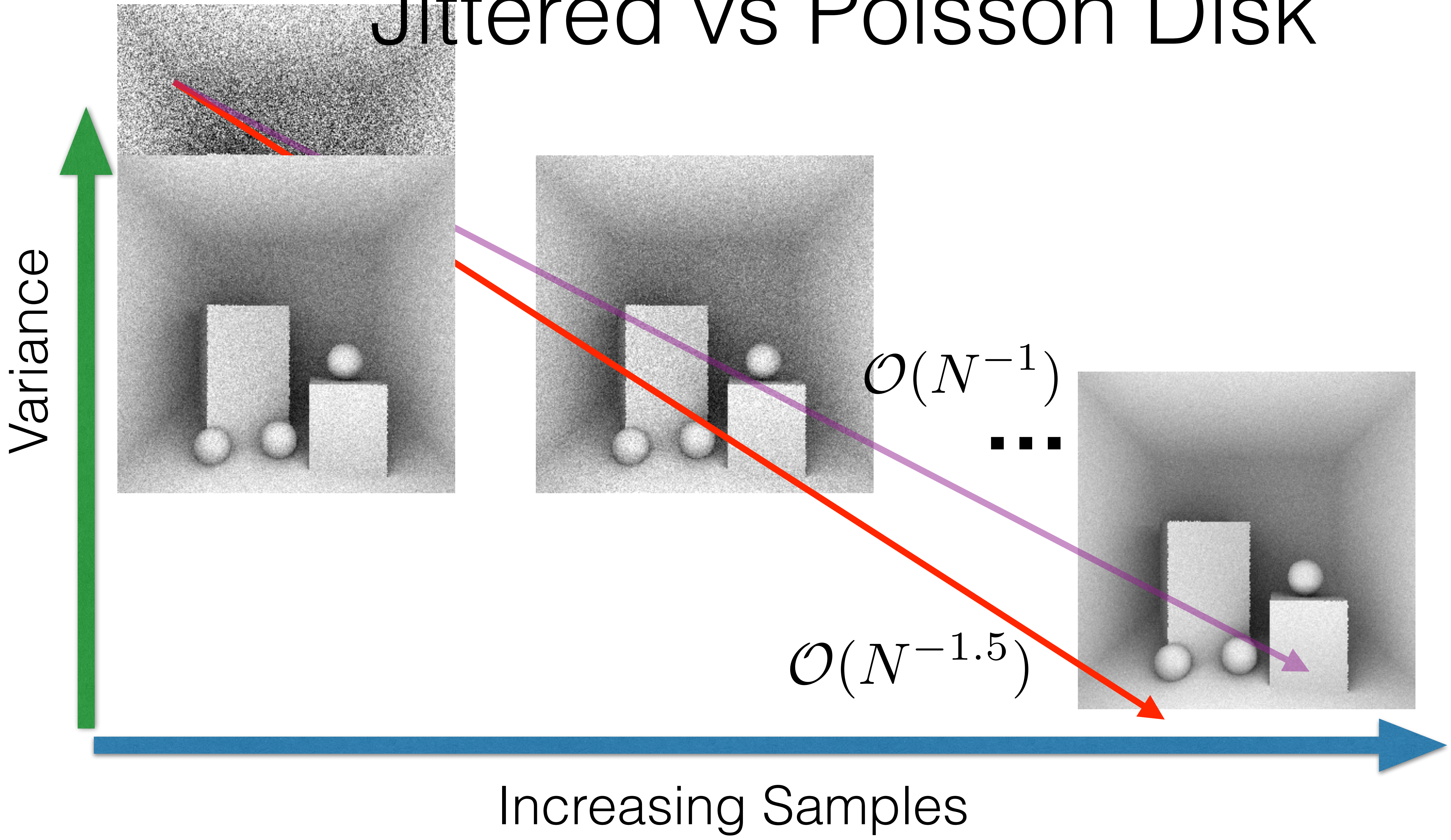

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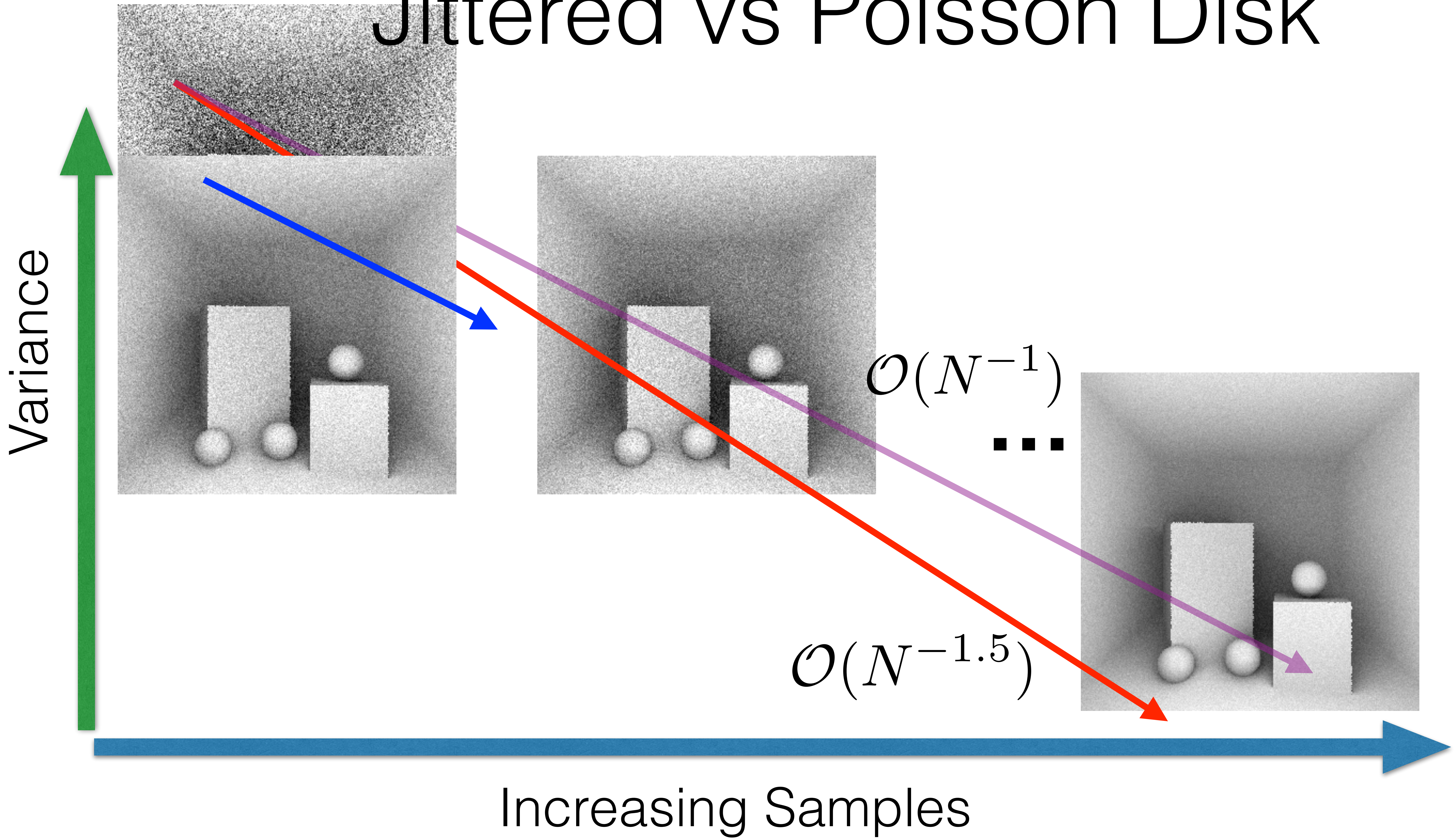

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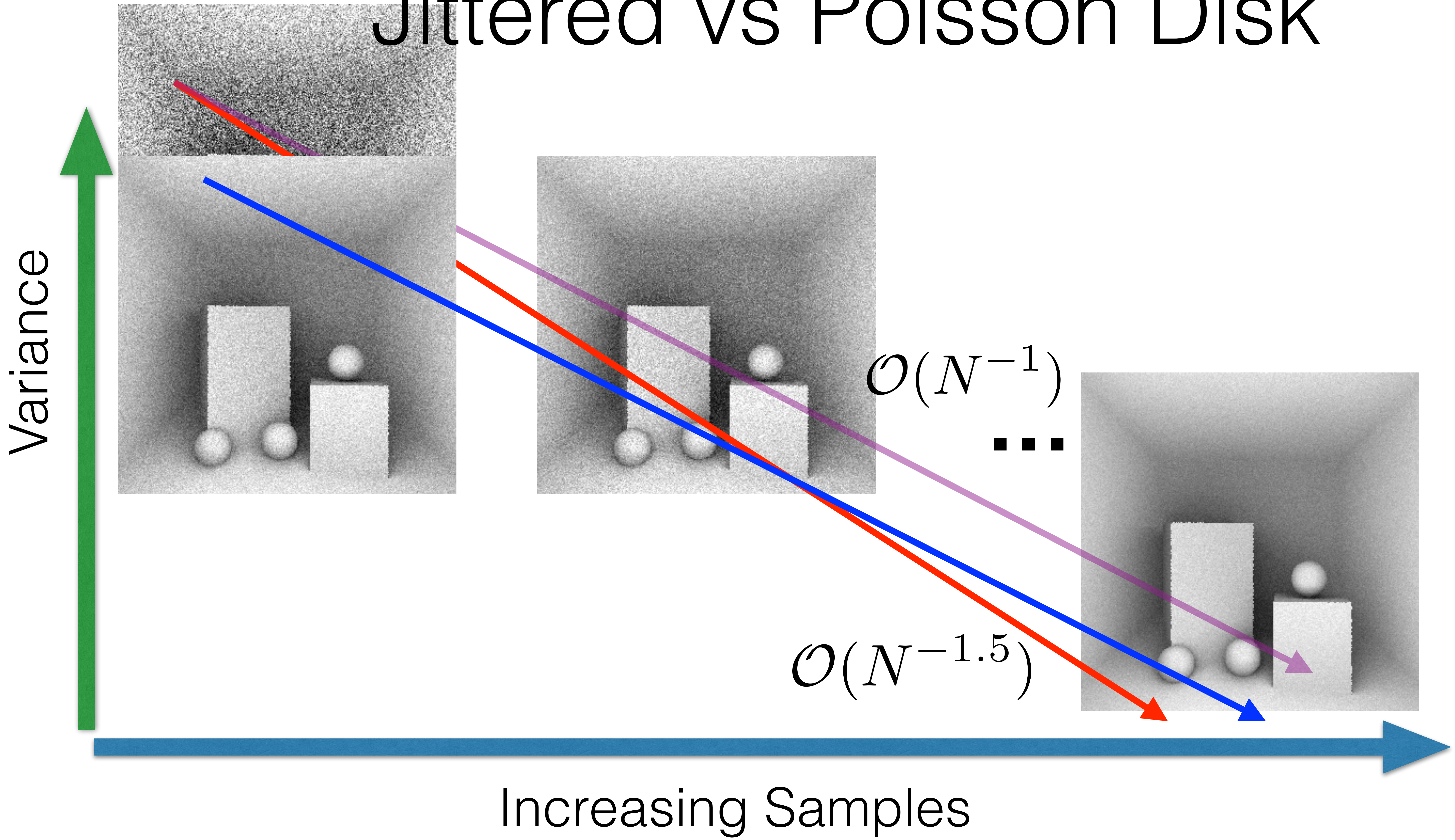
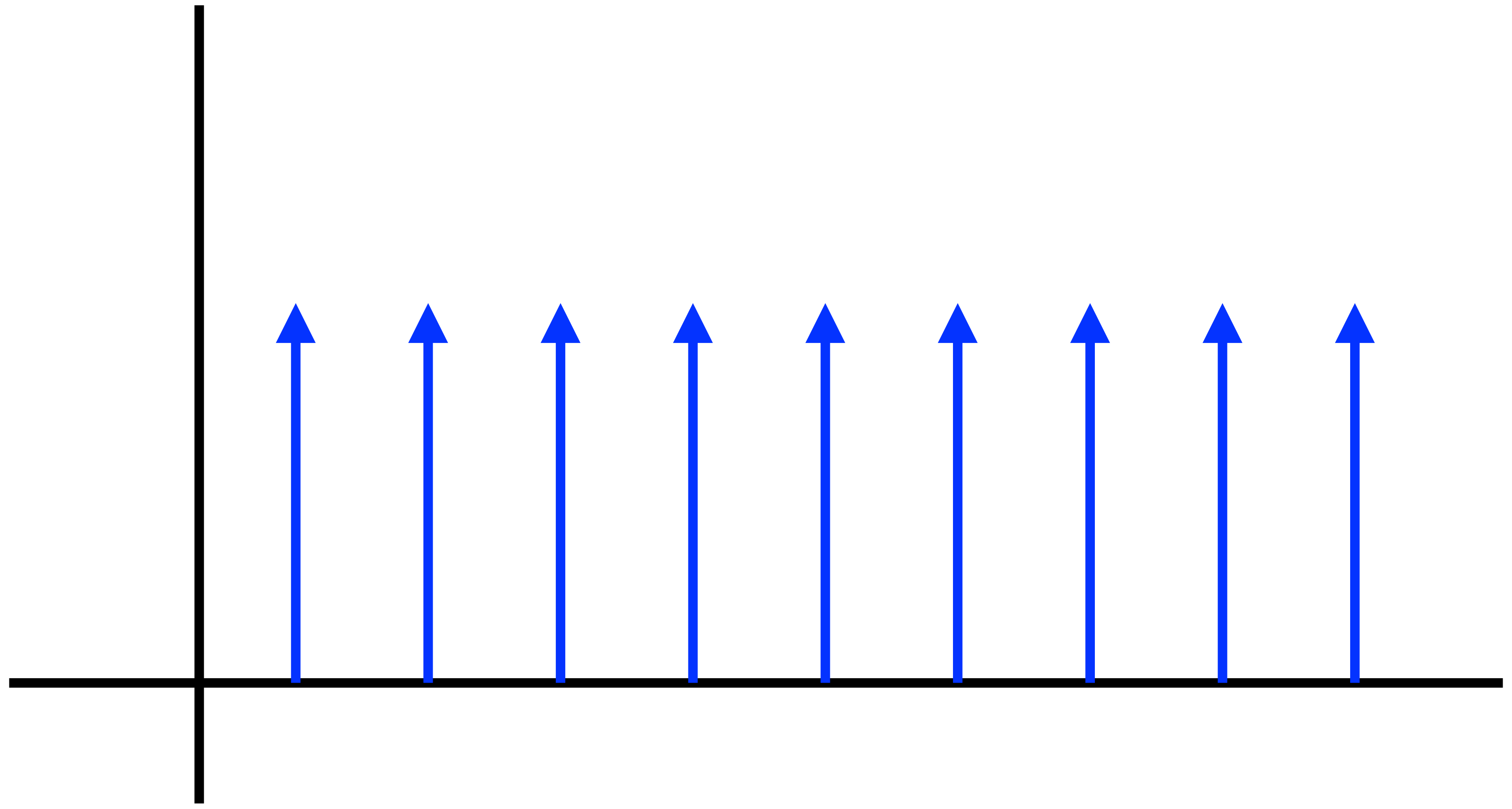


Samples and function in Fourier Domain

Spatial Domain

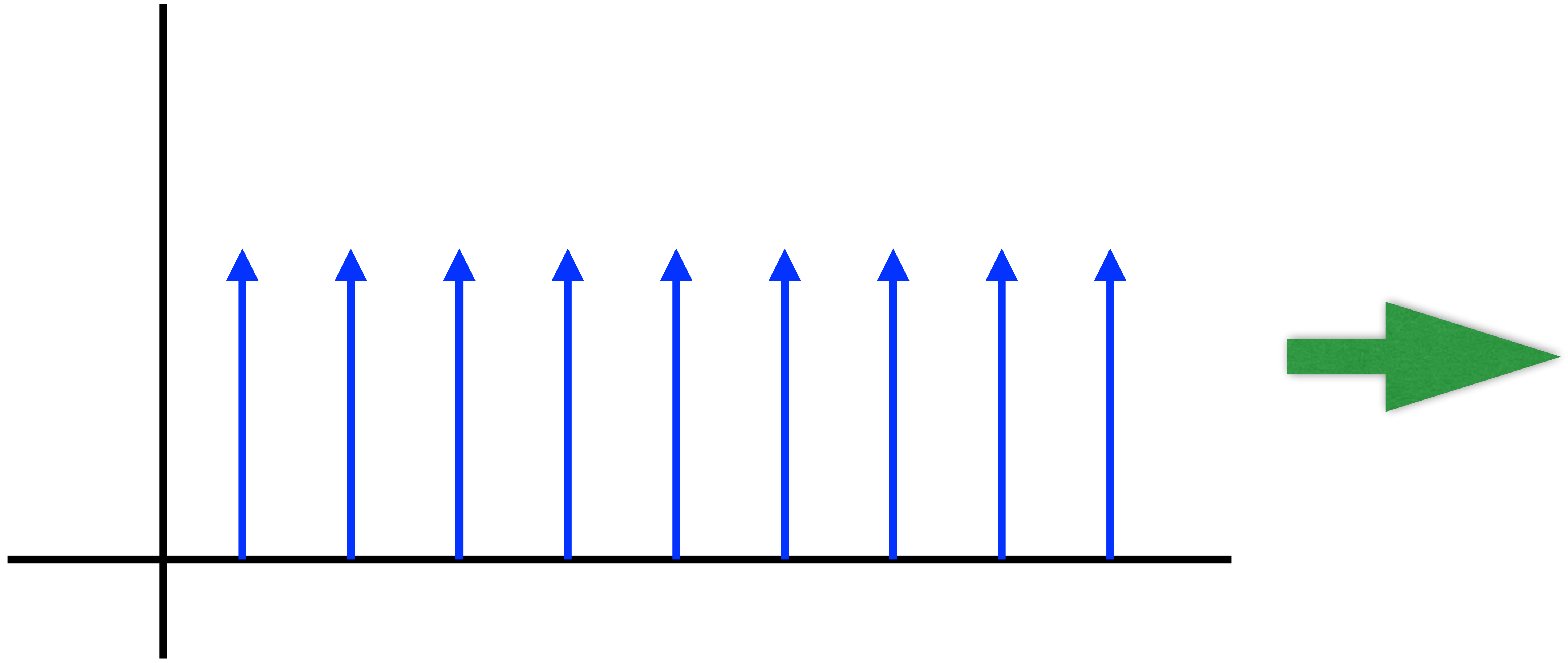


Fourier Domain

Samples and function in Fourier Domain

Spatial Domain

Fourier Domain



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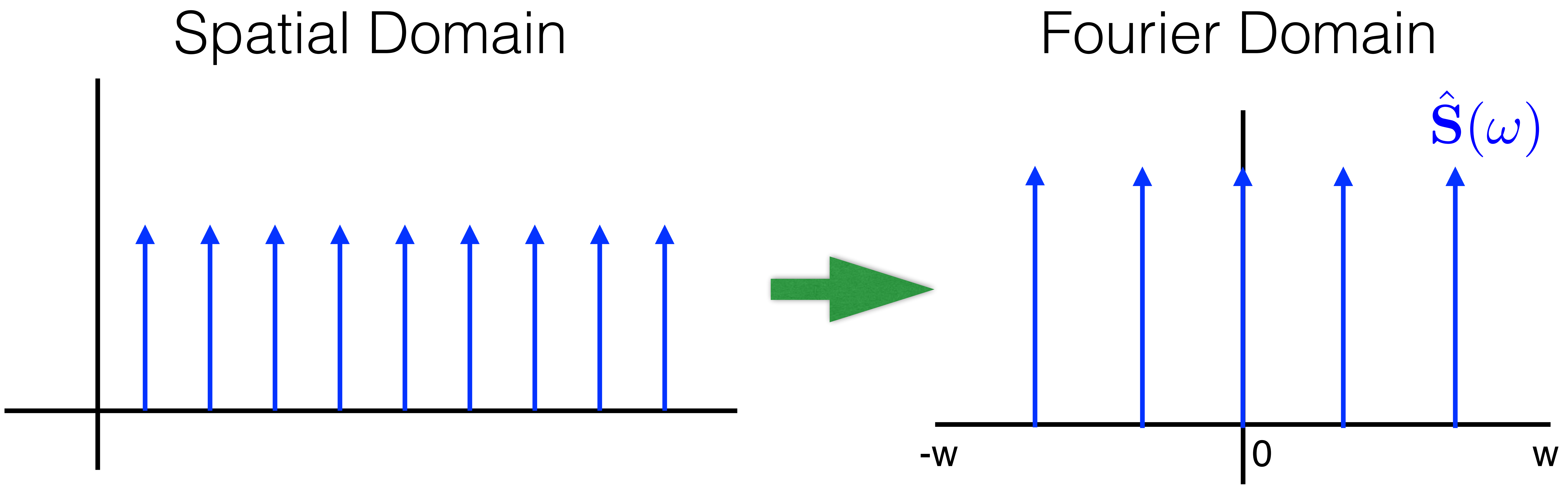

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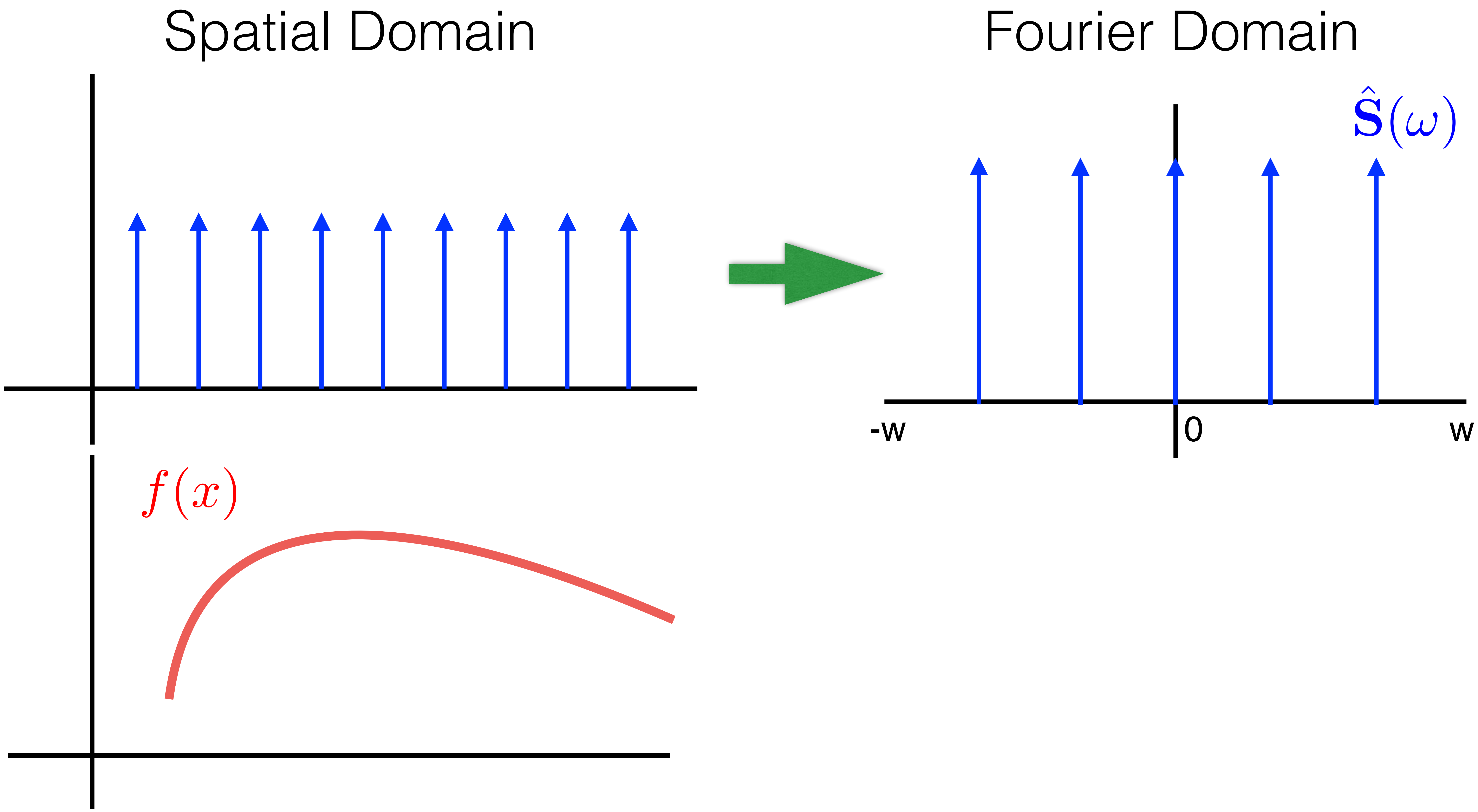
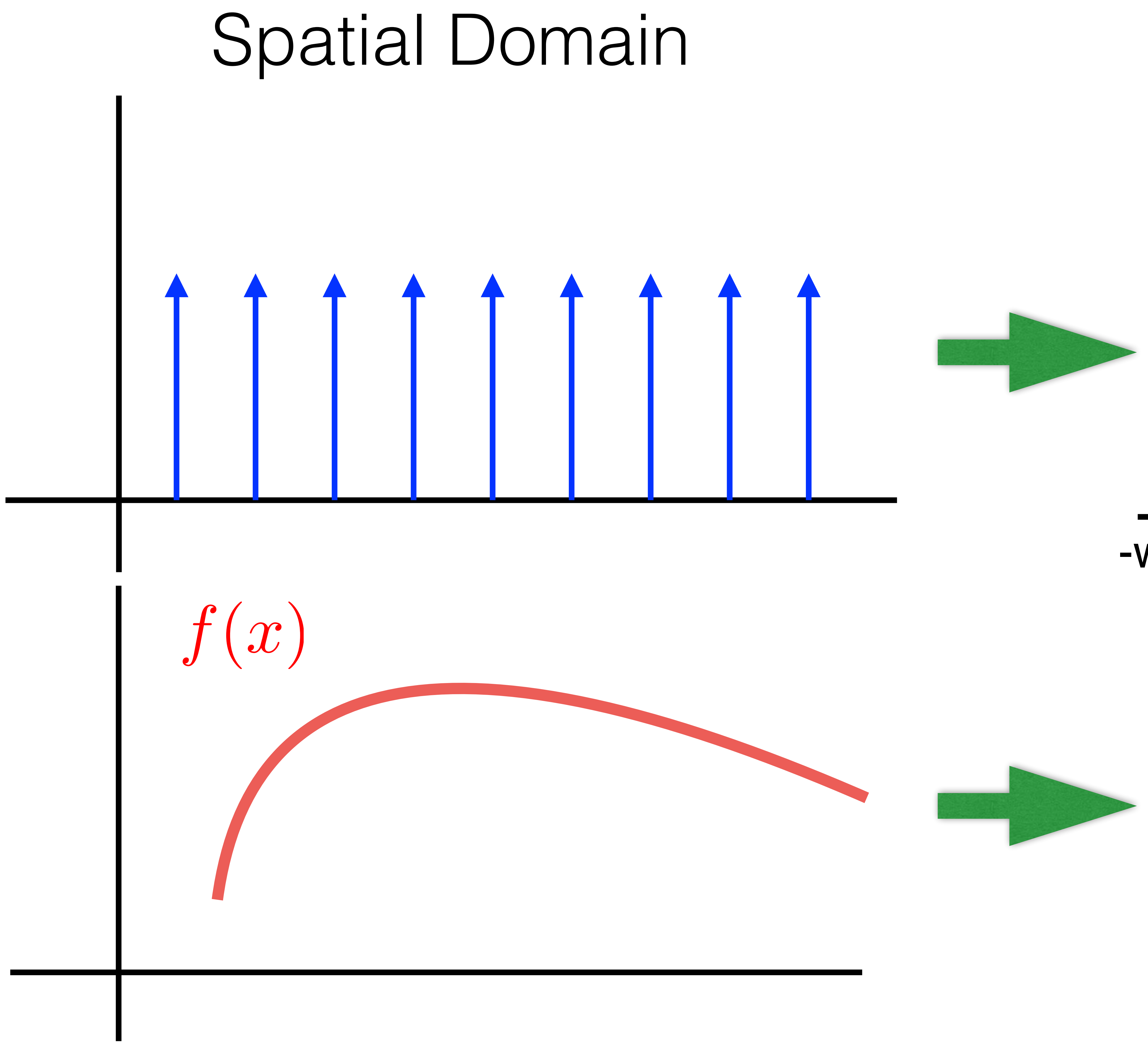
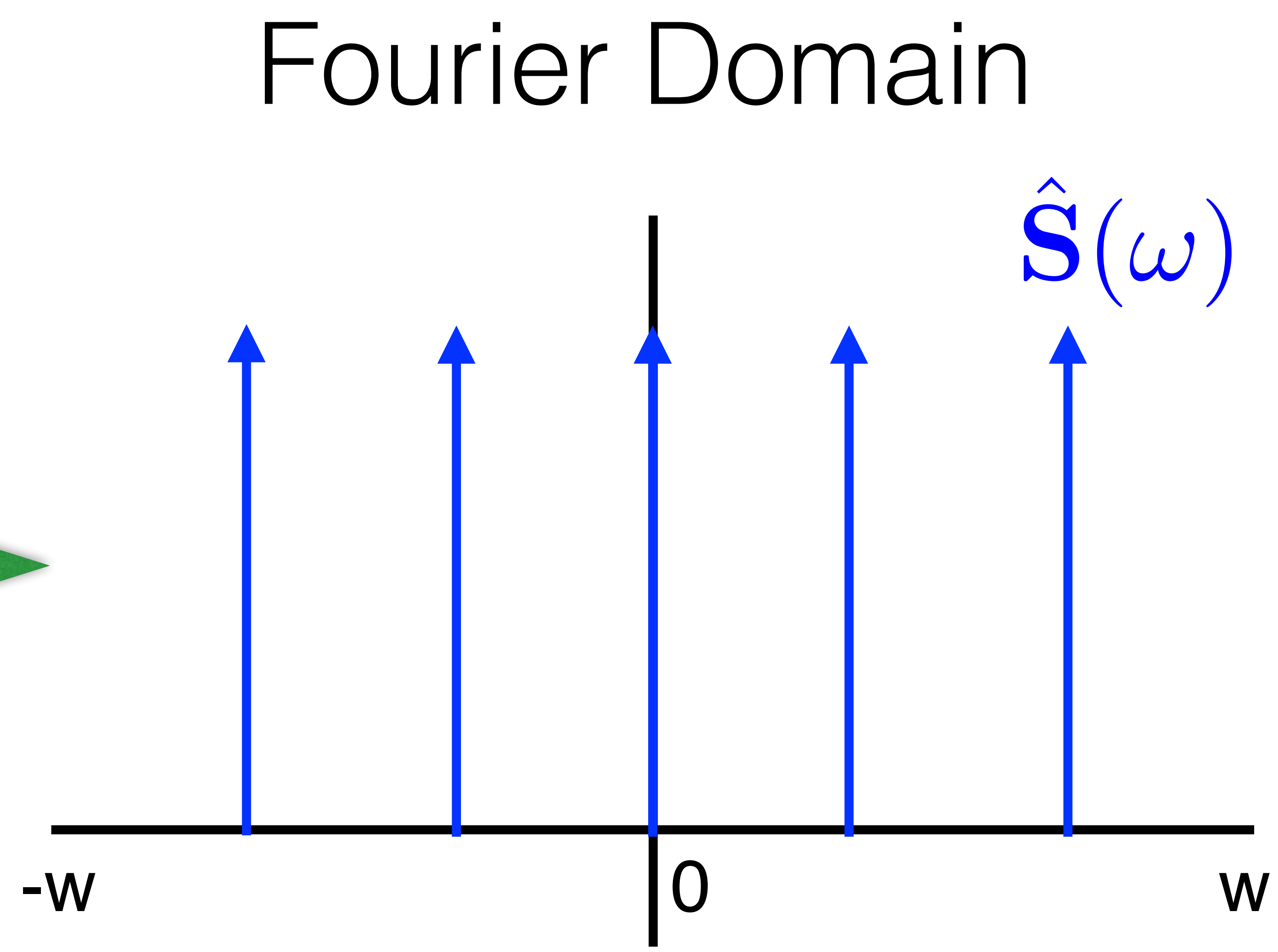
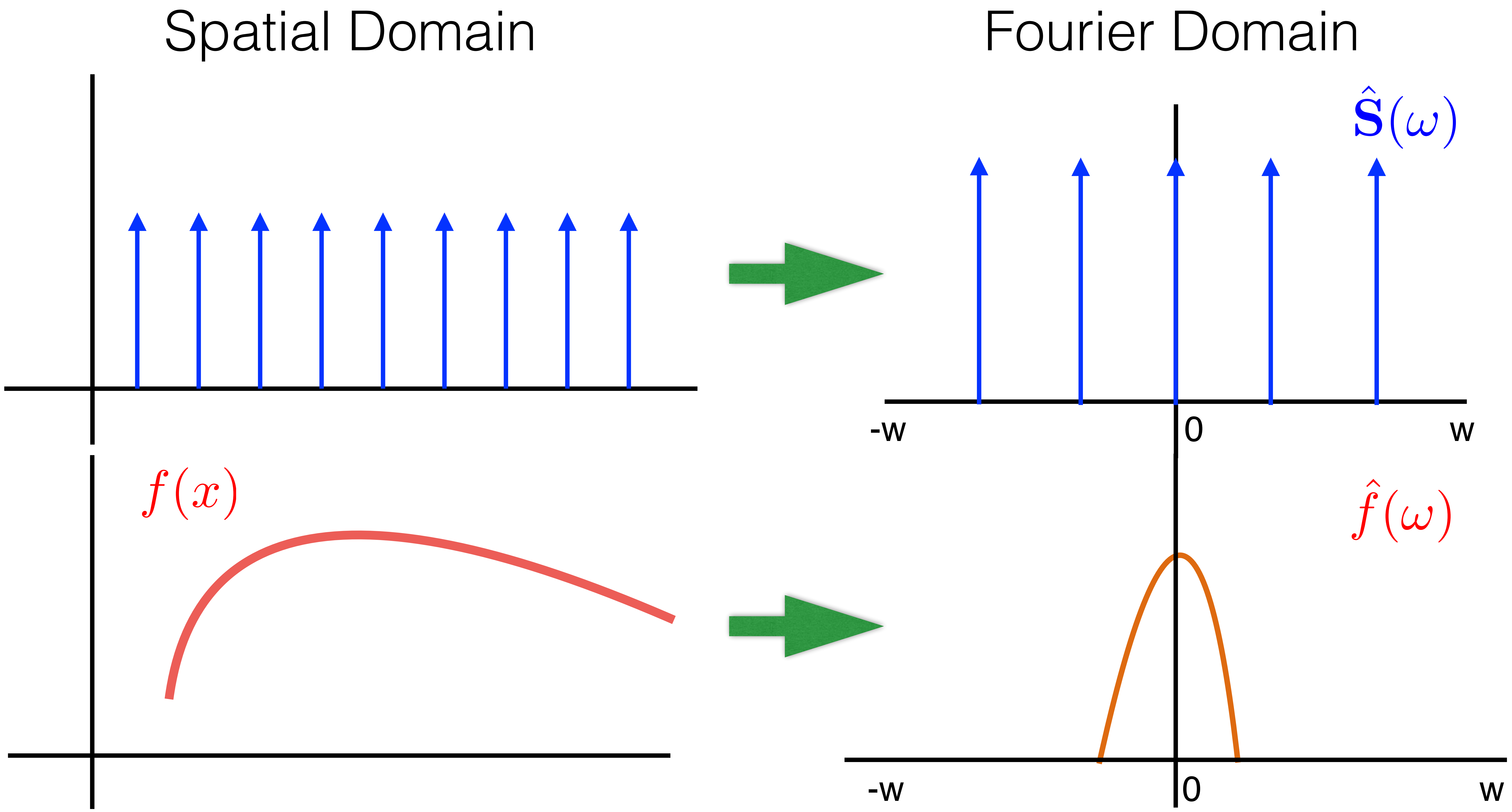


Samples and function in Fourier Domain

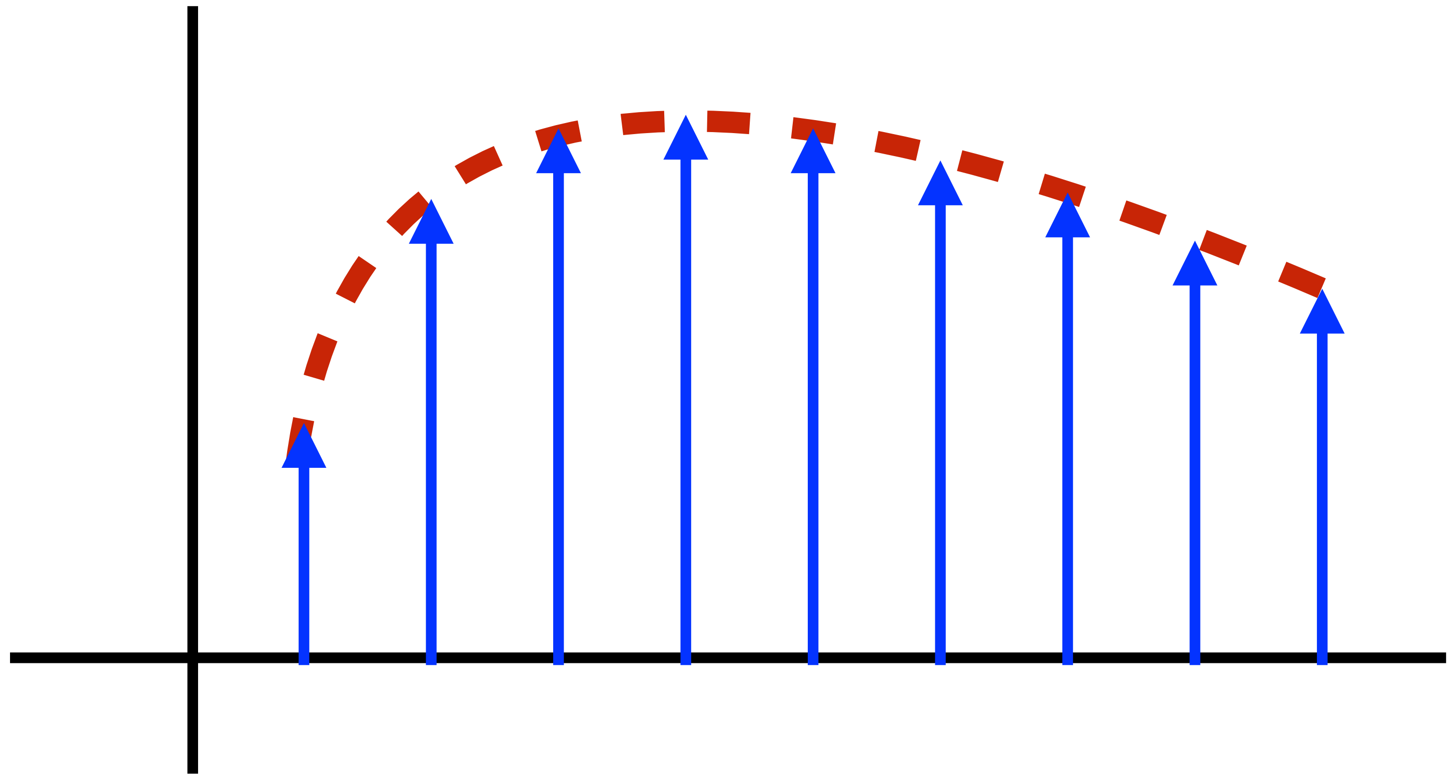




Samples and function in Fourier Domain



Sampling in Primal Domain is Convolution in Fourier Domain



$$f(x)\,\mathbf{S}(x)$$

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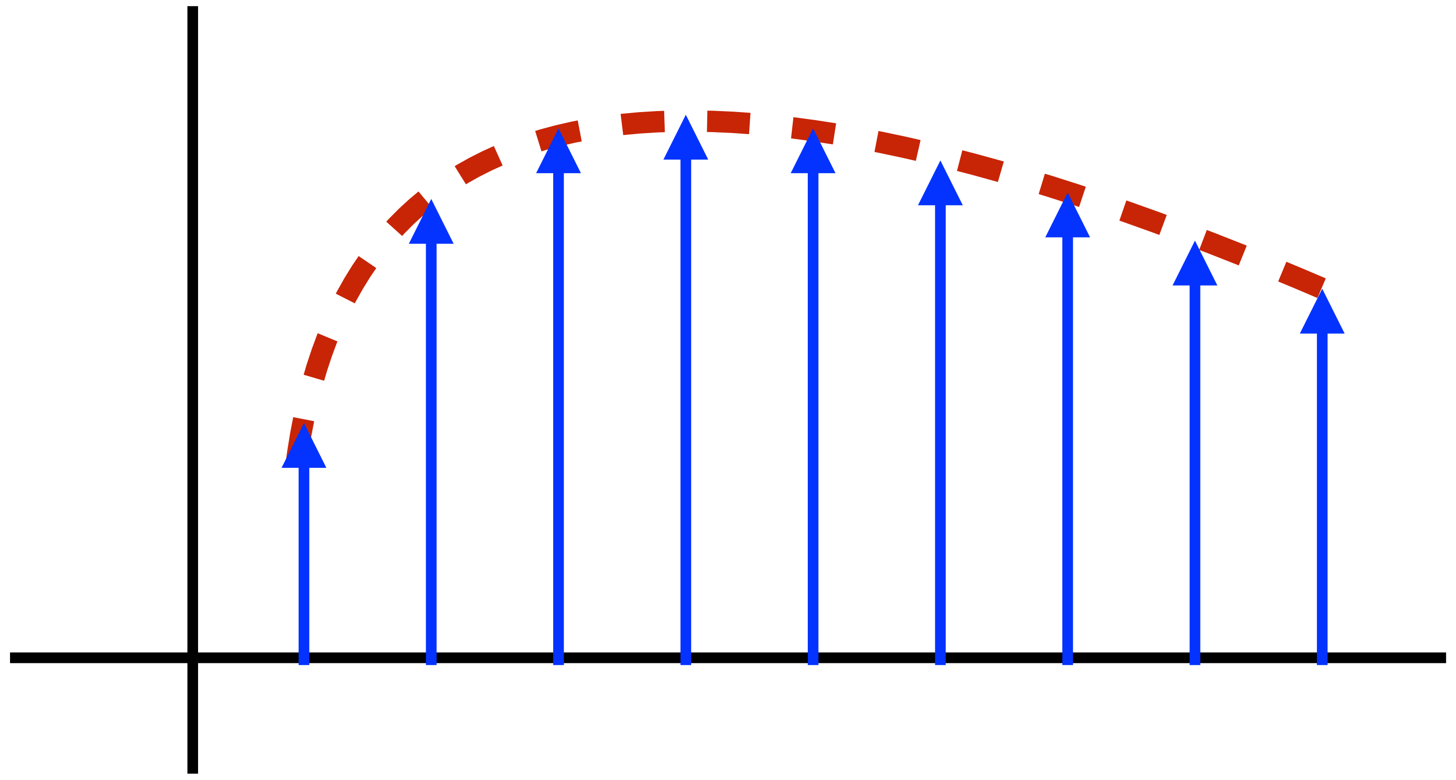
Sampling in Primal Domain is Convolution in Fourier Domain



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Fredo Durand [2011]

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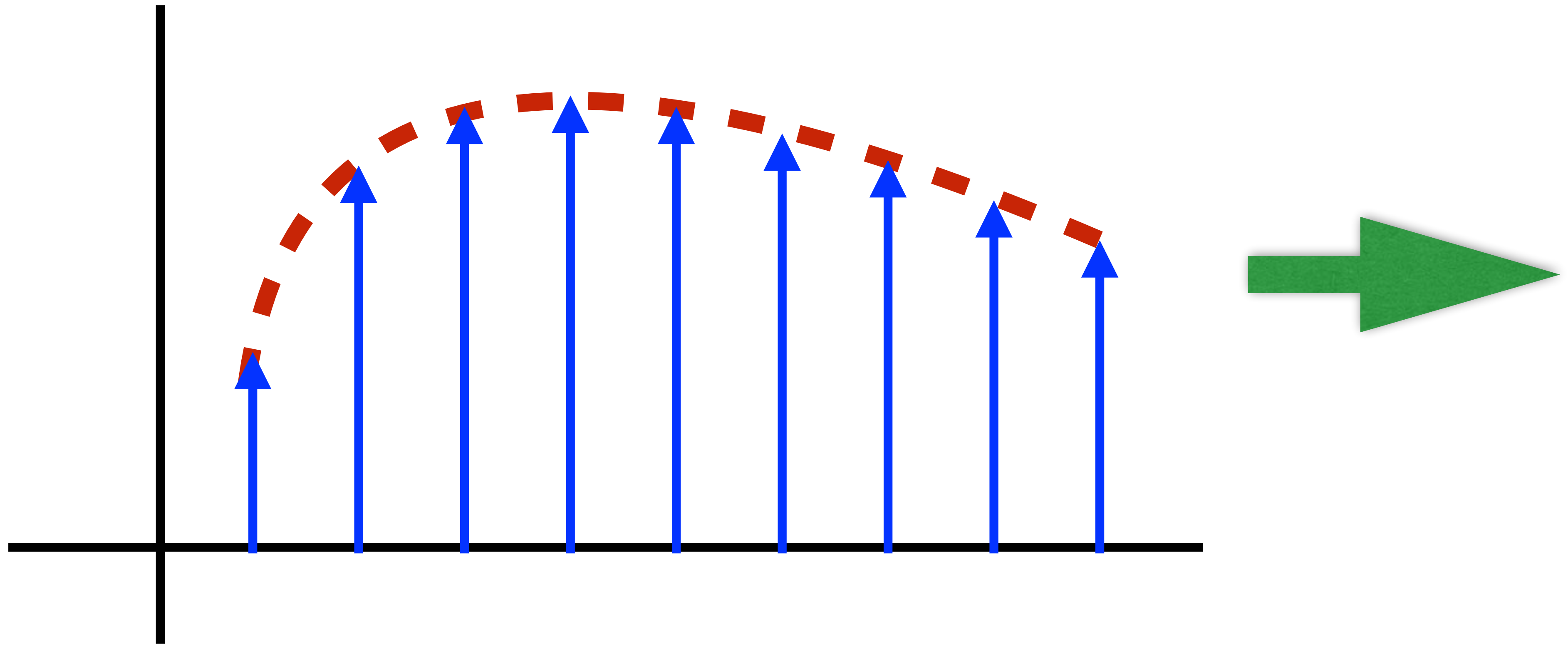

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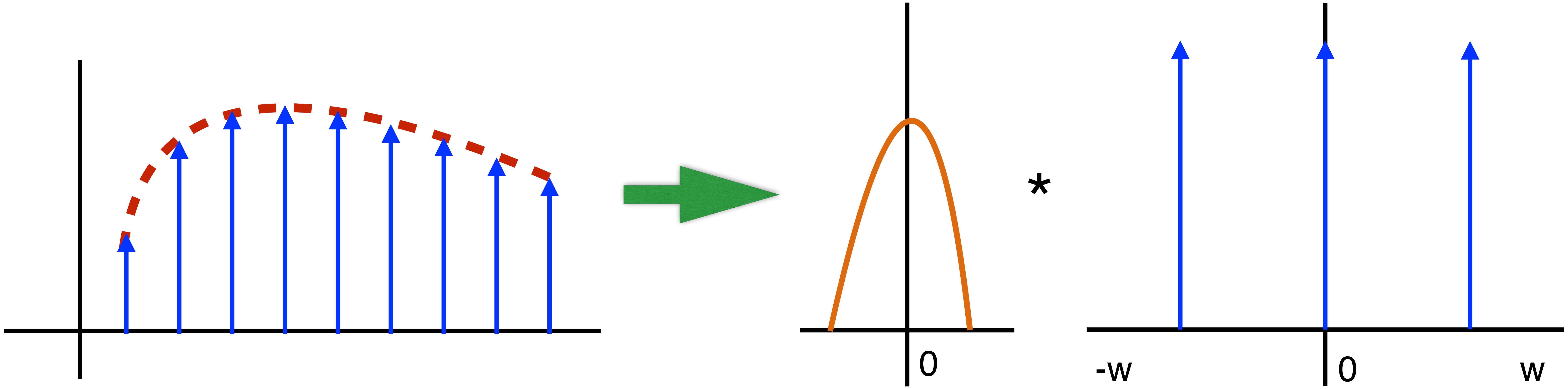

$$f(x)\,\mathbf{S}(x)$$

$$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$$

Fredo Durand [2011]

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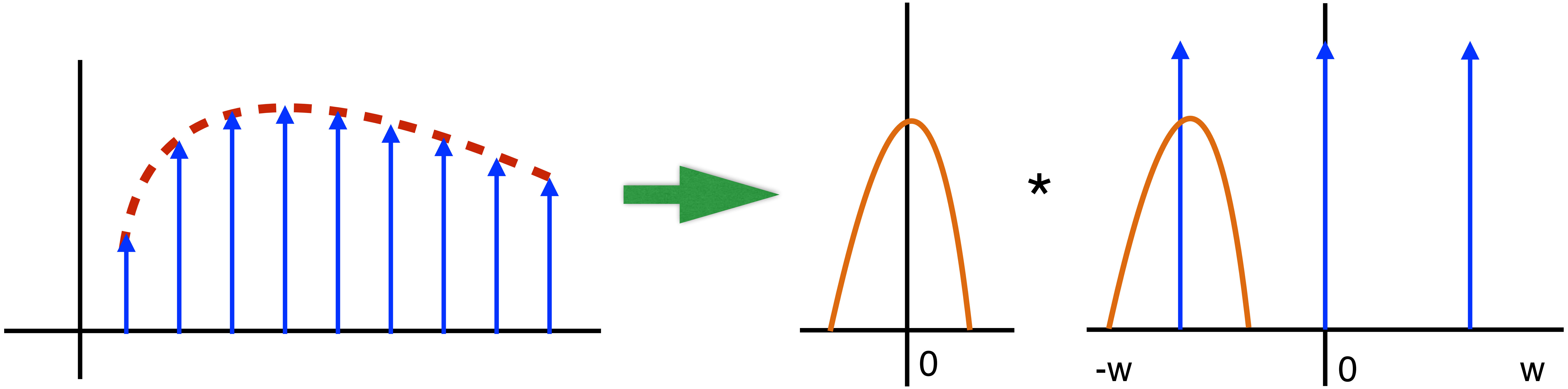

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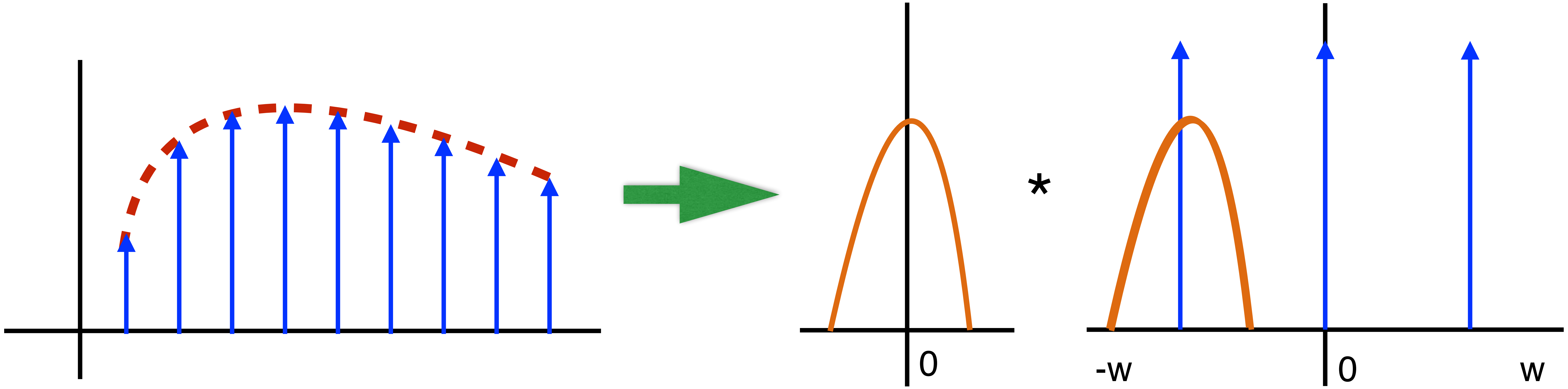

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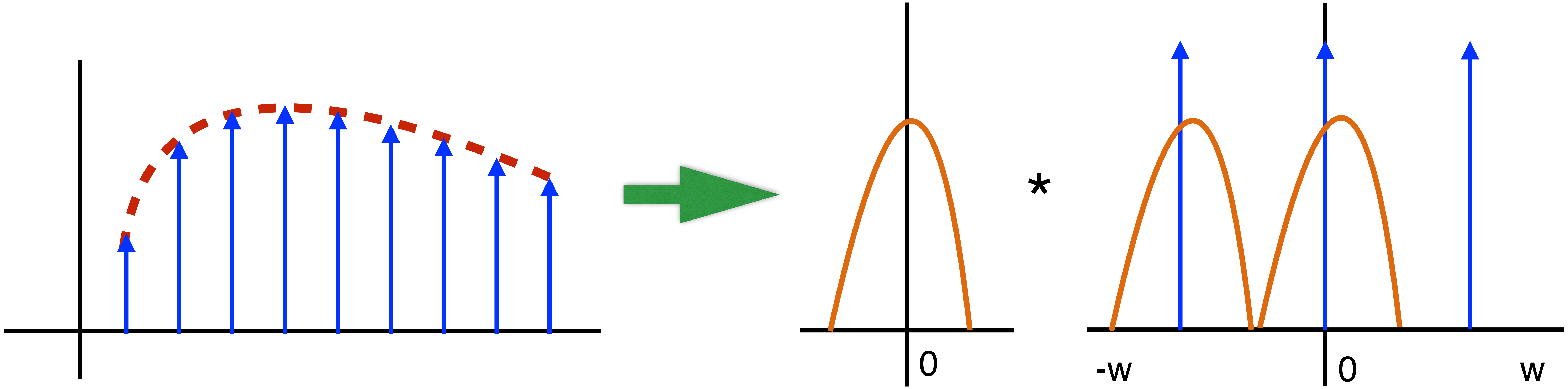

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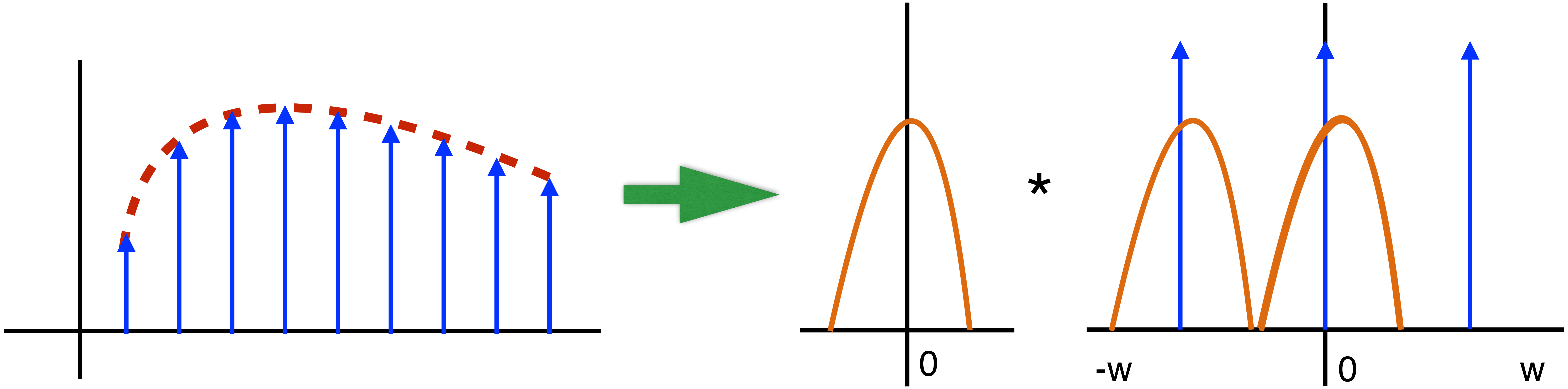

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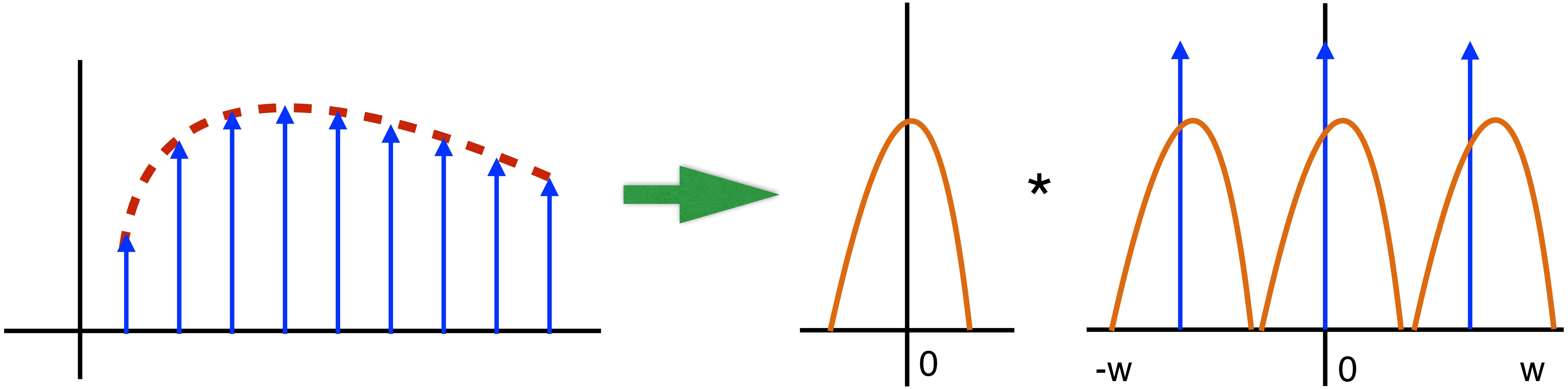

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Fredo Durand [2011]

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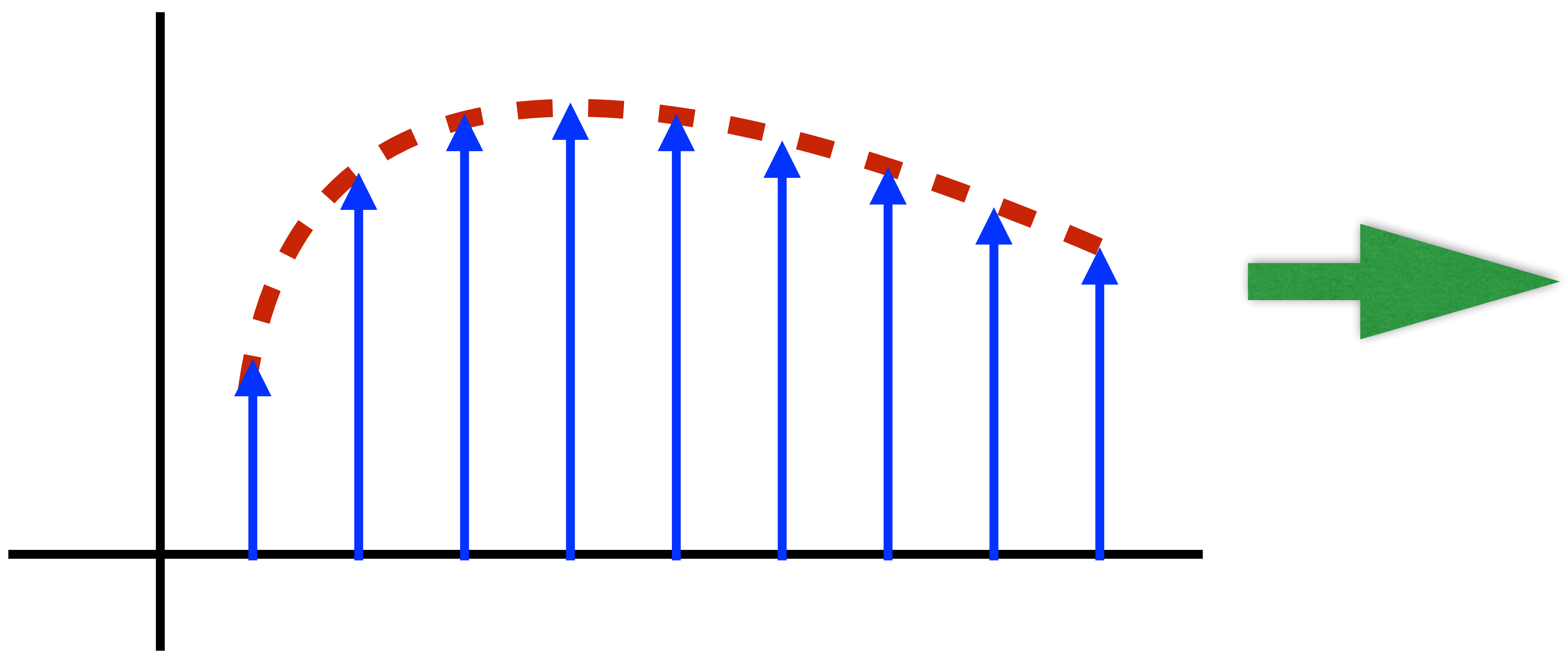

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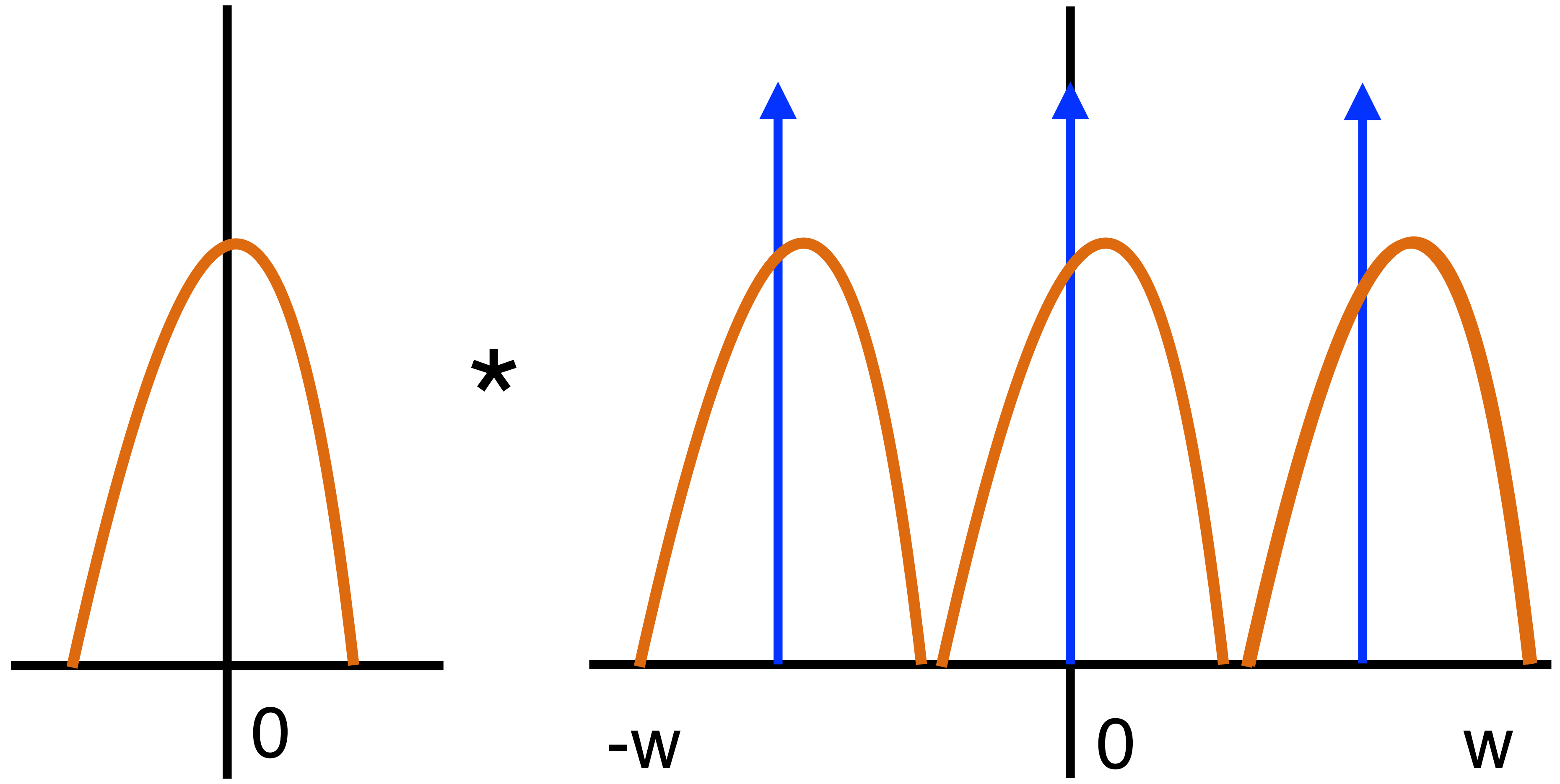
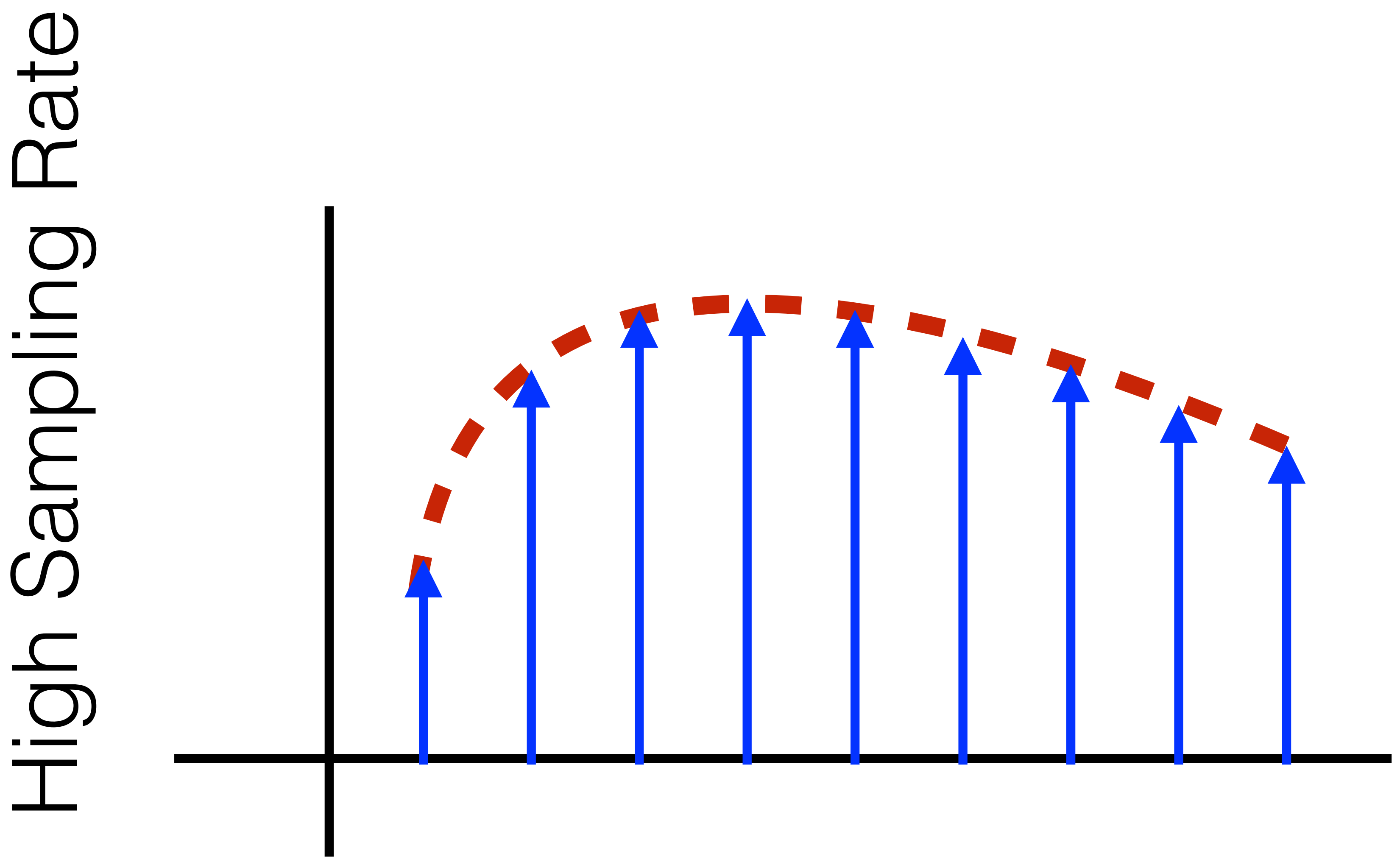
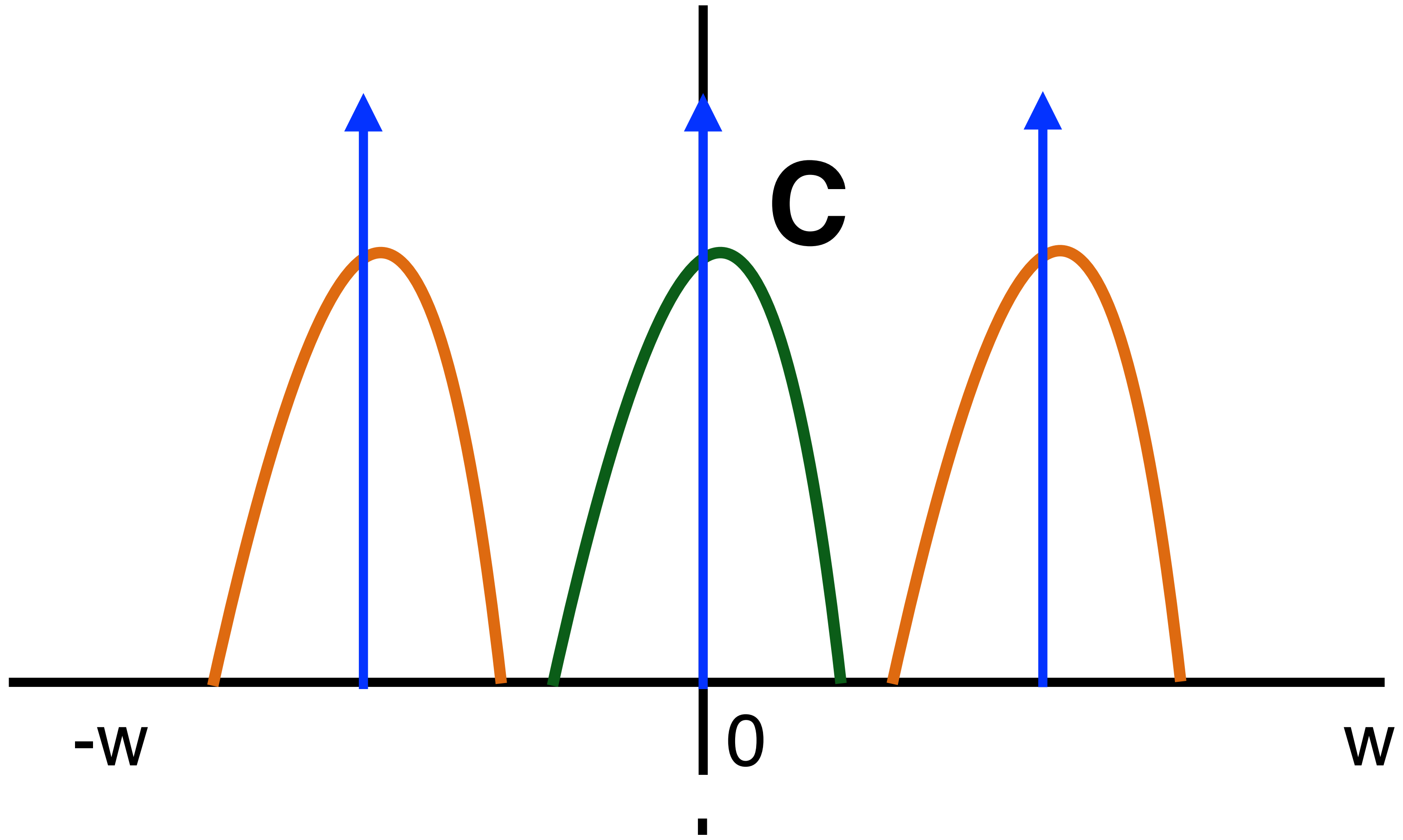


$\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Aliasing in Reconstruction





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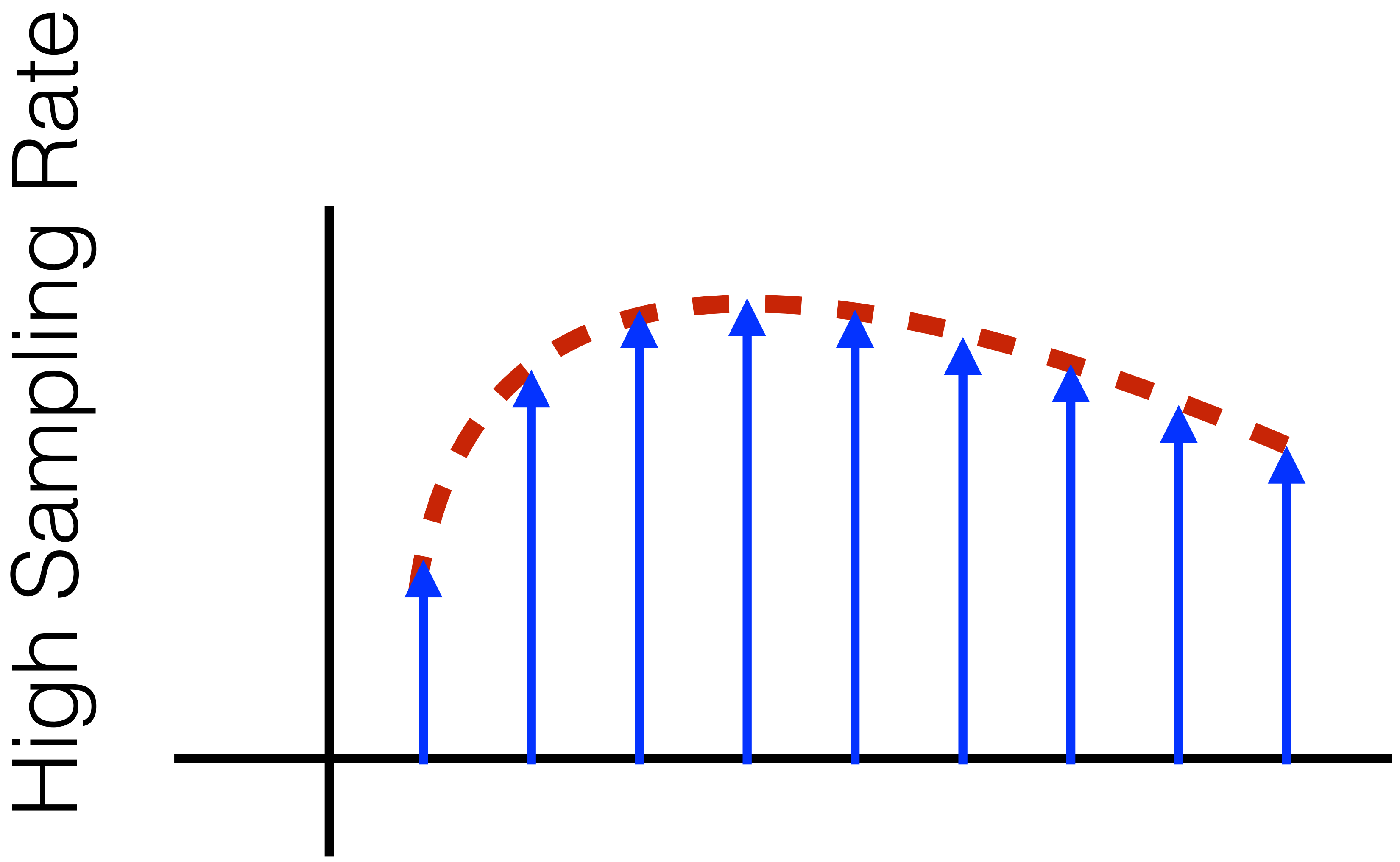

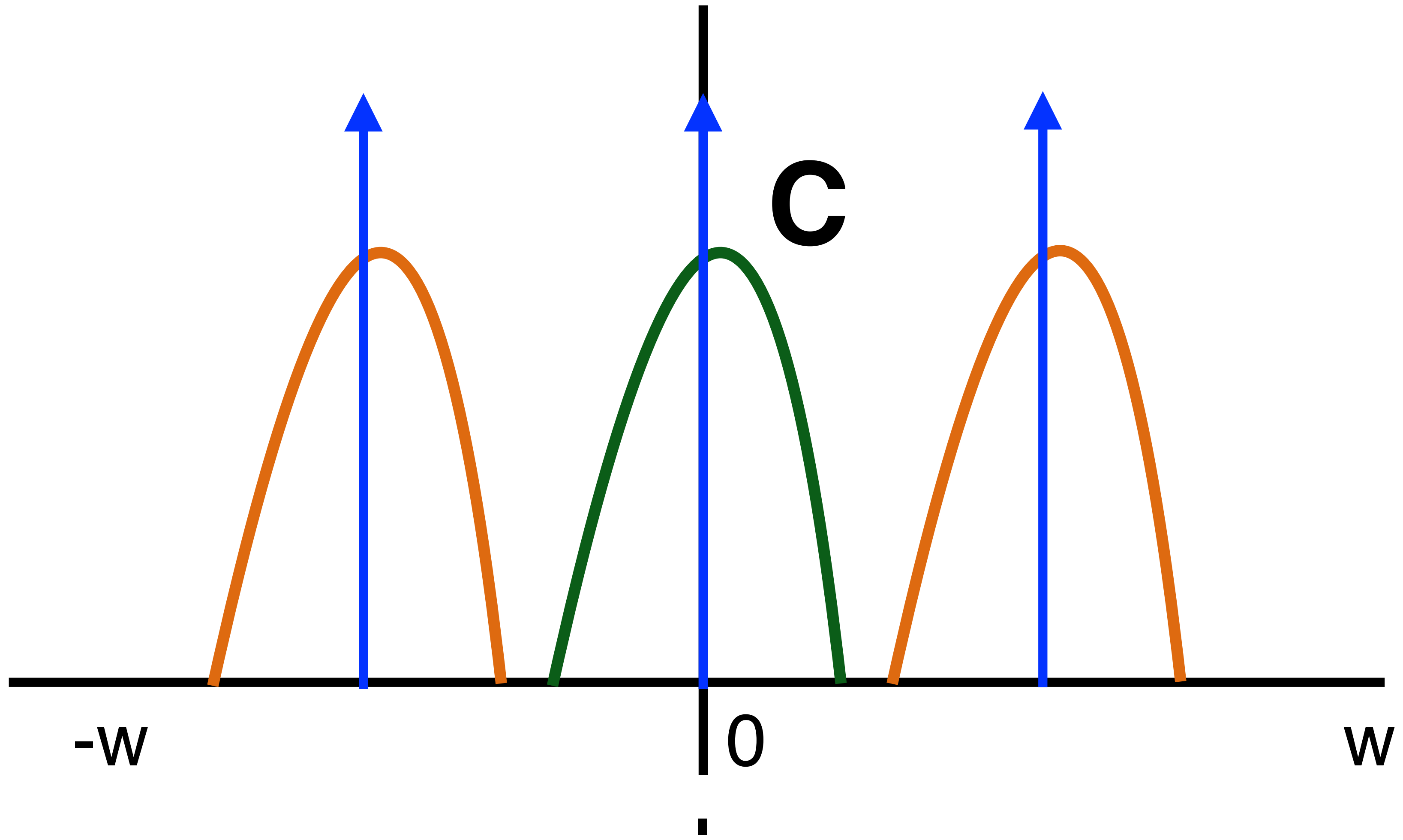

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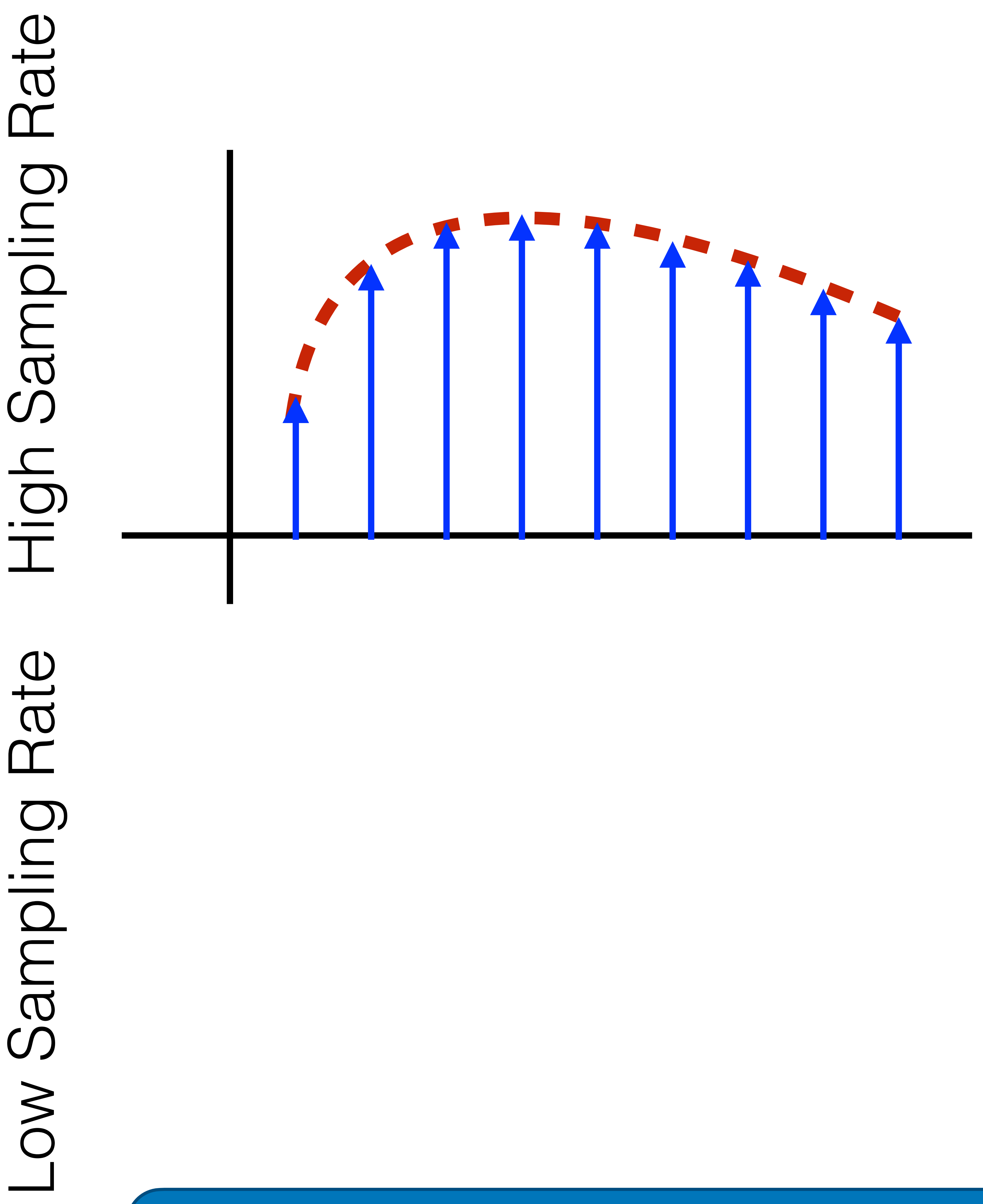
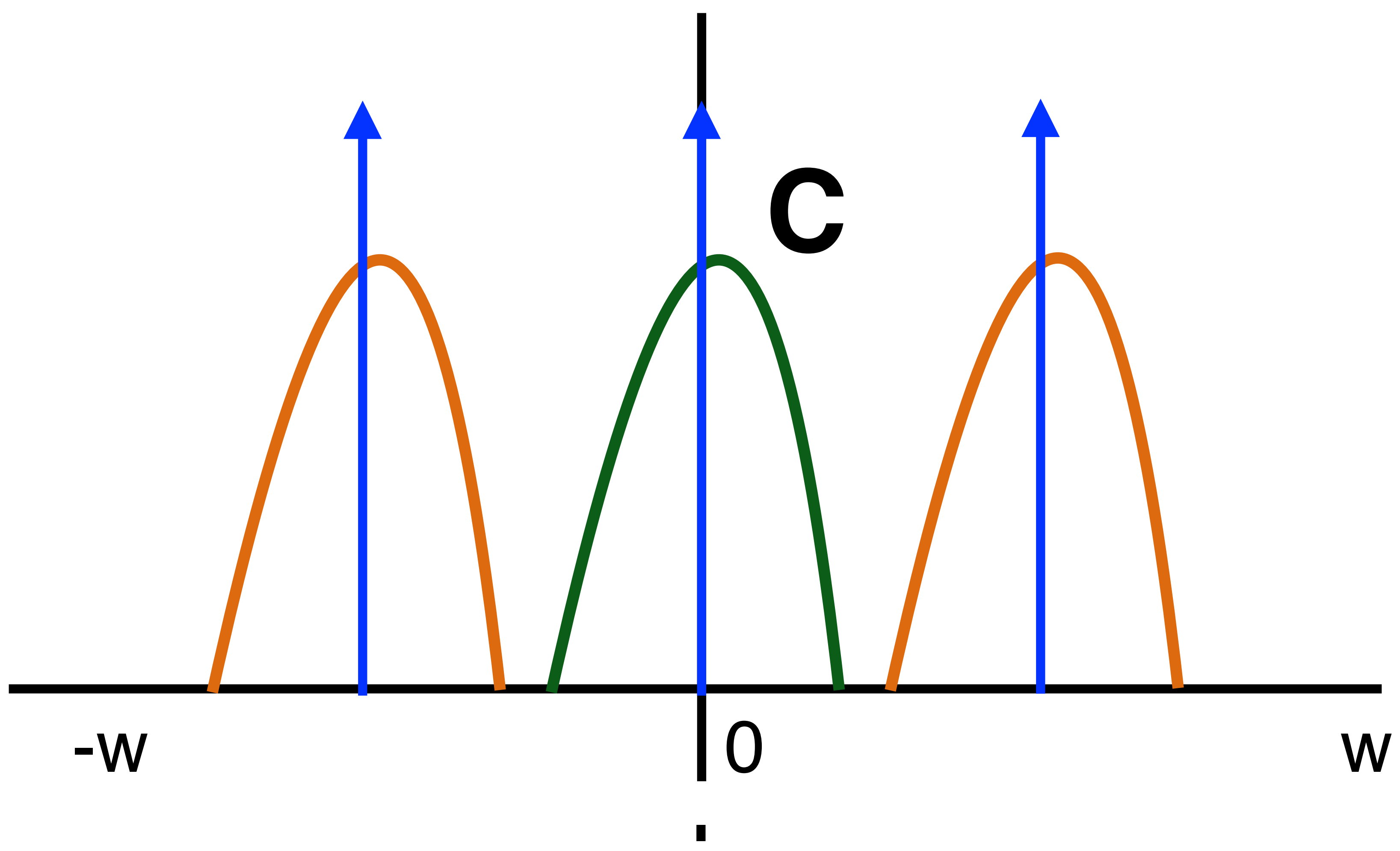
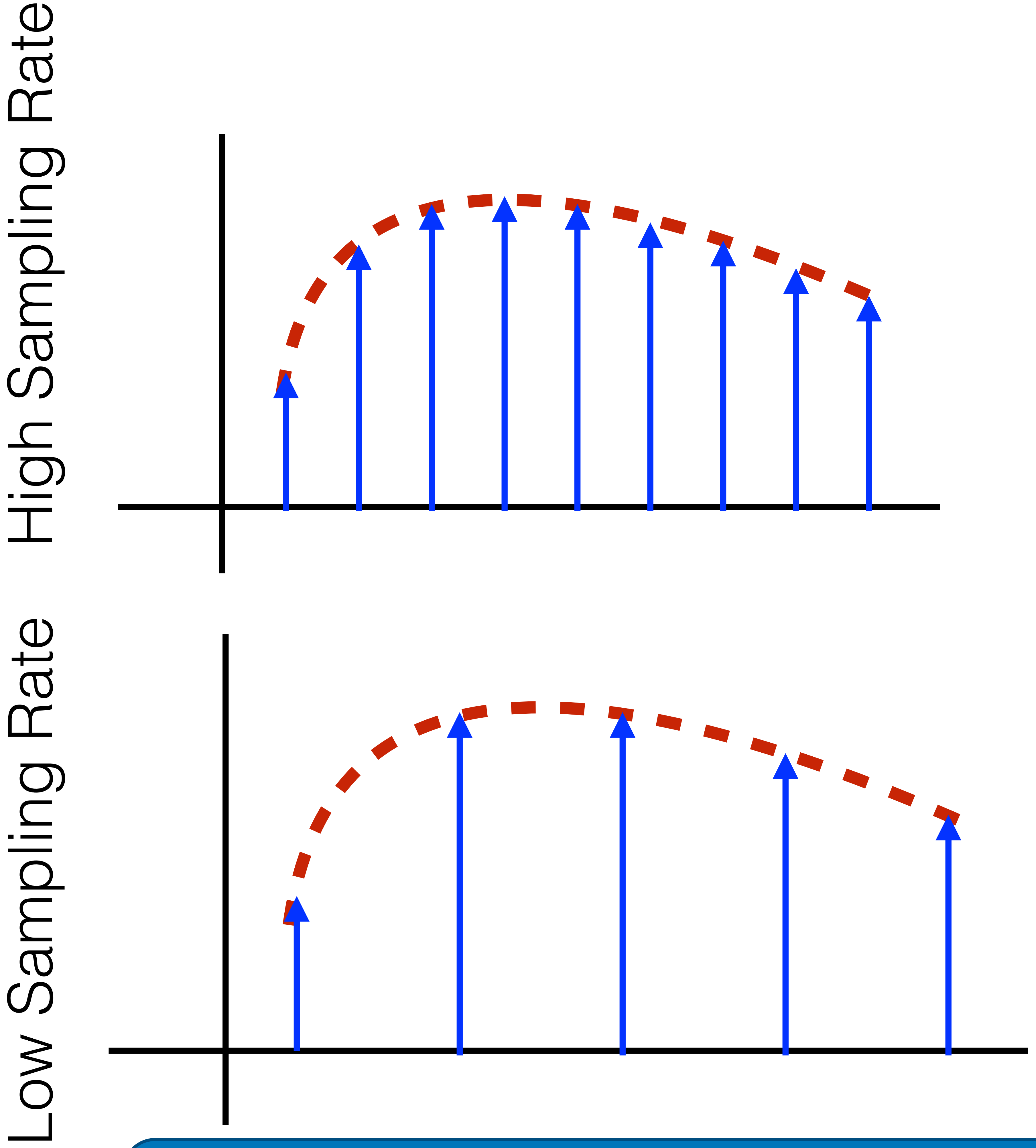
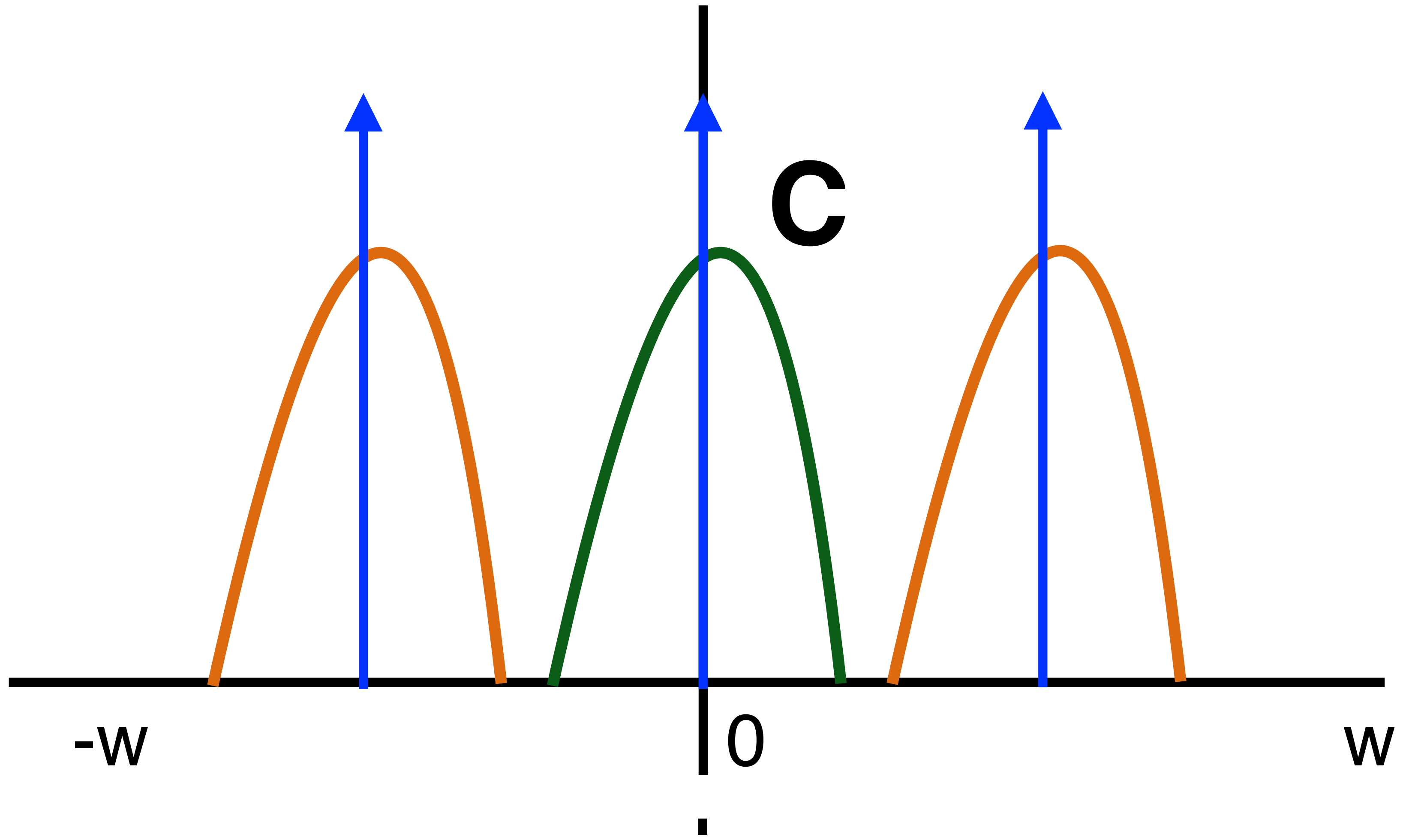




Aliasing in Reconstruction





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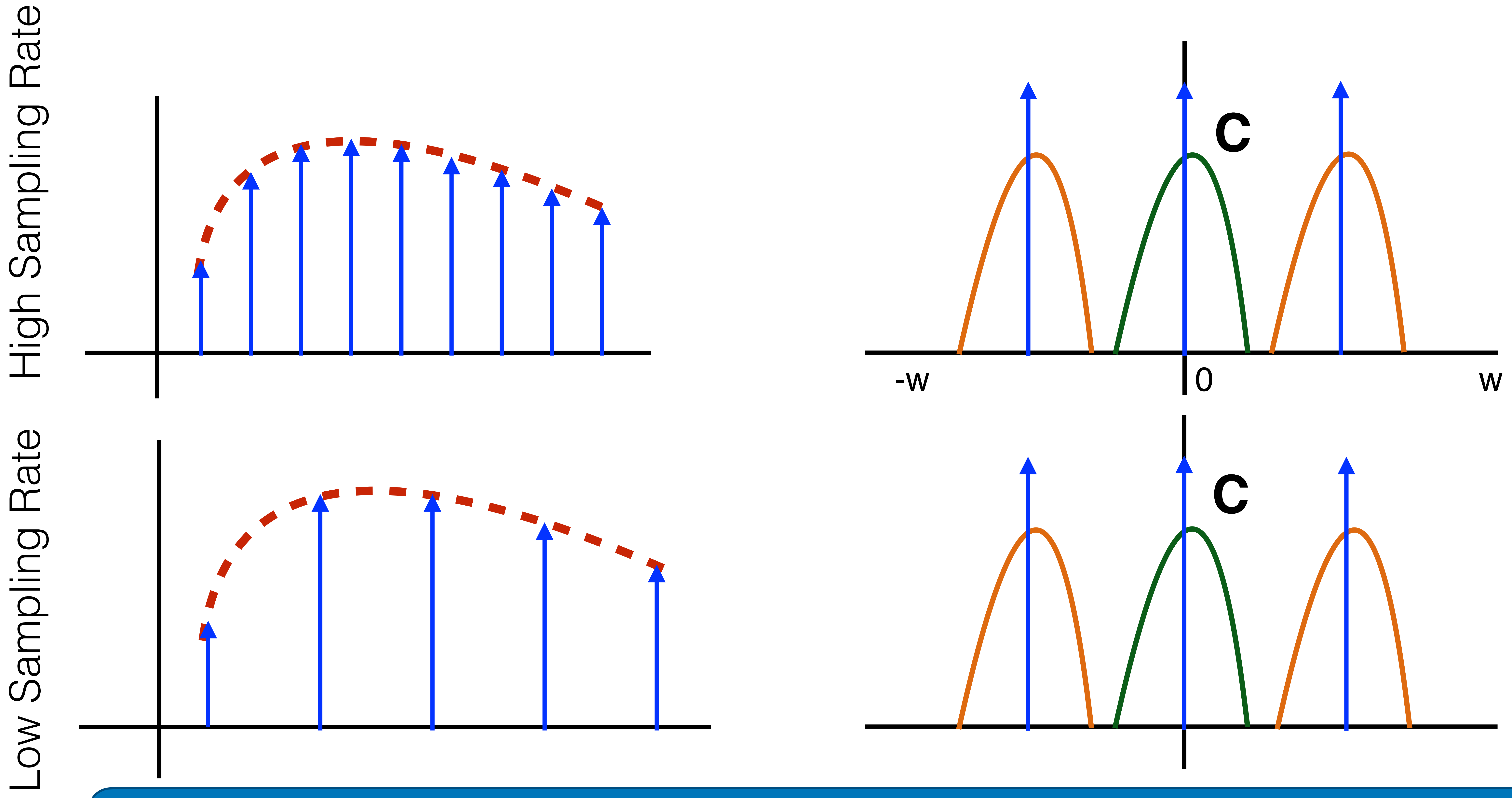

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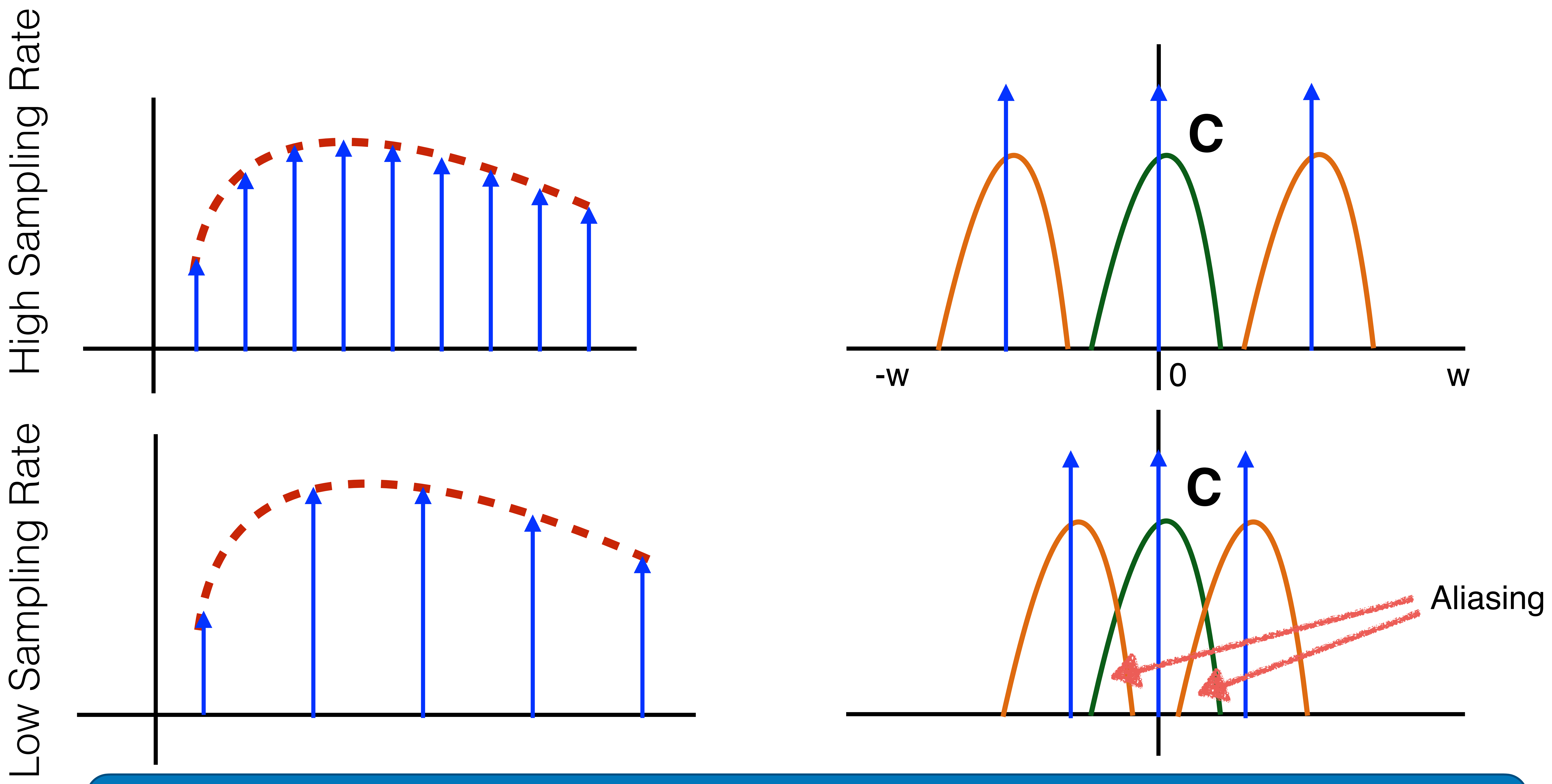

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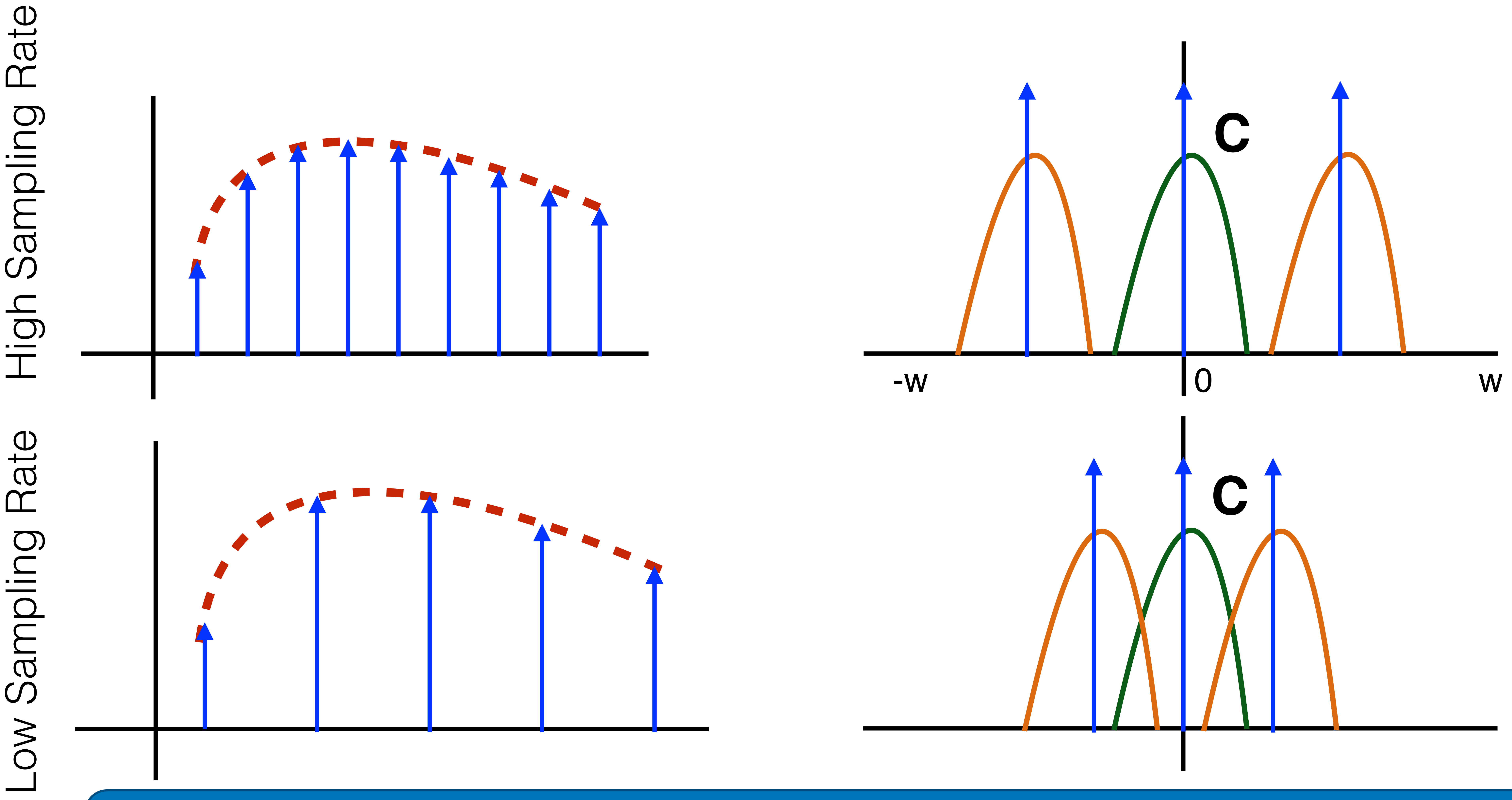
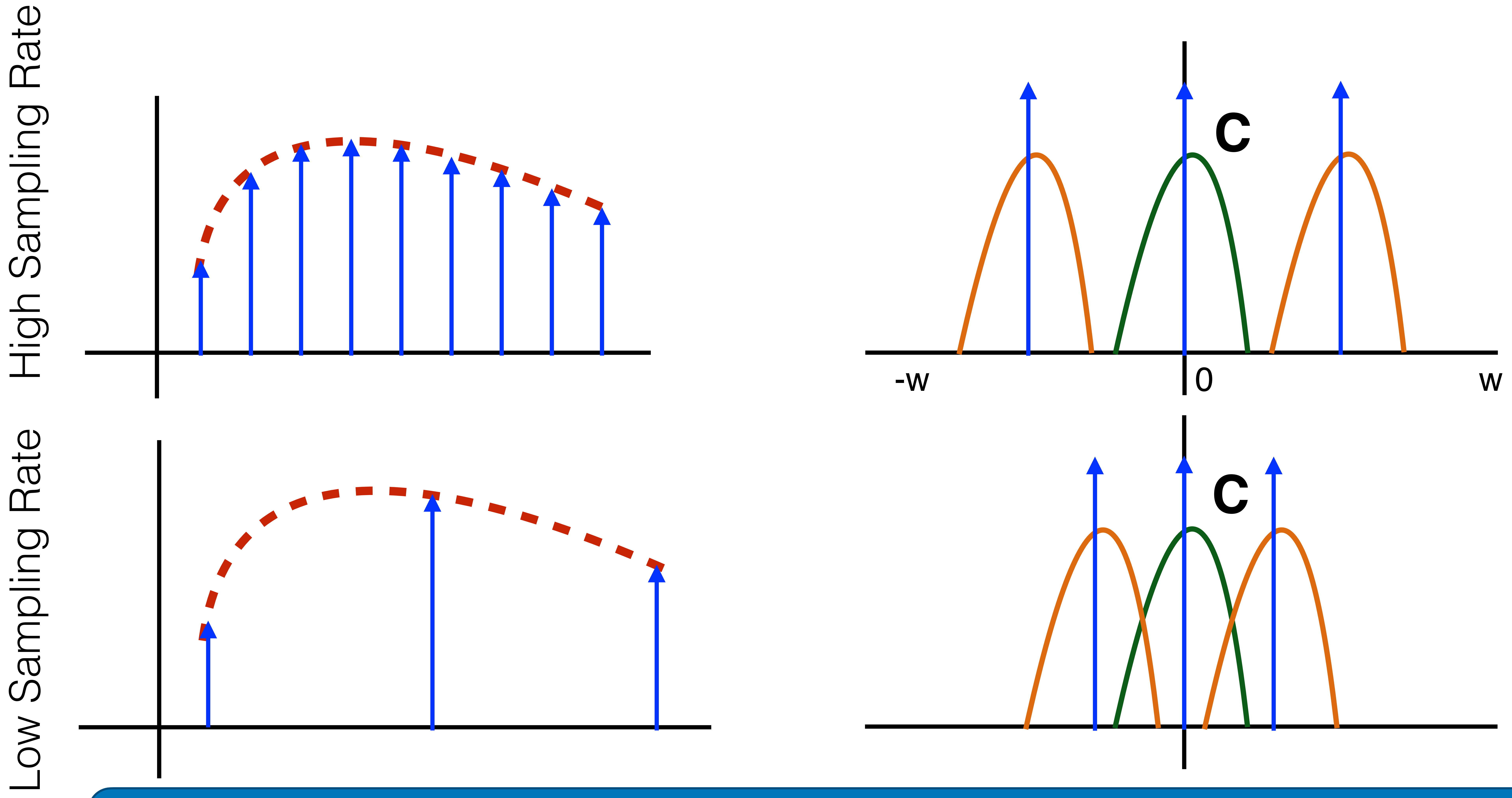


Error in Monte Carlo Integration



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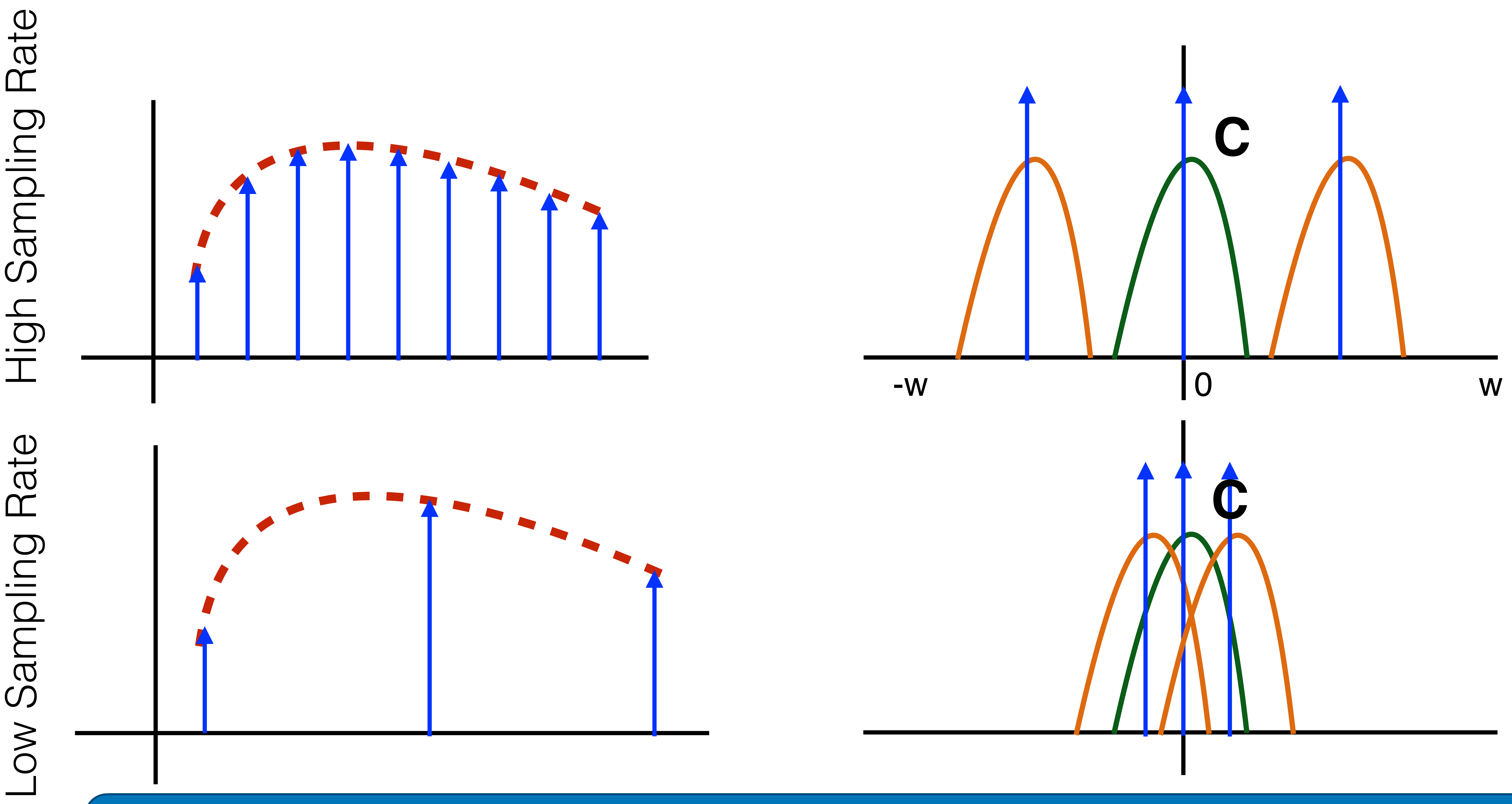

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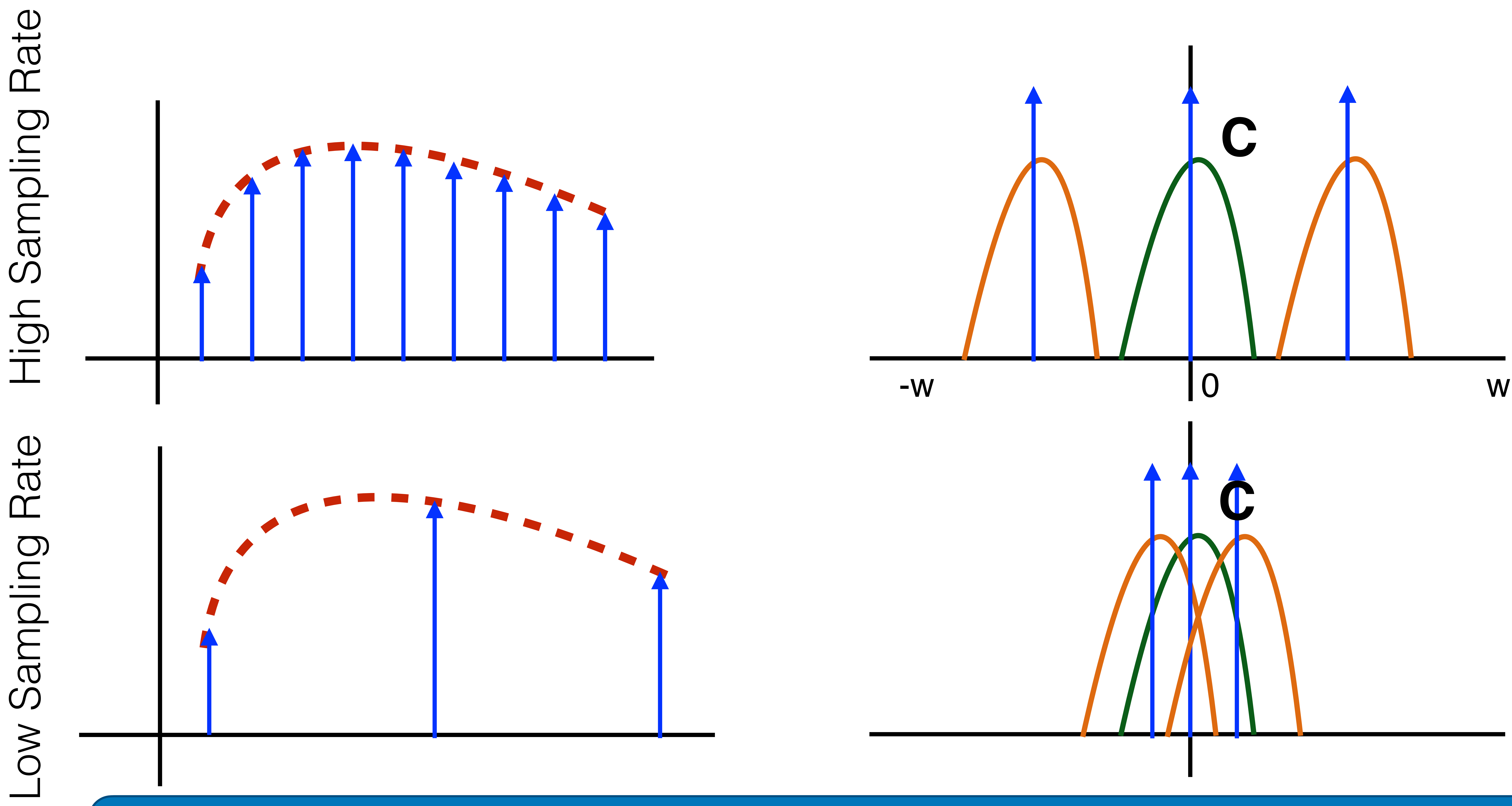

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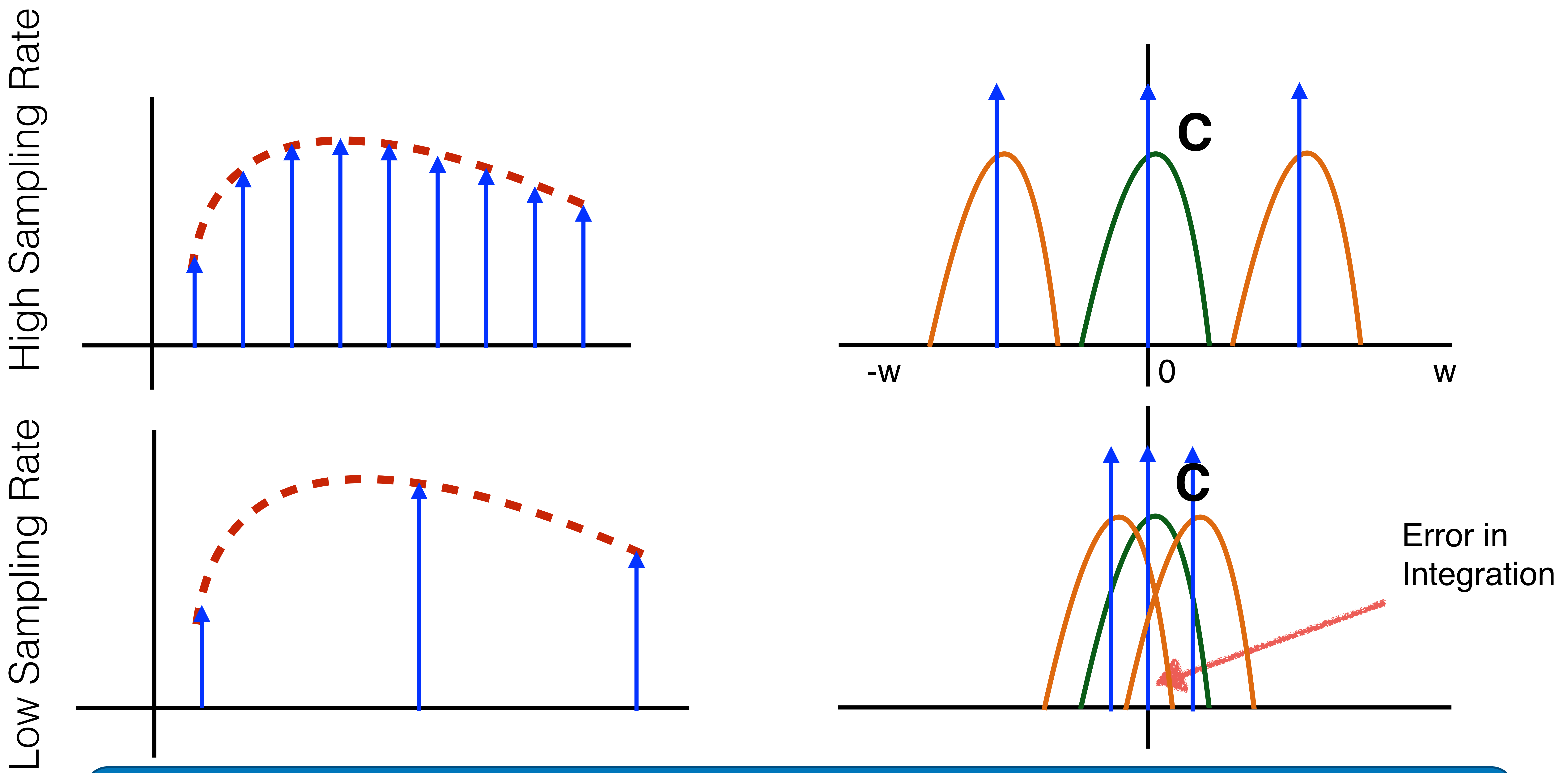
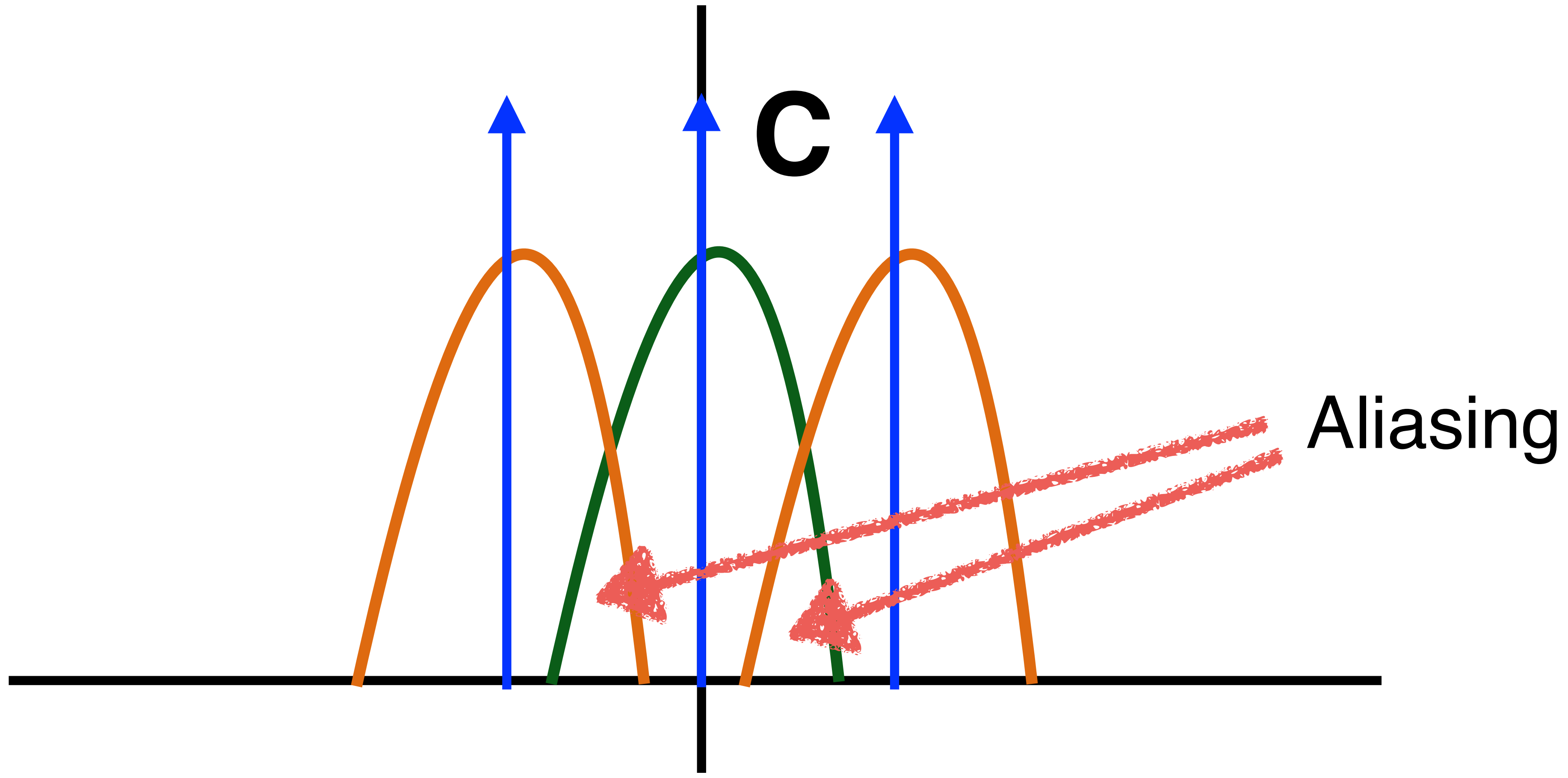
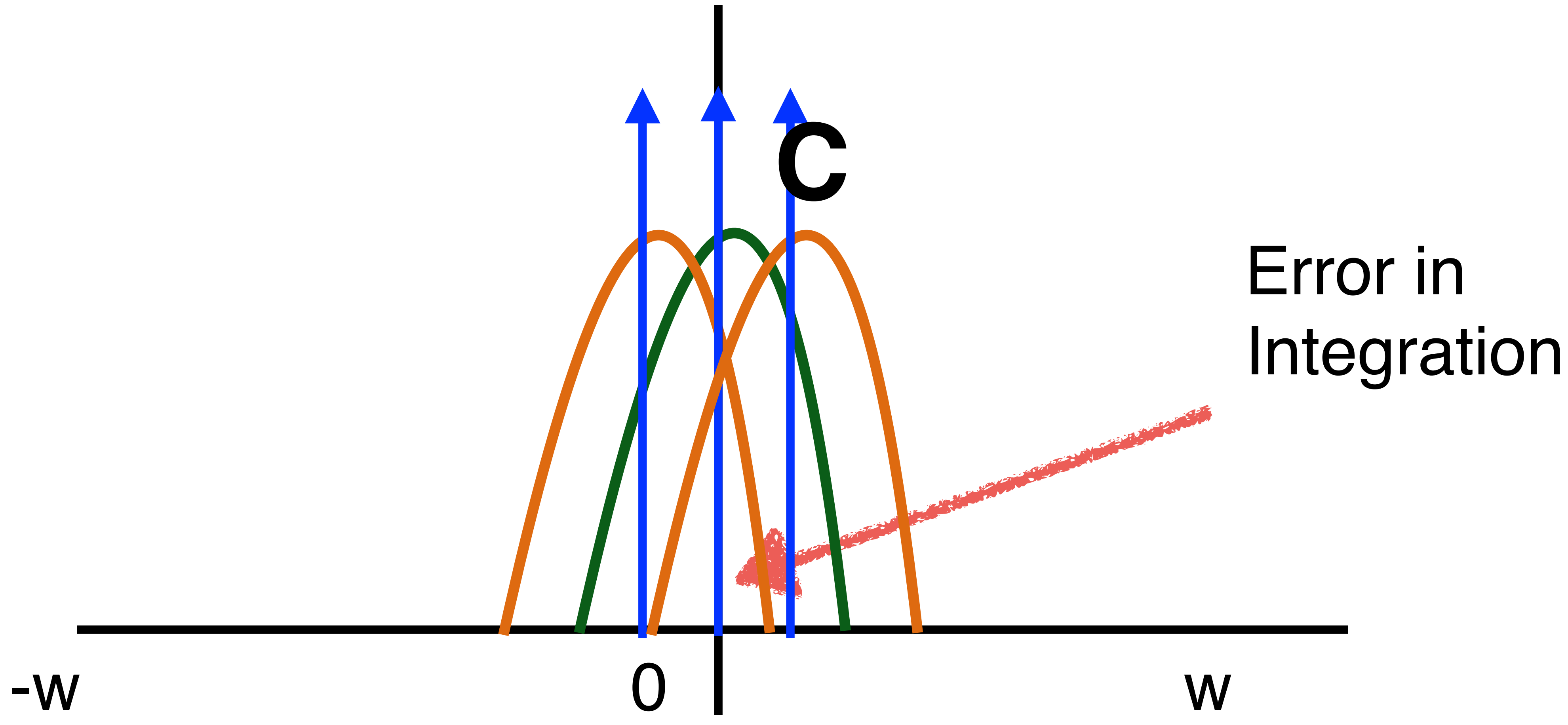


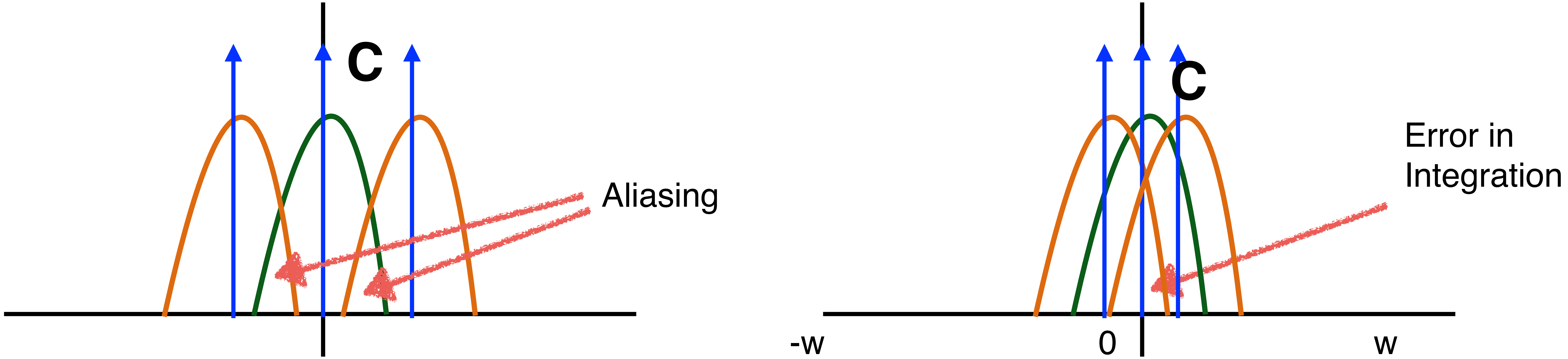
Aliasing (Reconstruction) vs. Error (Integration)





Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]



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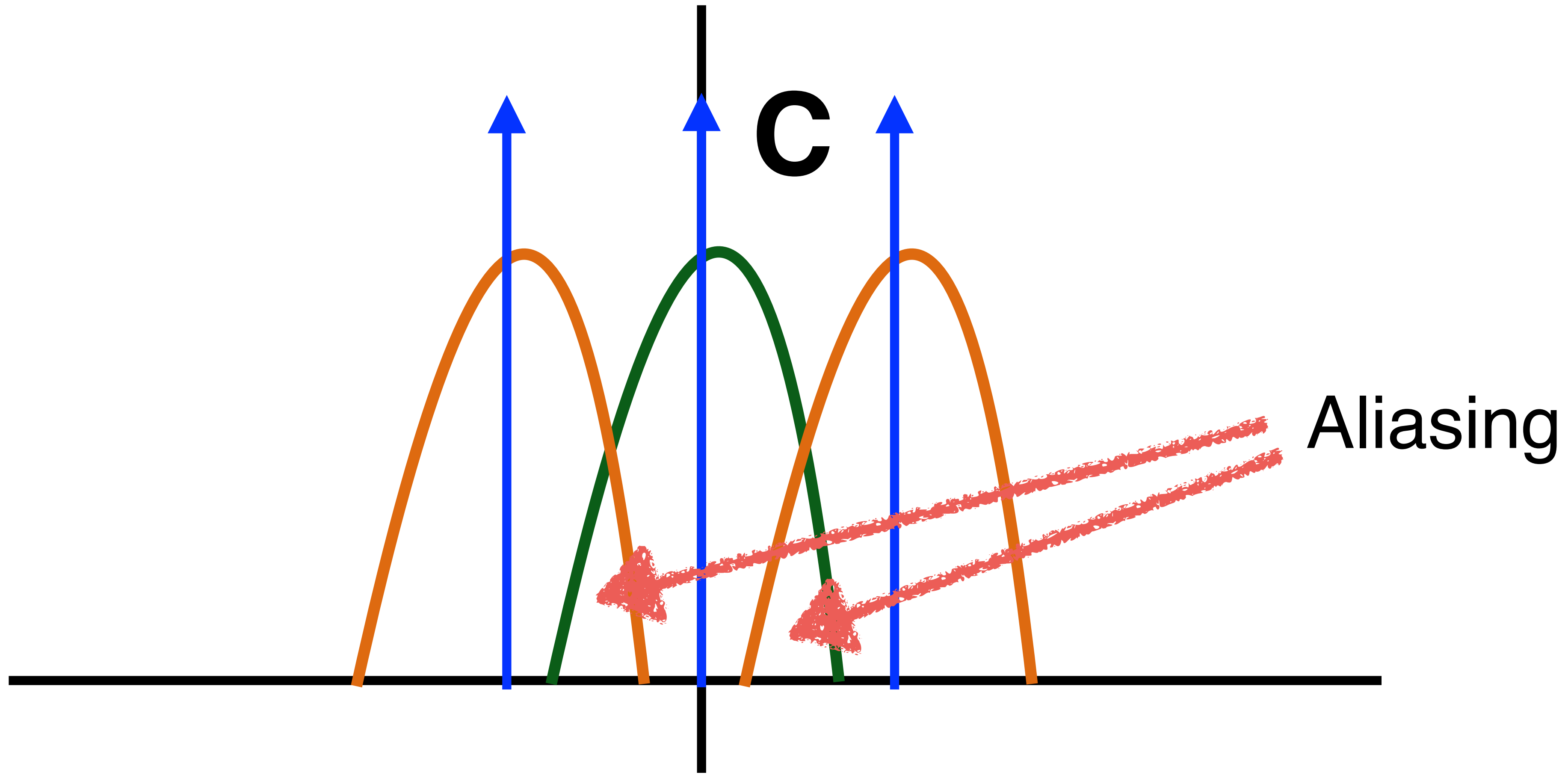

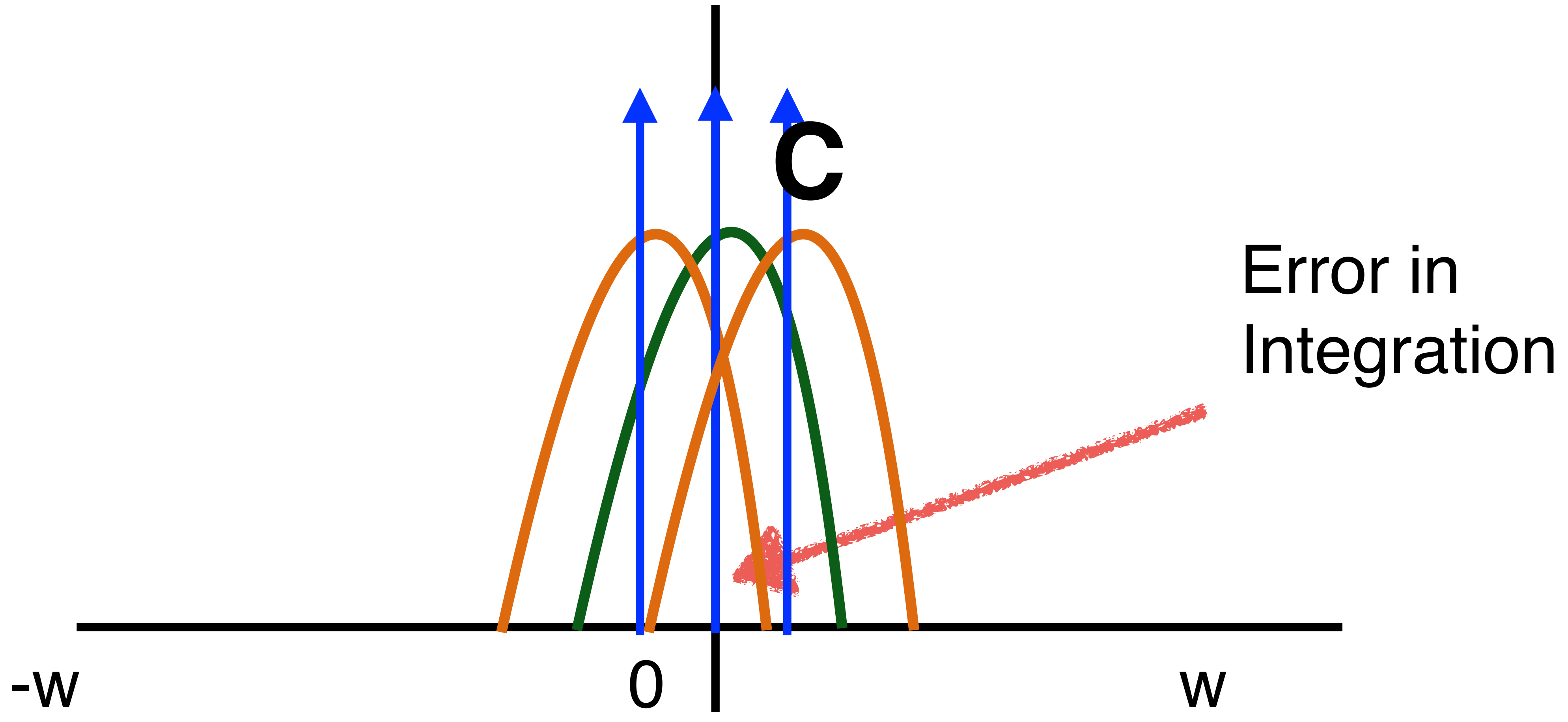

Integration in the Fourier Domain

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x) dx$$

Integration is the DC term in the Fourier Domain

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Fourier Domain:

Integration is the DC term in the Fourier Domain

Spatial Domain:

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Fourier Domain:

$$\hat{f}(0)$$

Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

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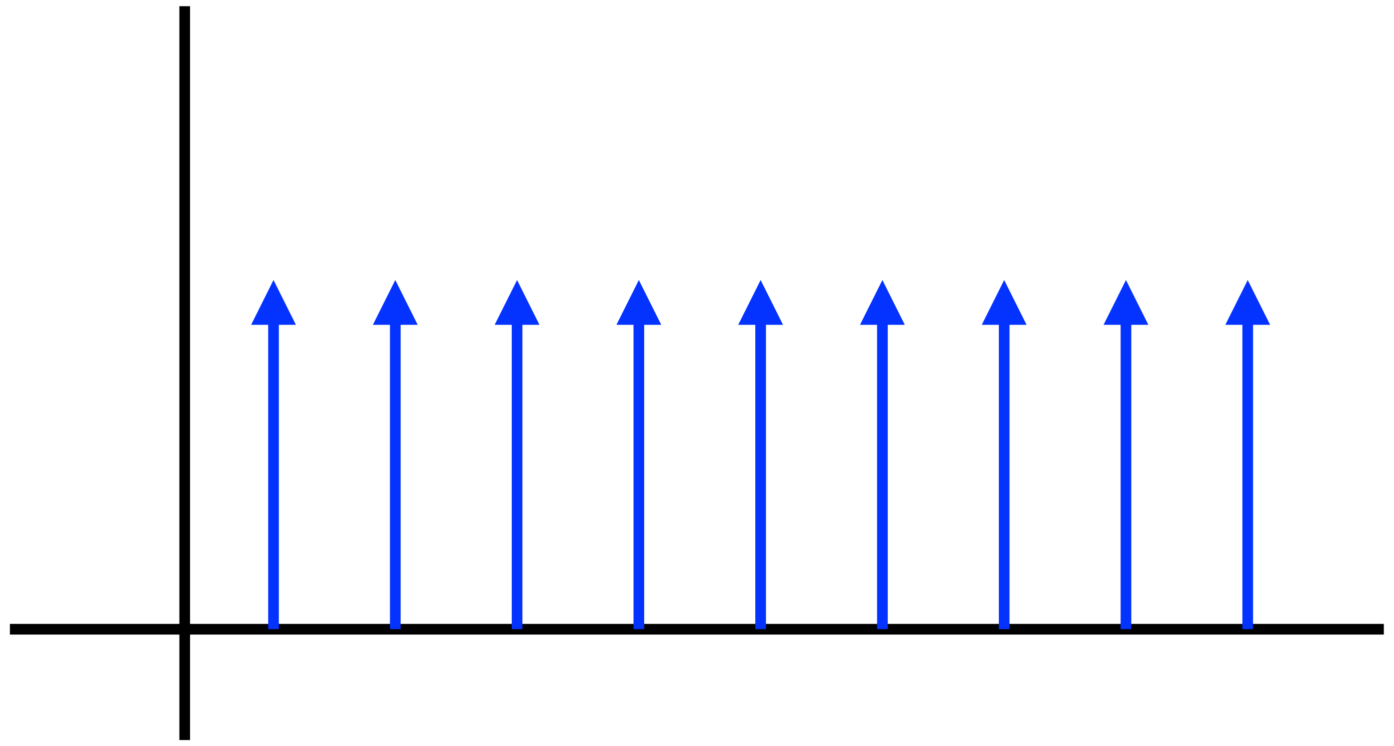
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



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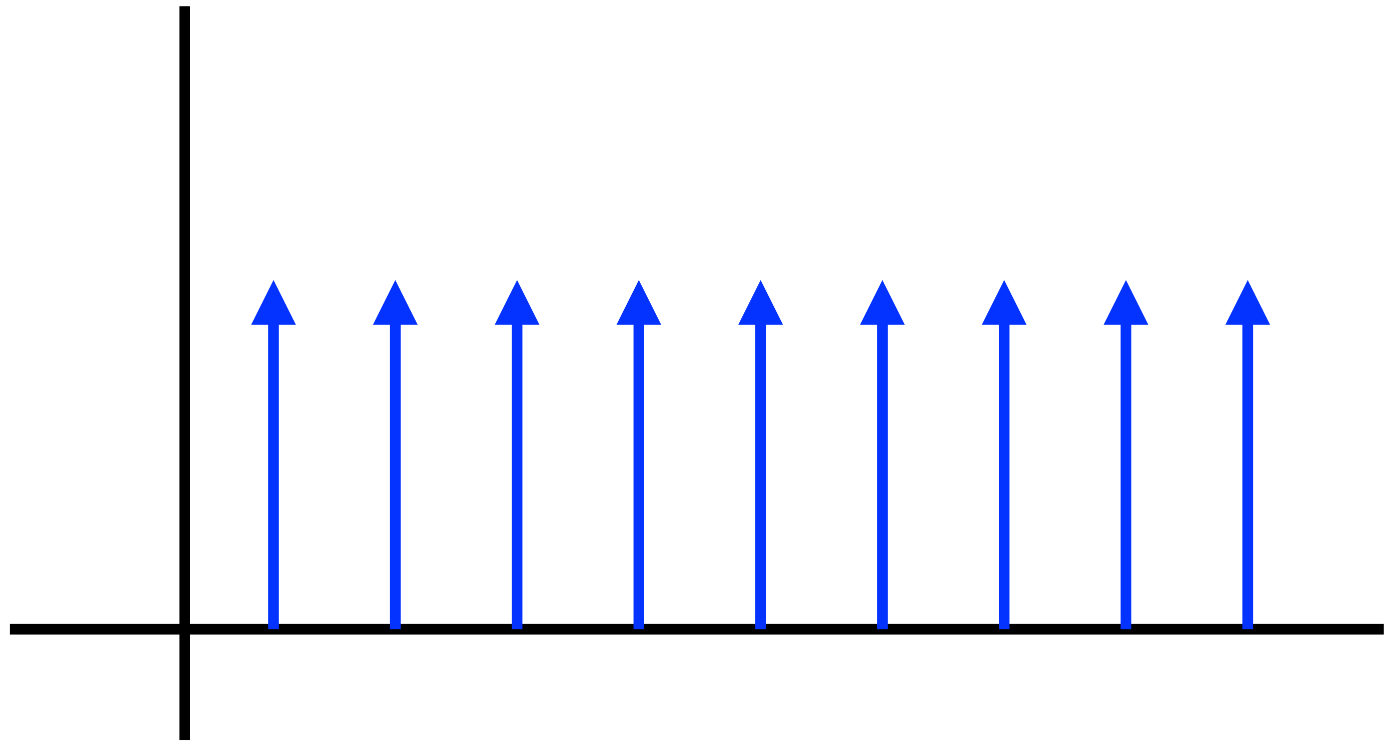

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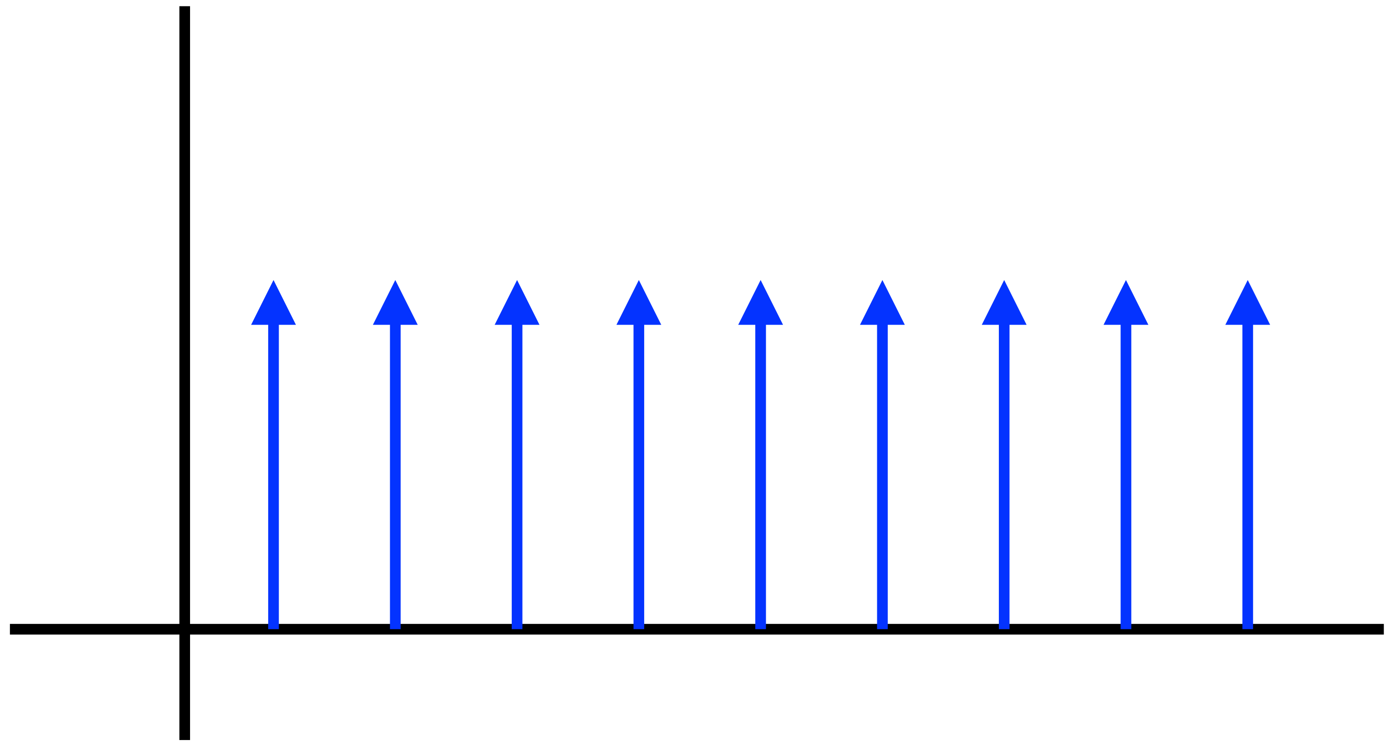

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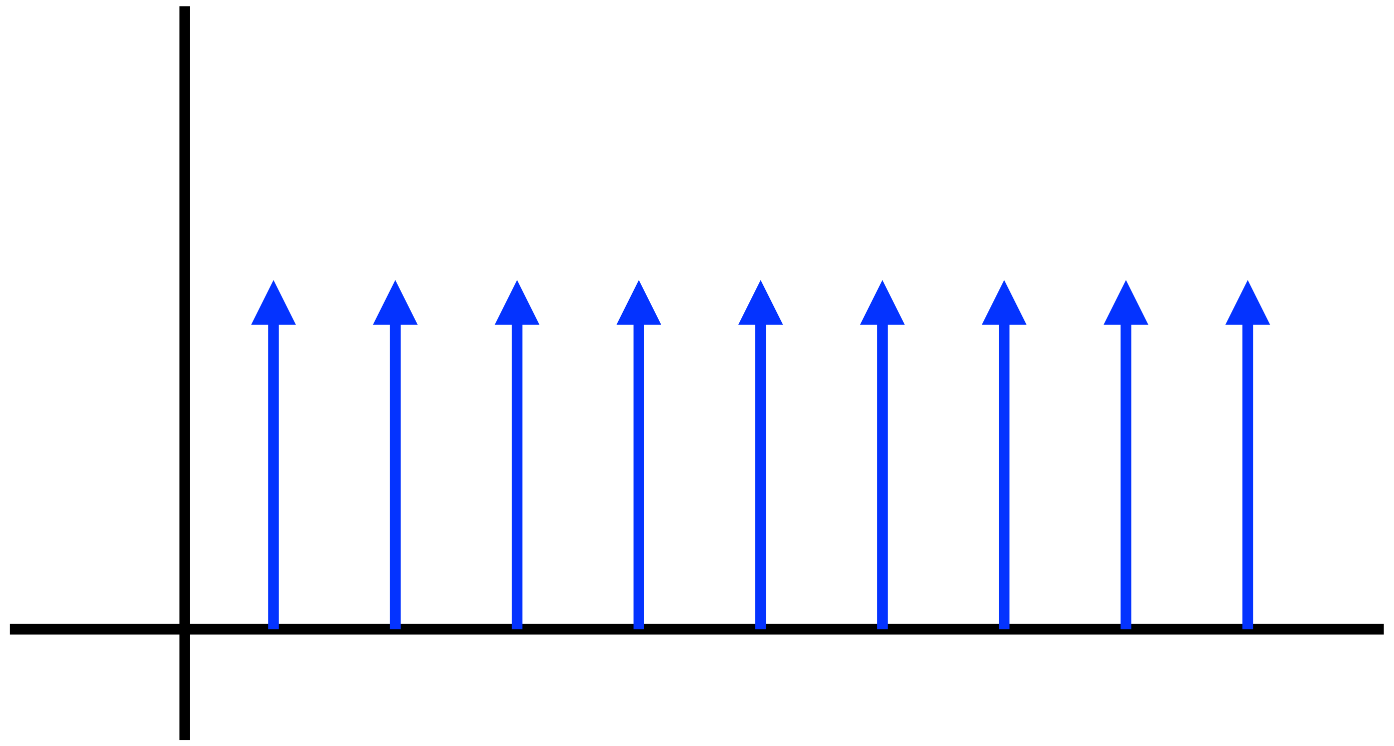

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$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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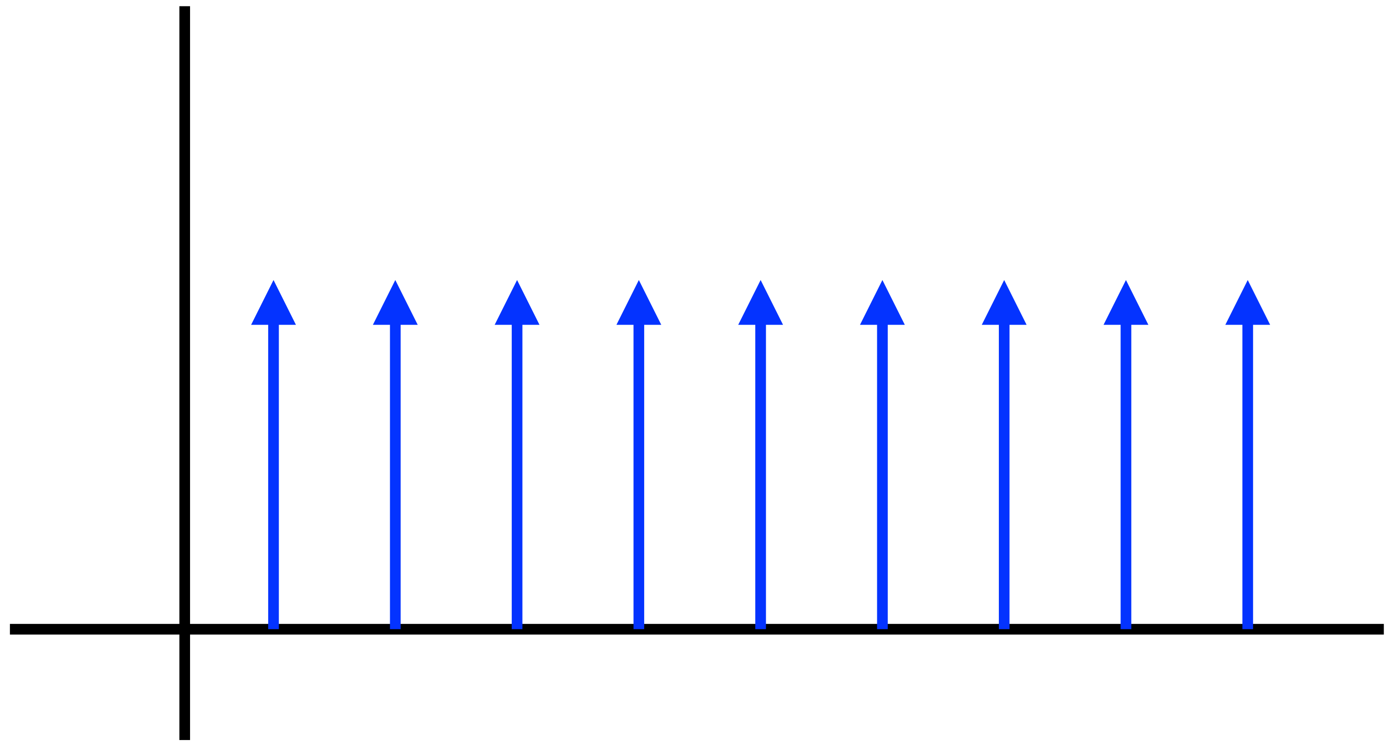

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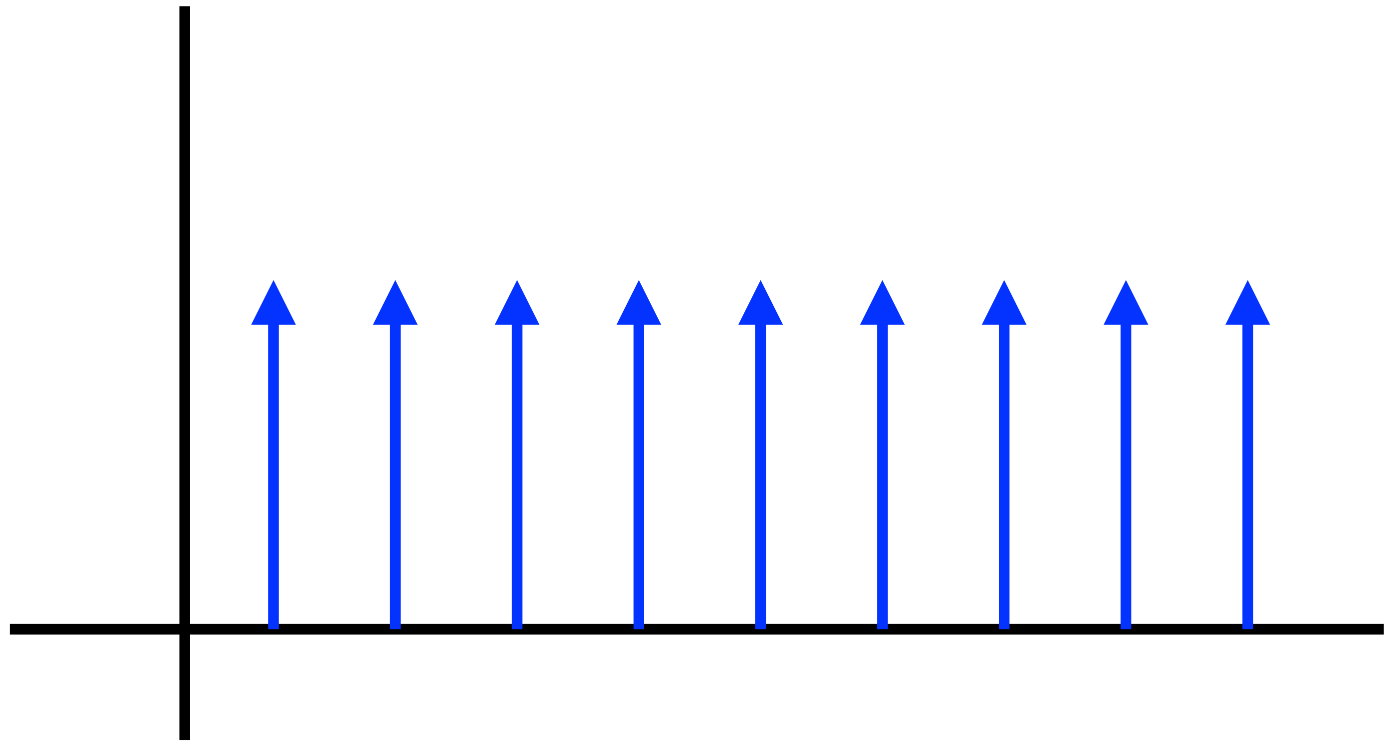

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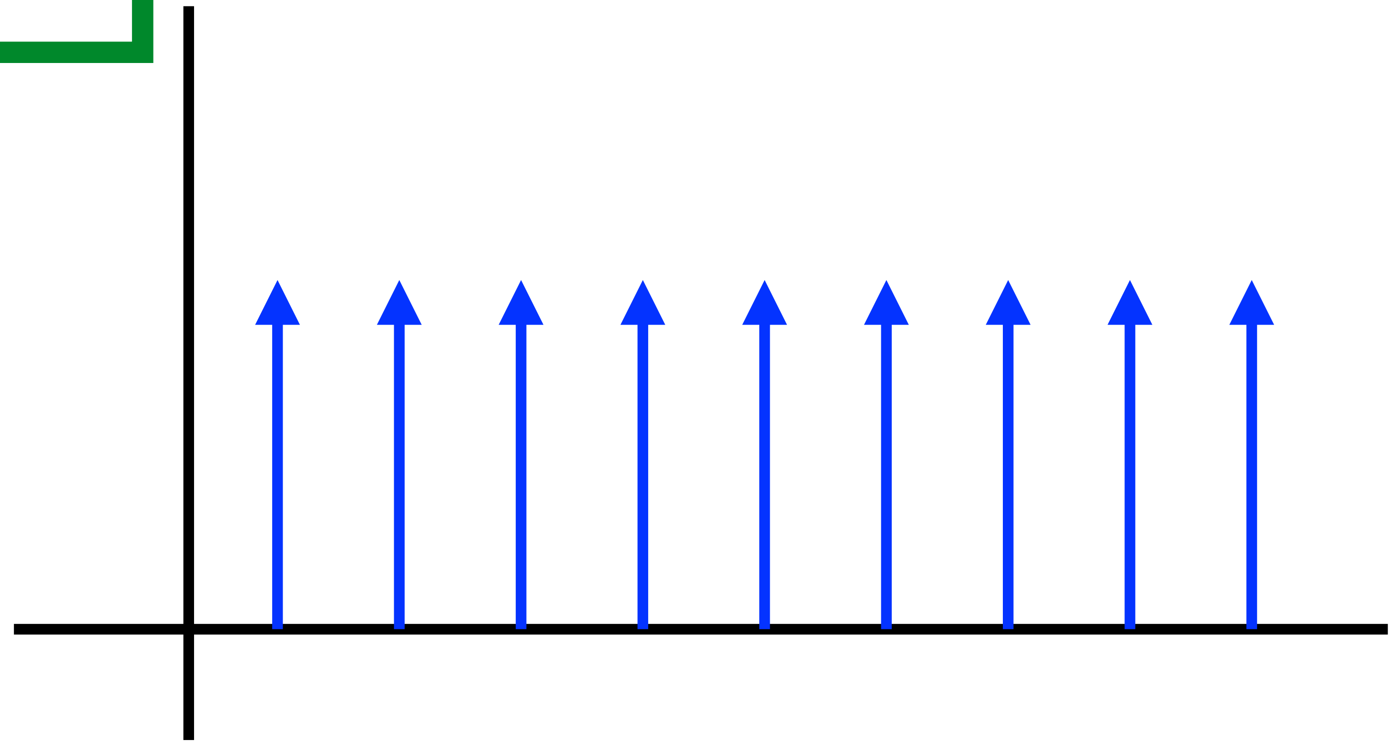

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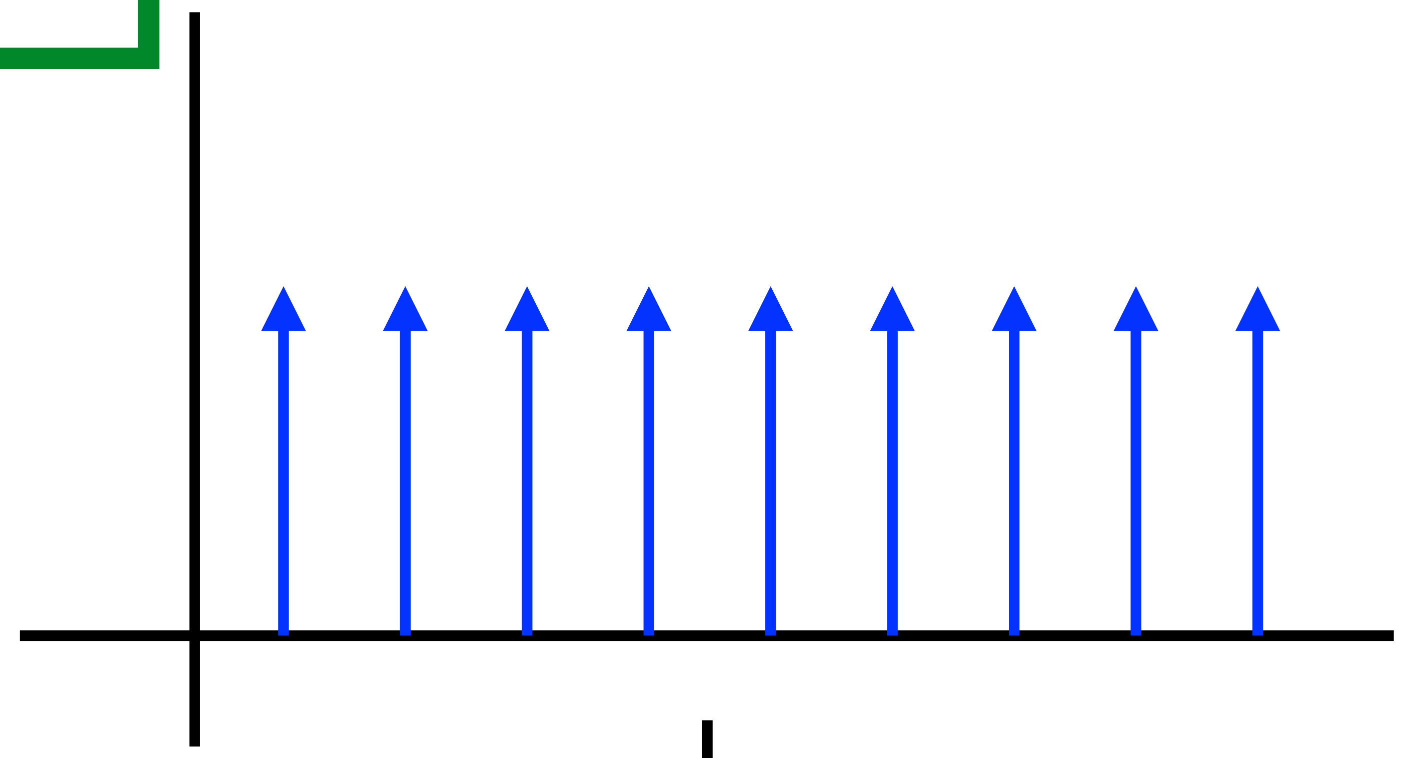
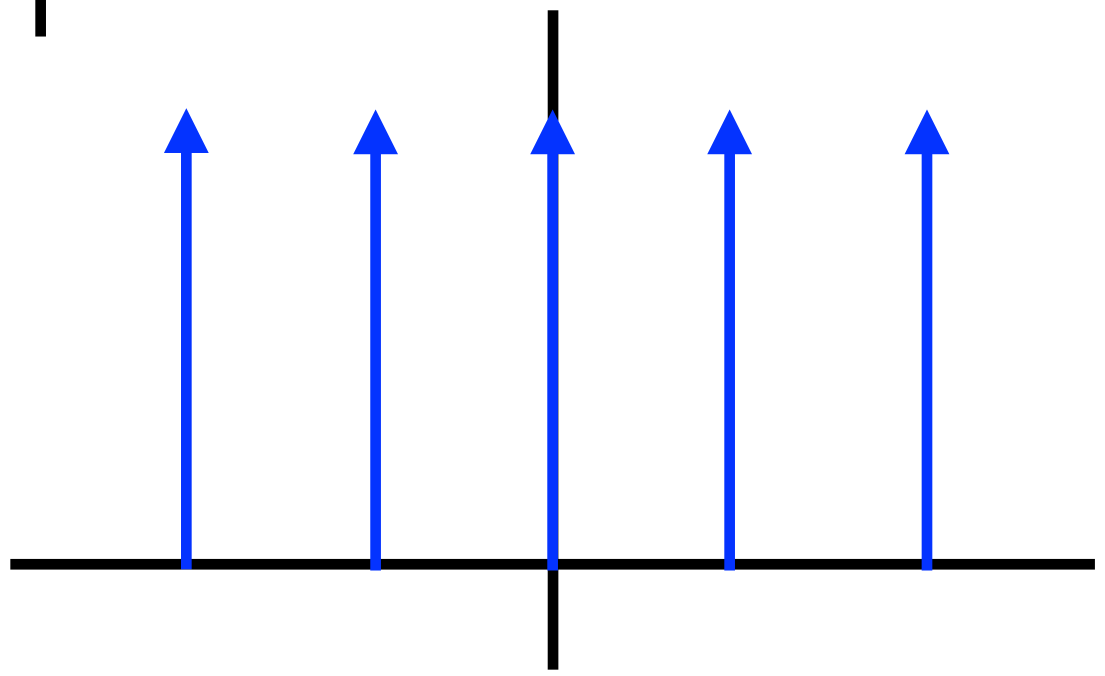


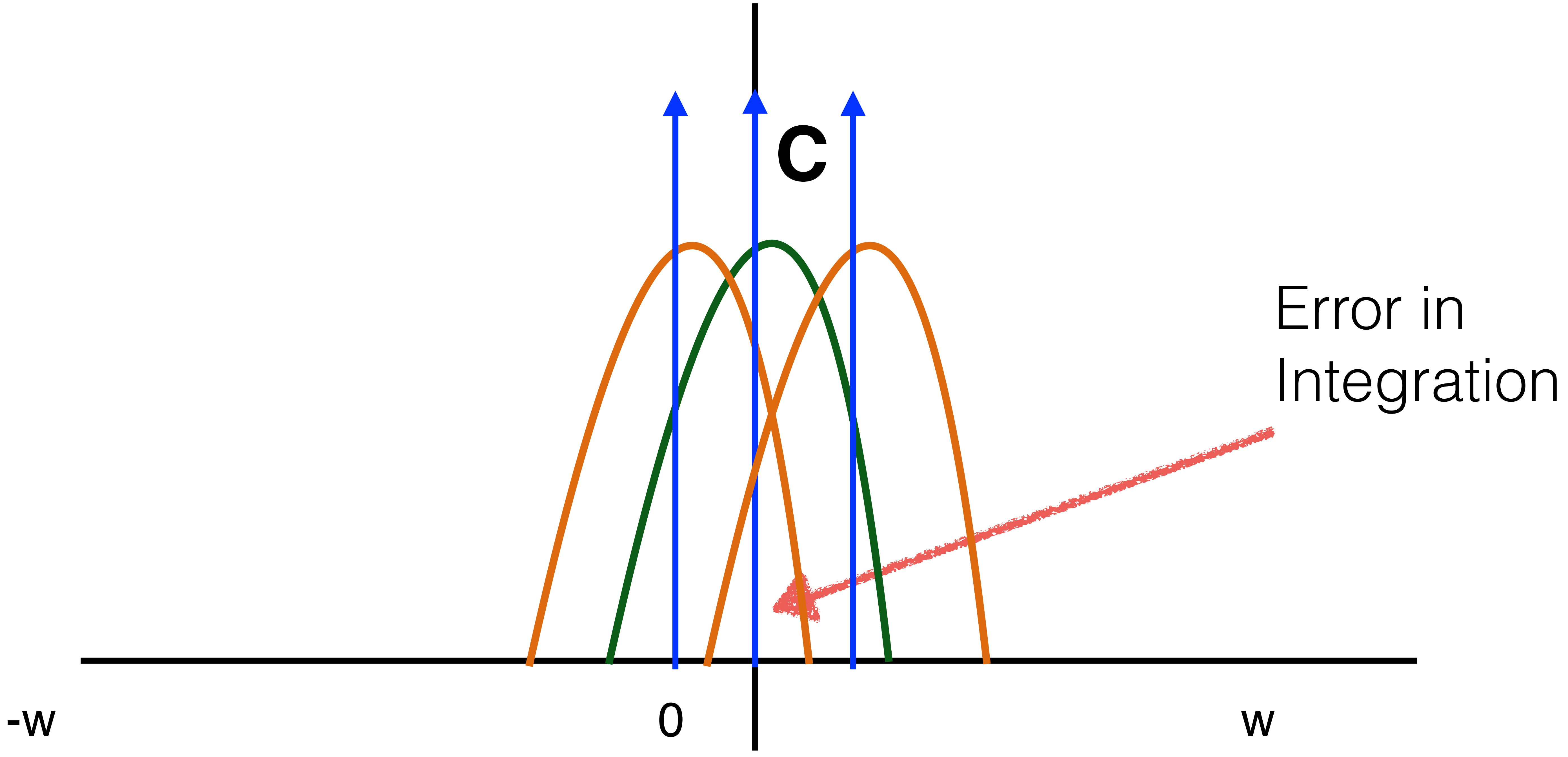
$$\hat{\mathbf{S}}(\omega) = \frac{1}{N}\sum_{k=1}^{N} e^{-i2\pi\omega x_k}$$



How to Formulate Error in Fourier Domain ?

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$



Fredo Durand [2011]

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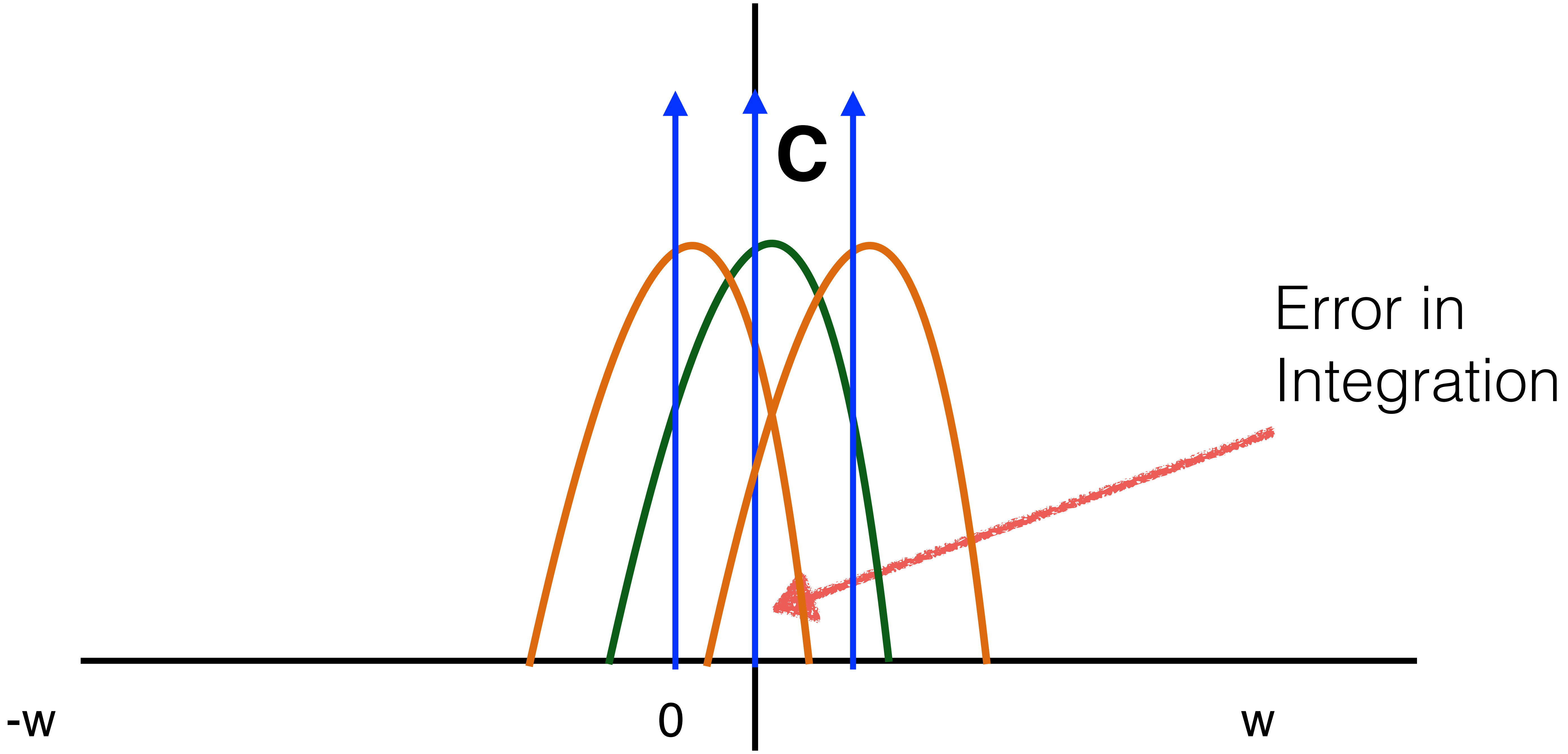

Fredo Durand [2011]

Error in Spatial Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

Error in Spatial Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

True Integral

$$I - \tilde{\mu}_N = \boxed{\int_D f(x)dx} - \int_D f(x)\mathbf{S}(x)dx$$

Error in Spatial Domain

$$I = \hat{f}(0)$$

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True Integral

$$I - \tilde{\mu}_N = \int_D f(x)dx - \int_D f(x)\mathbf{S}(x)dx$$

Monte Carlo Estimator

Error in Spatial Domain

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Error in Fourier Domain

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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Fredo Durand [2011]

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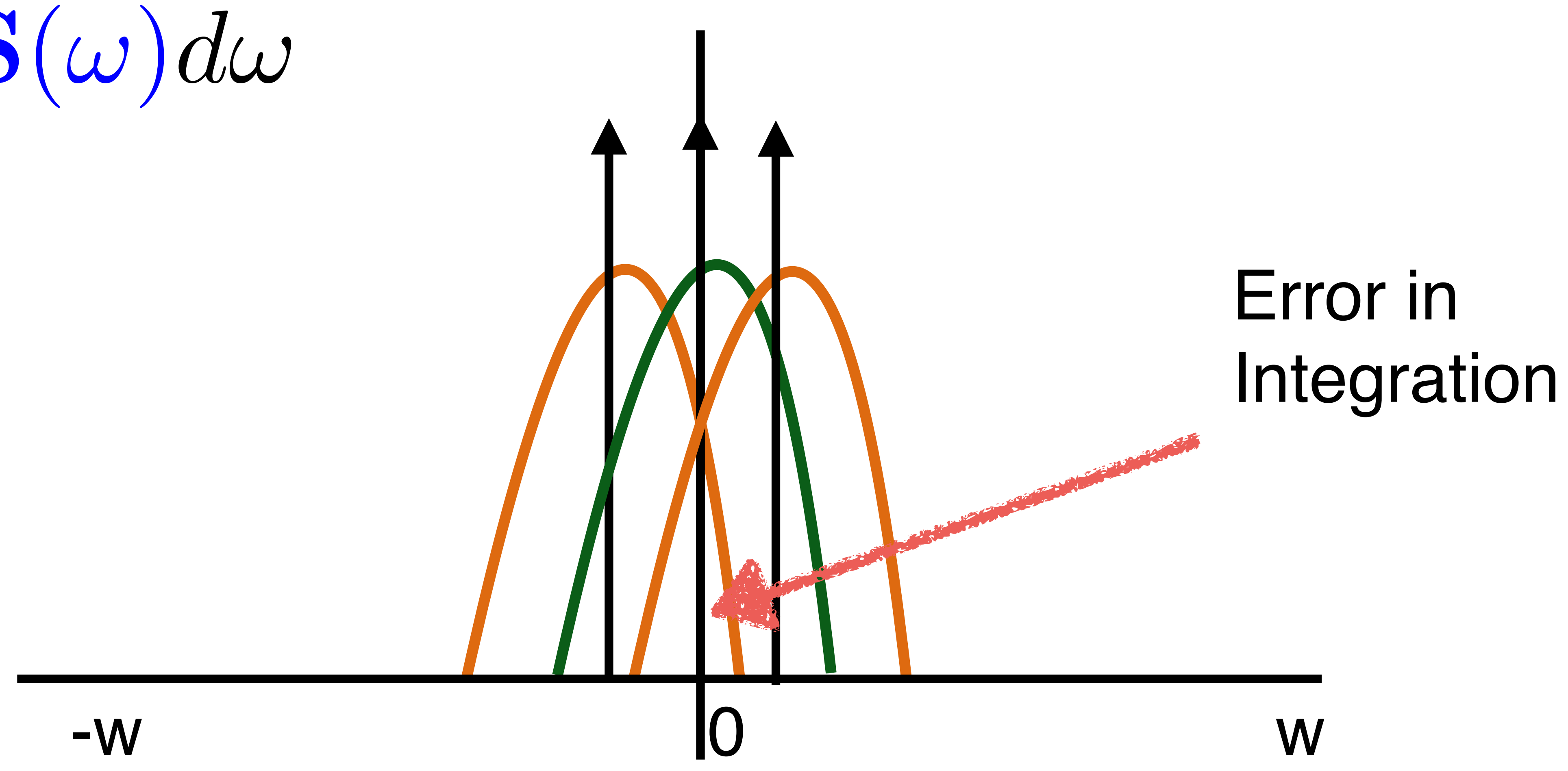

Fredo Durand [2011]

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

Properties of Error

- Bias
- Variance

Properties of Error

- Bias: Expected value of the Error
- Variance

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- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
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Properties of Error

- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
- Variance: $\mathrm{Var}(I - \mu_N)$

Subr and Kautz [2013]

Bias in the Monte Carlo Estimator

Bias in Fourier Domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

Bias in Fourier Domain

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Bias: $$\langle I - \tilde{\mu}_N \rangle$$

Bias in Fourier Domain

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Subr and Kautz [2013]

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To obtain an unbiased estimator:

Subr and Kautz [2013]

Bias in Fourier Domain

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To obtain an unbiased estimator:

Subr and Kautz [2013]

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

for frequencies other than zero

How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$?

Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = |\langle \hat{\mathbf{S}}(\omega) \rangle| \, e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

Complex form in Amplitude and Phase

Amplitude

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|} e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

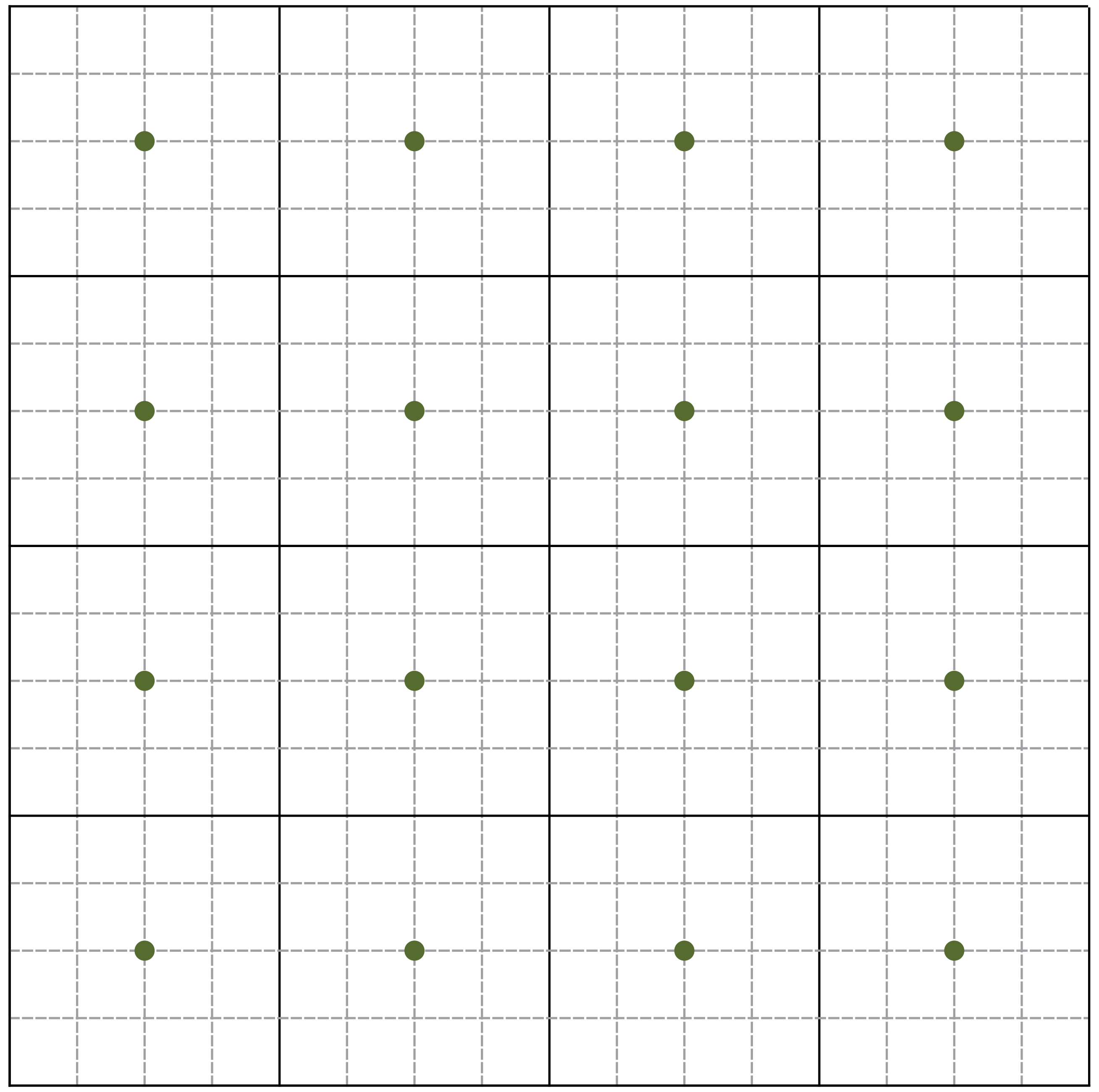
Complex form in Amplitude and Phase

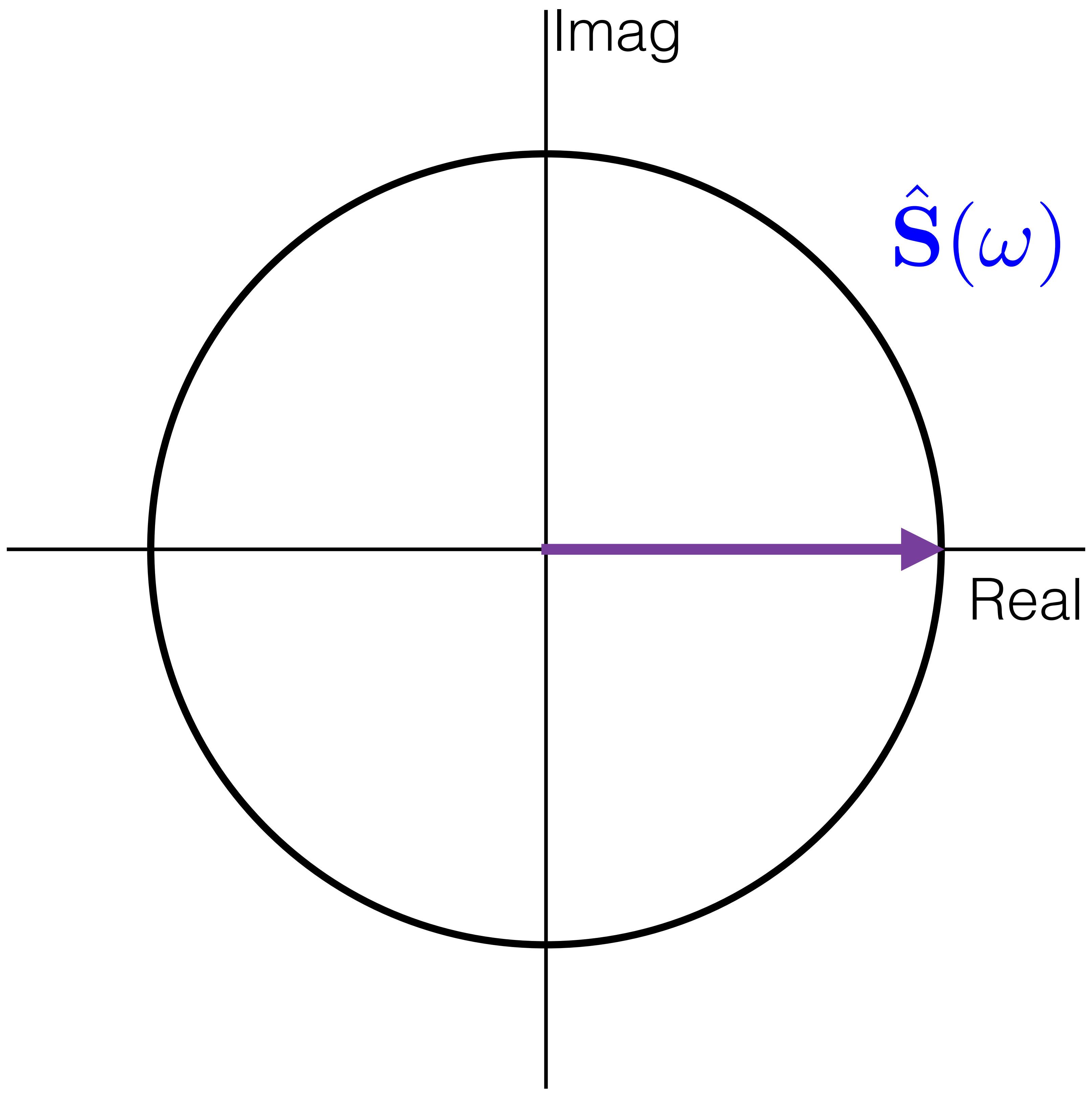
Amplitude Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|} e^{-\boxed{\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}}$$

Phase change due to Random Shift

For a given frequency ω





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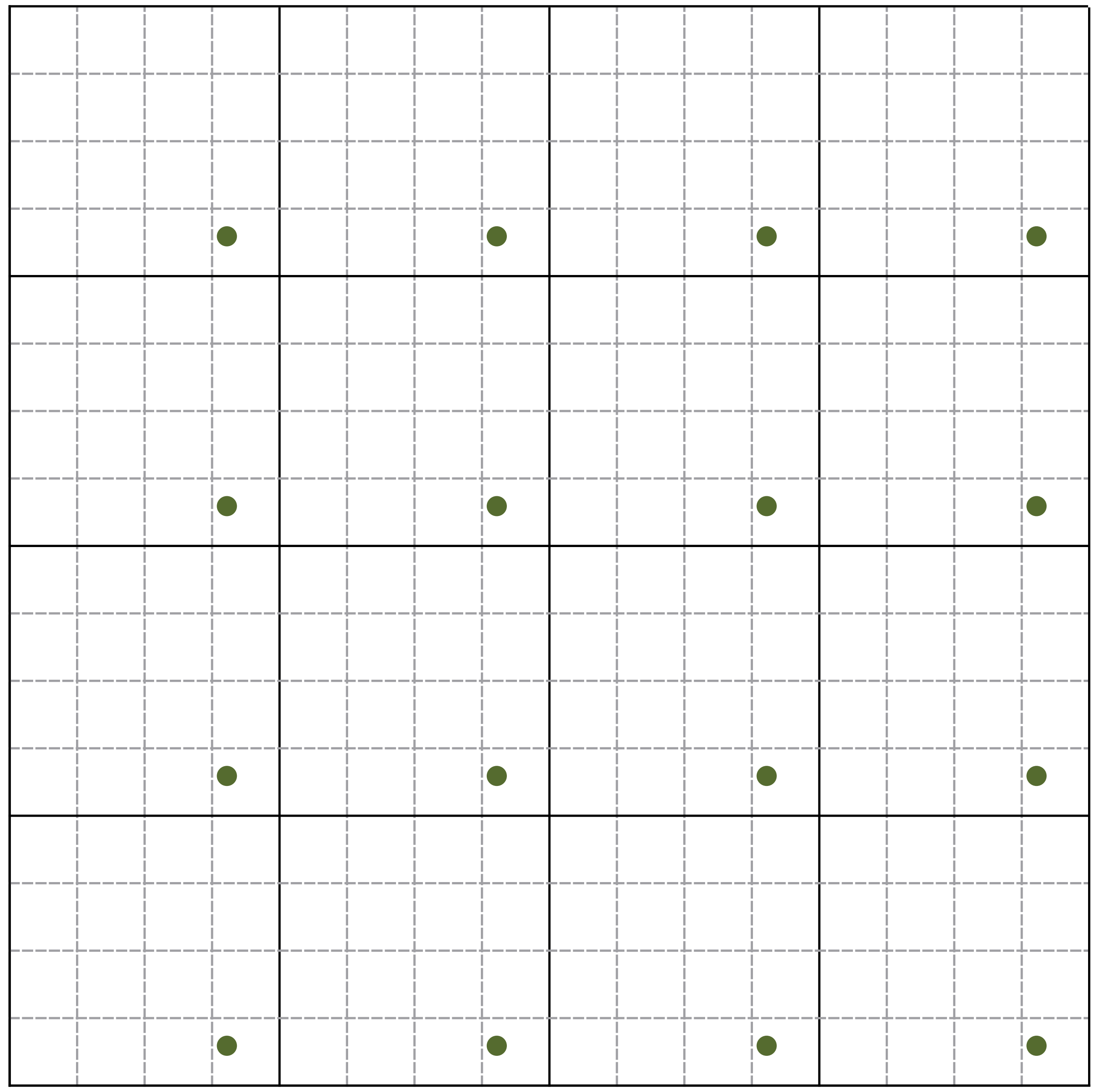

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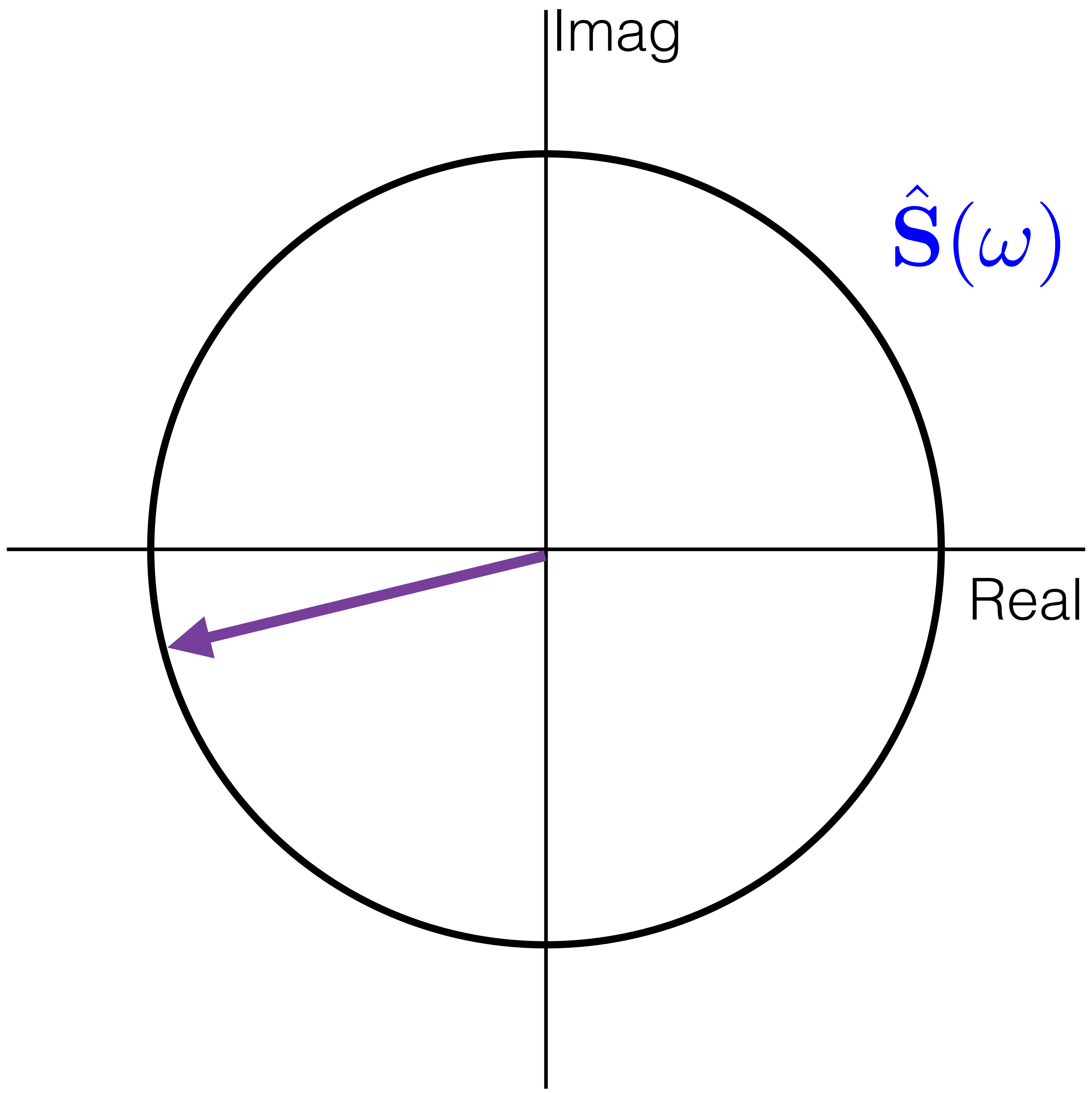

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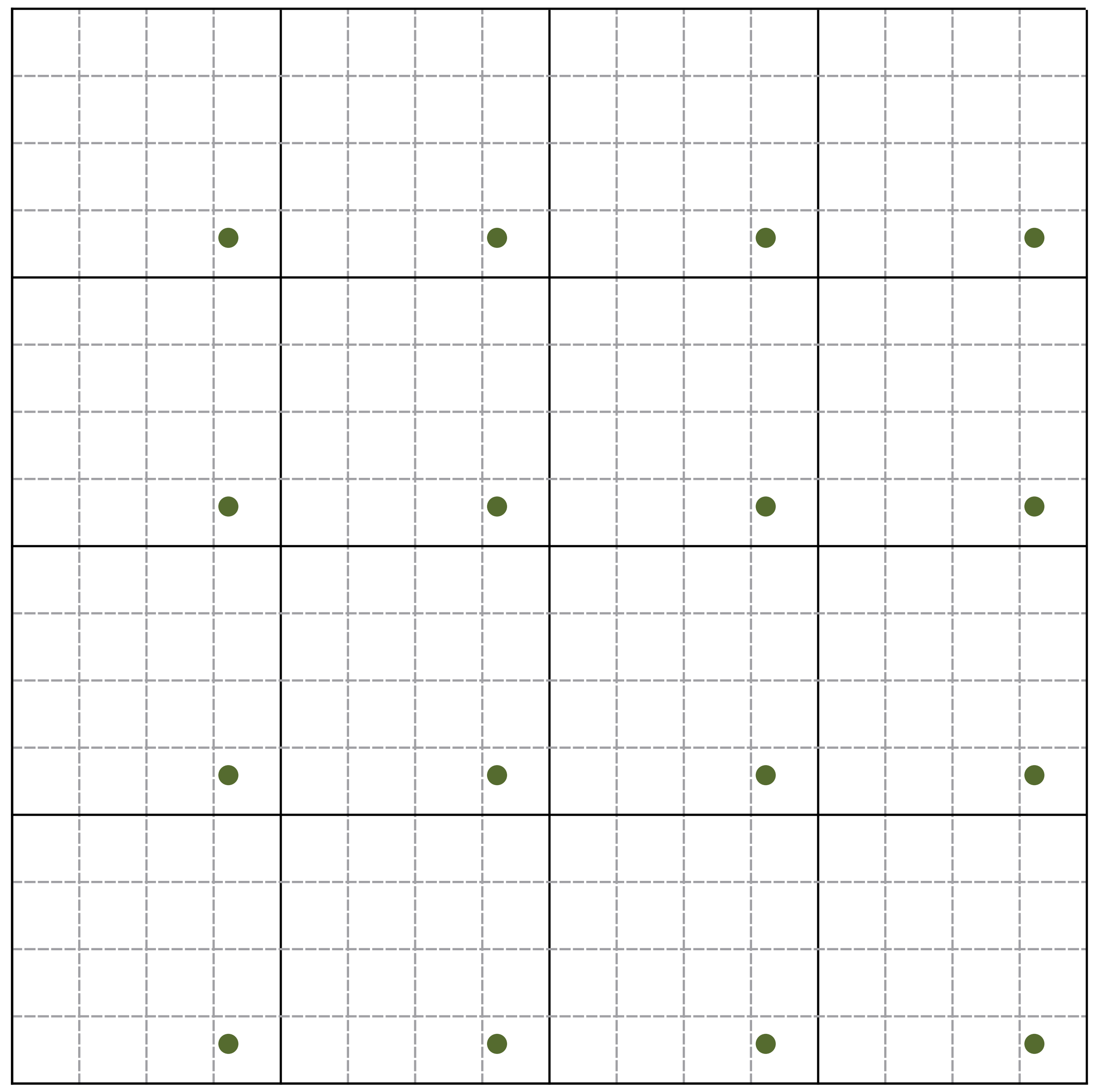

Pauly et al. [2000]
Ramamoorthi et al. [2012]

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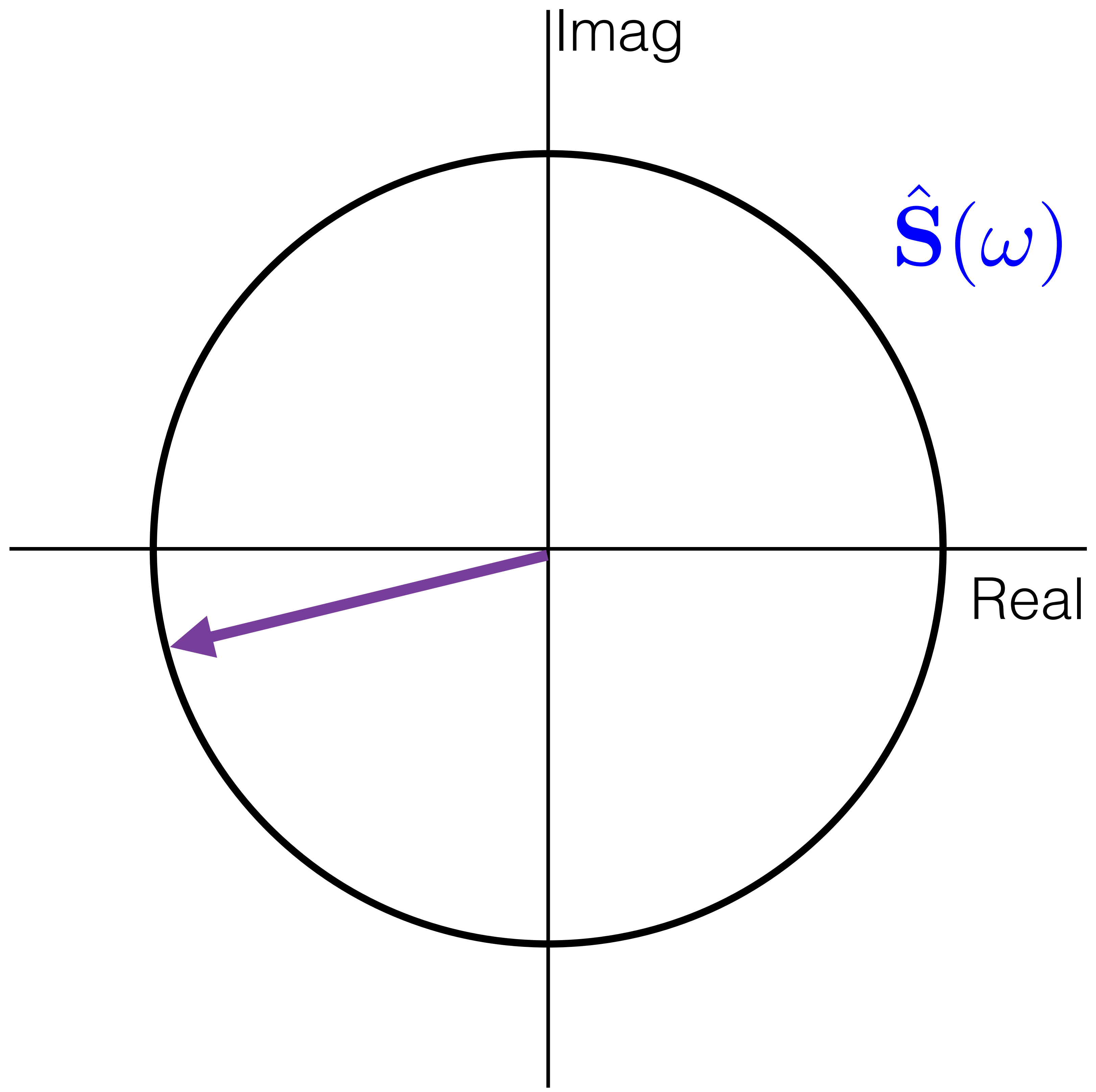

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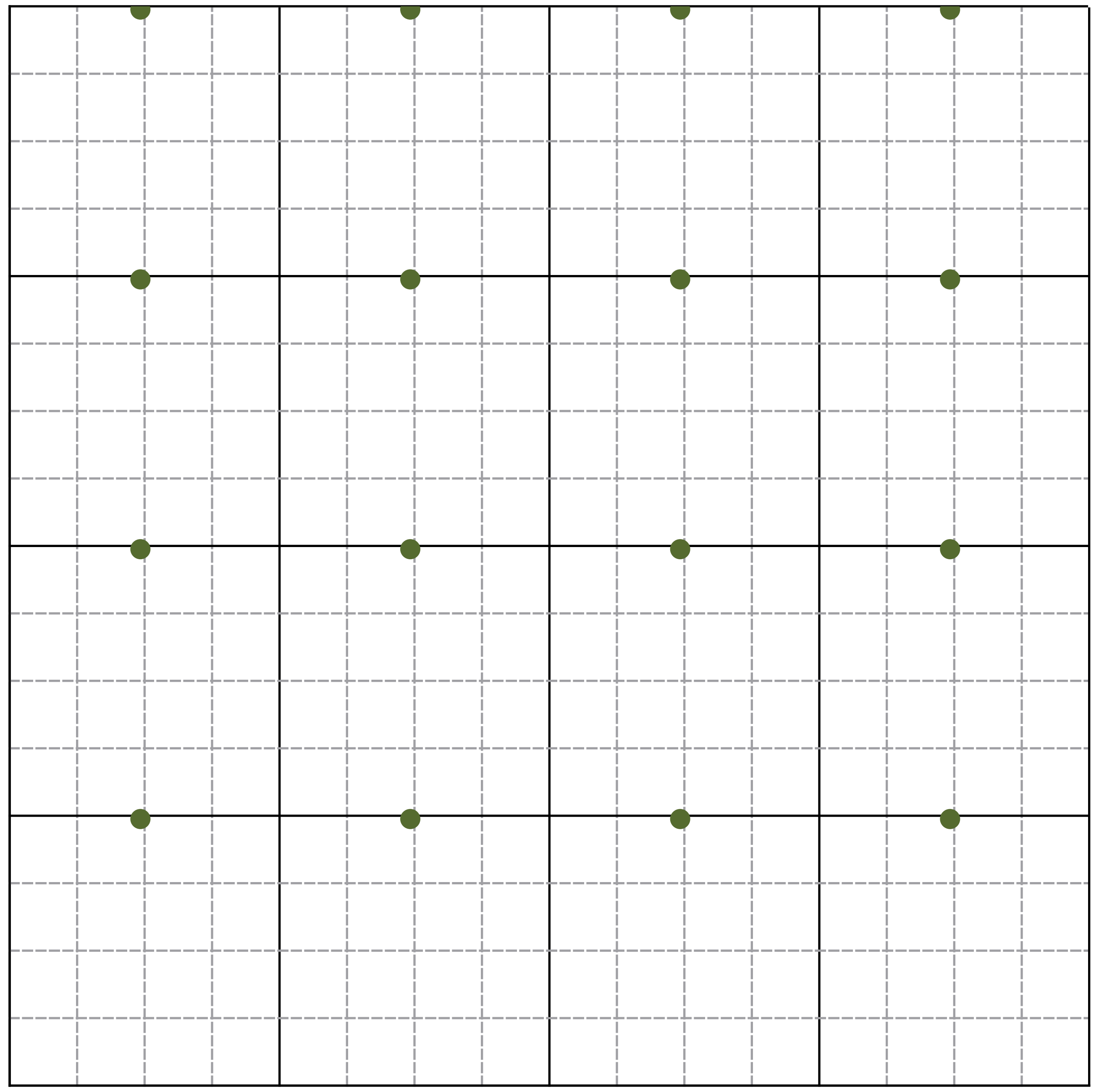

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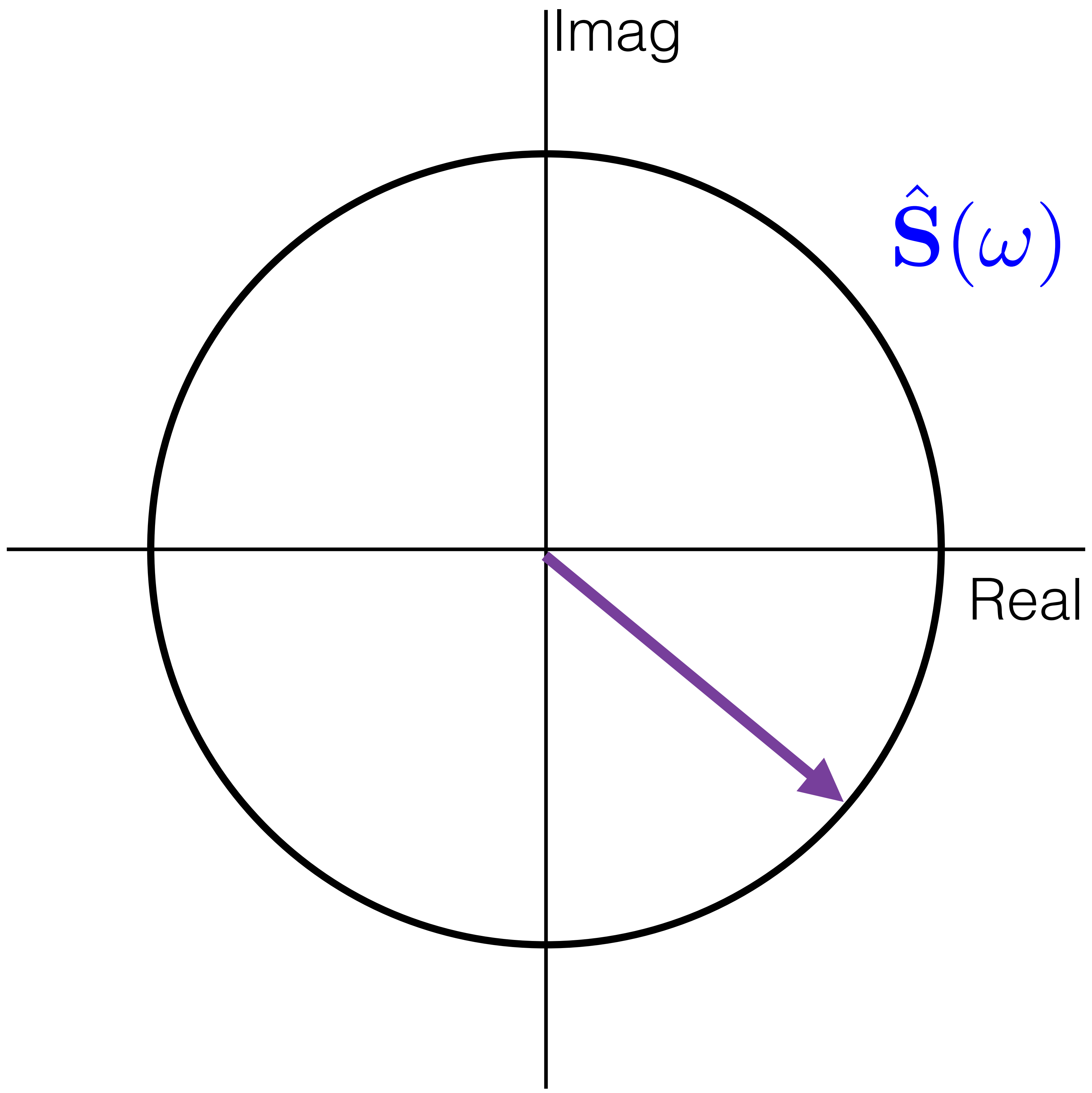
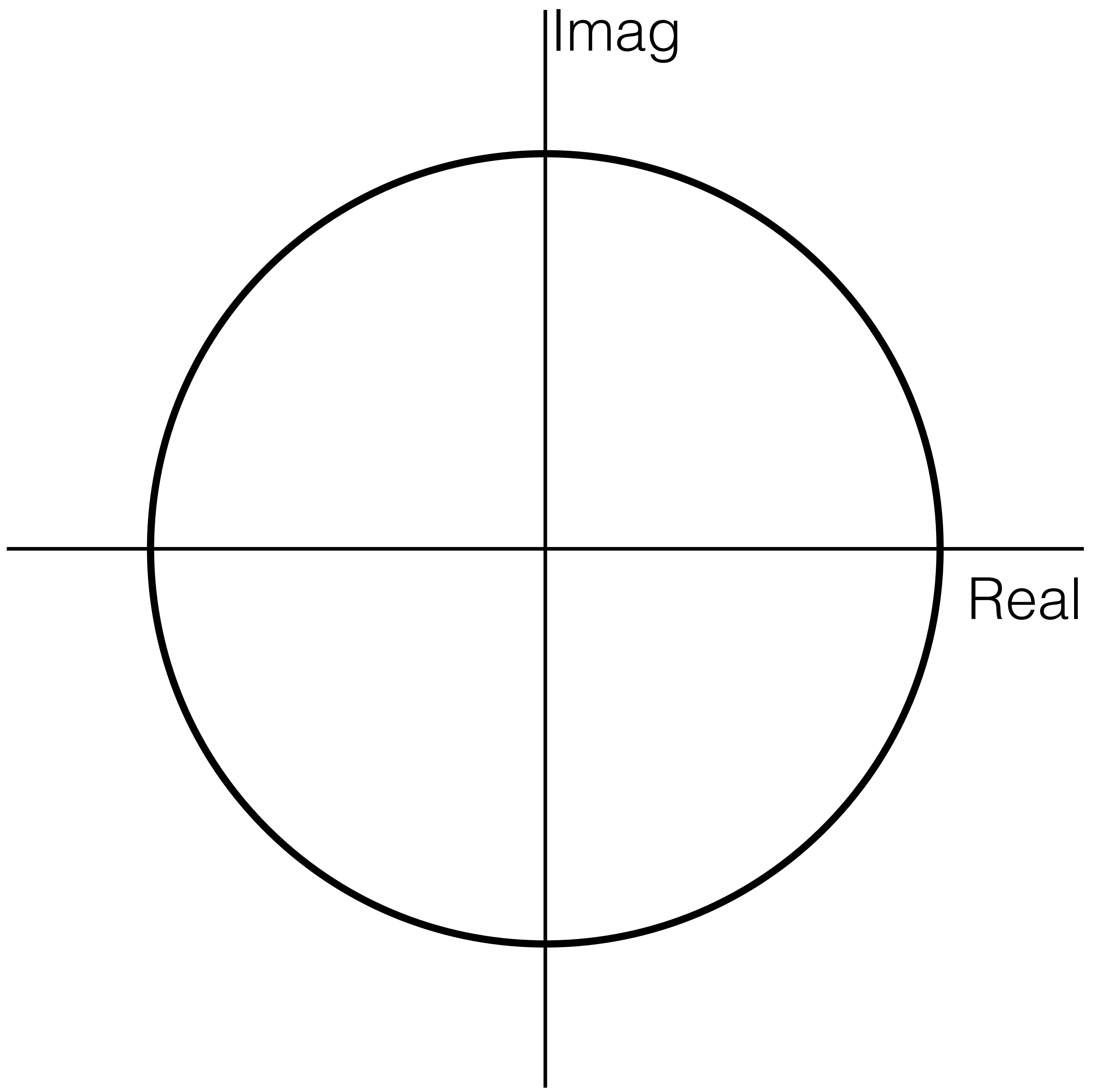


Phase change due to Random Shift

Multiple realizations

For a given frequency ω



Phase change due to Random Shift

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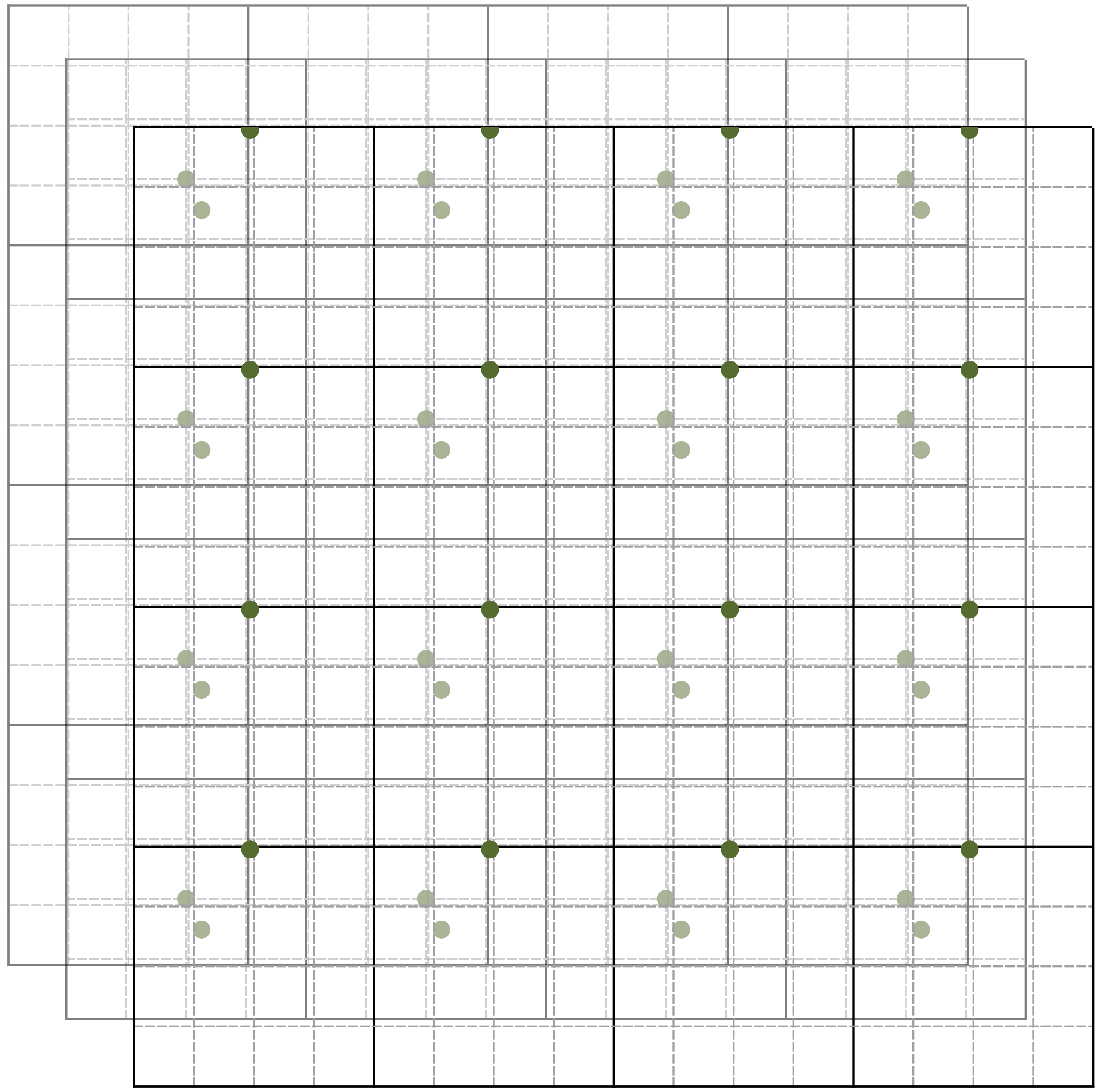

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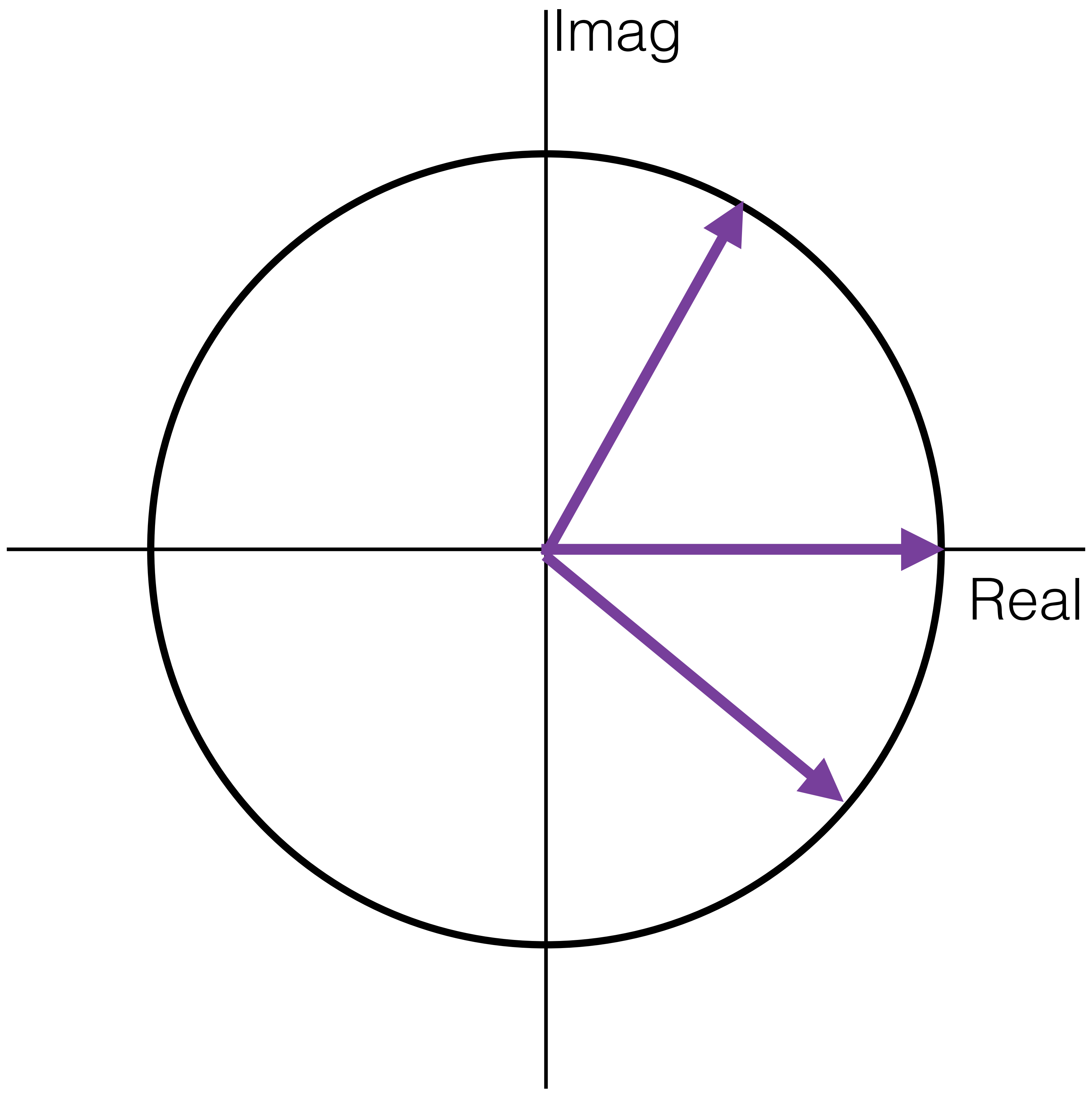

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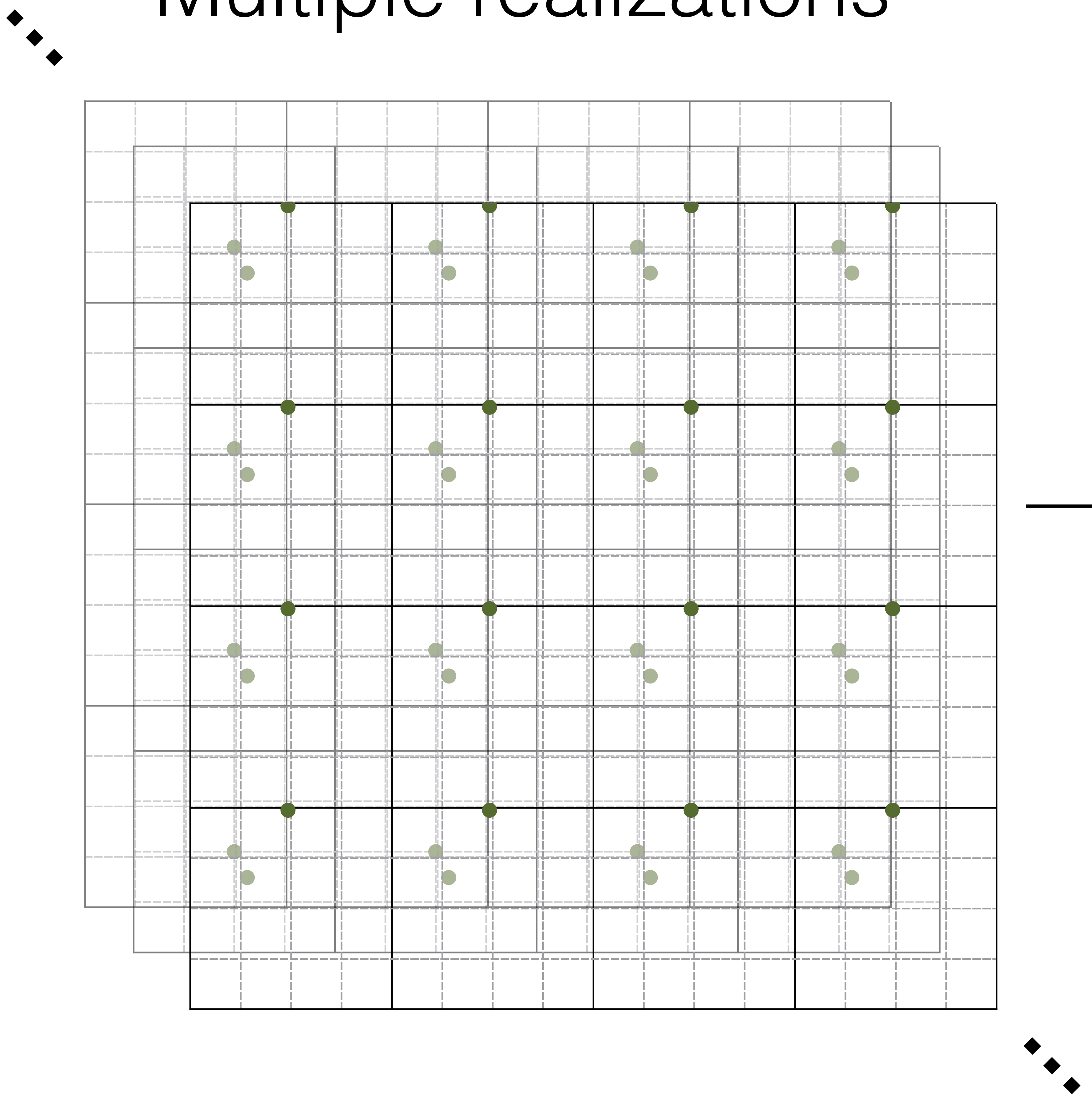

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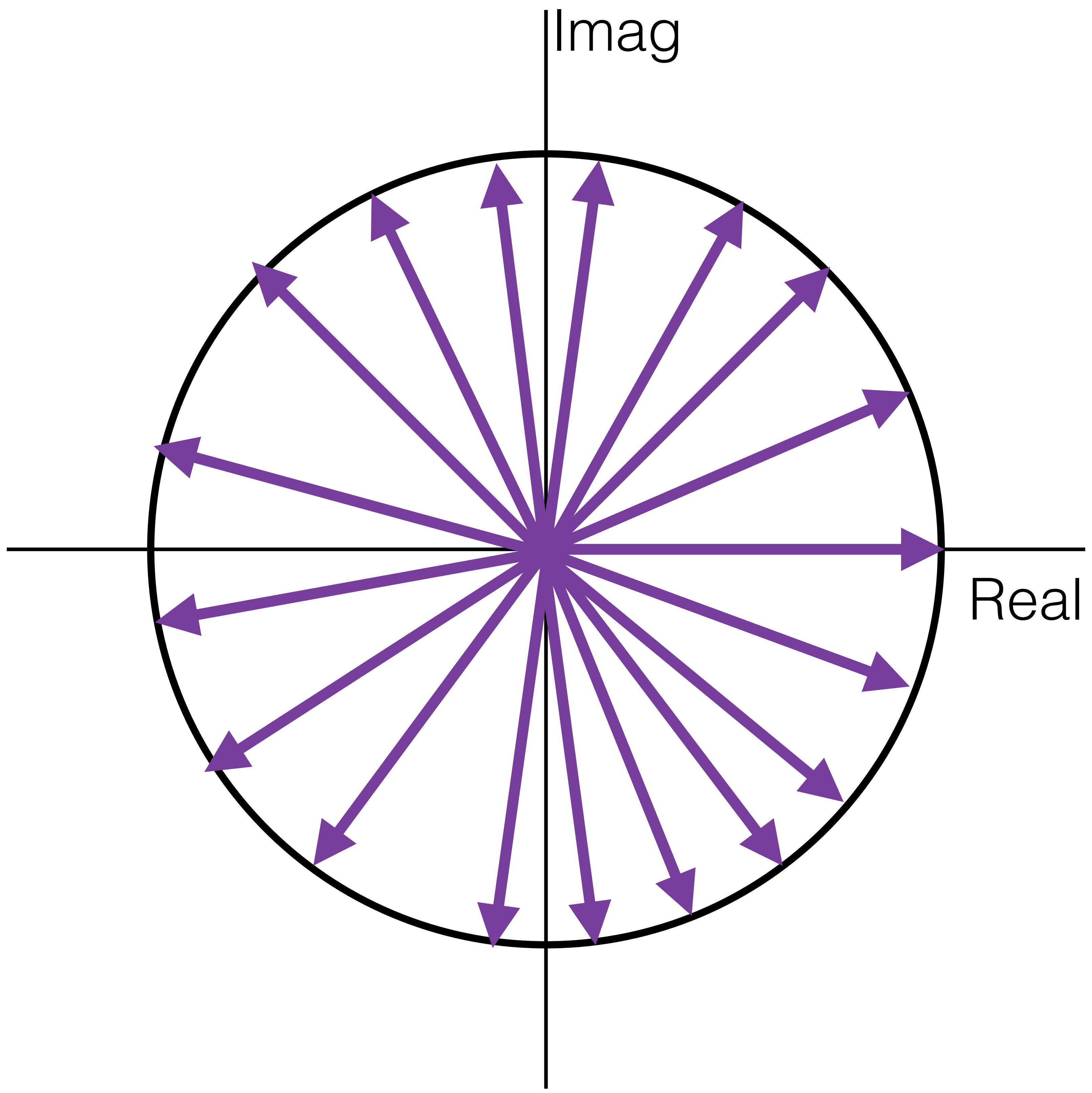

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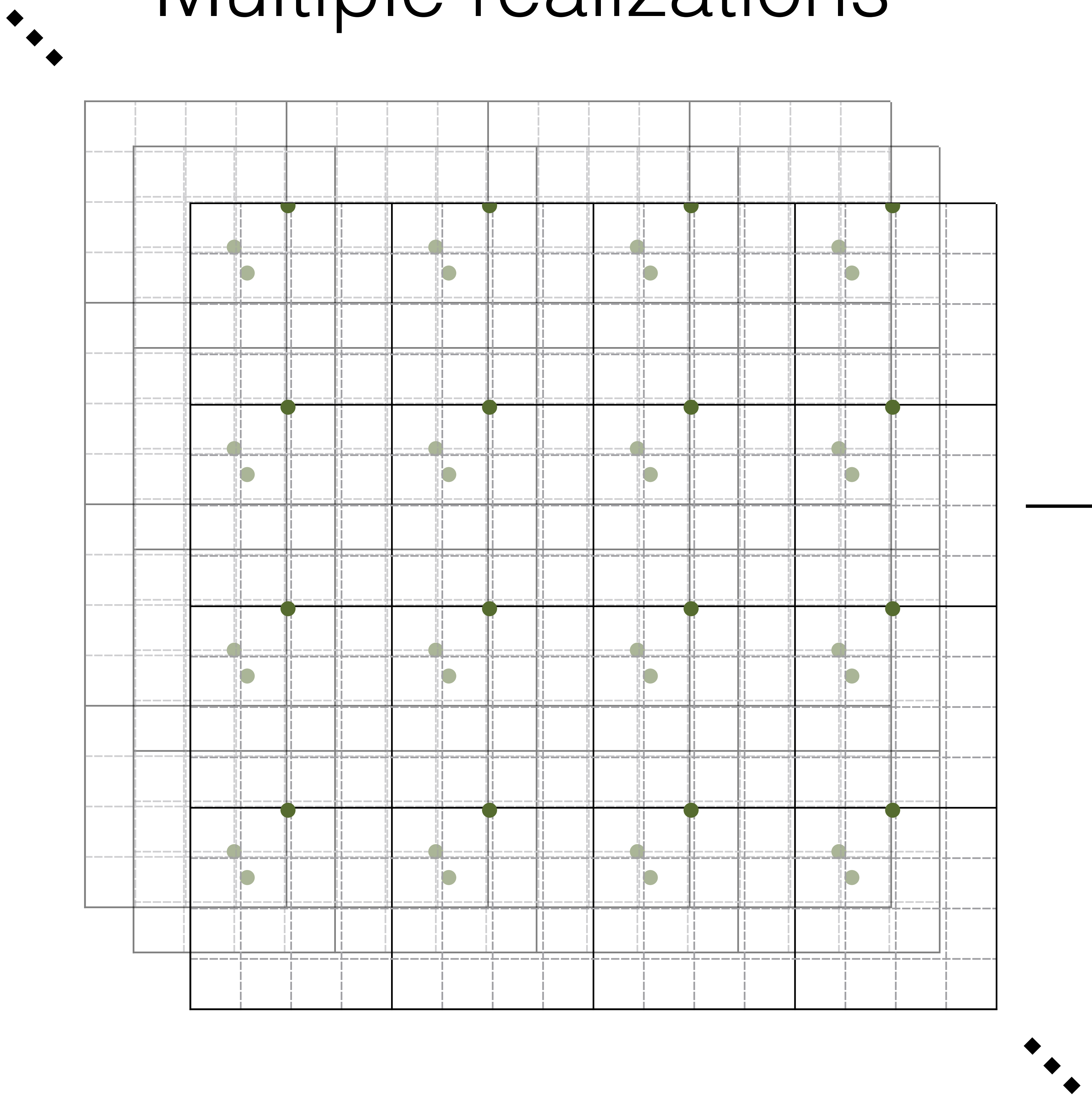

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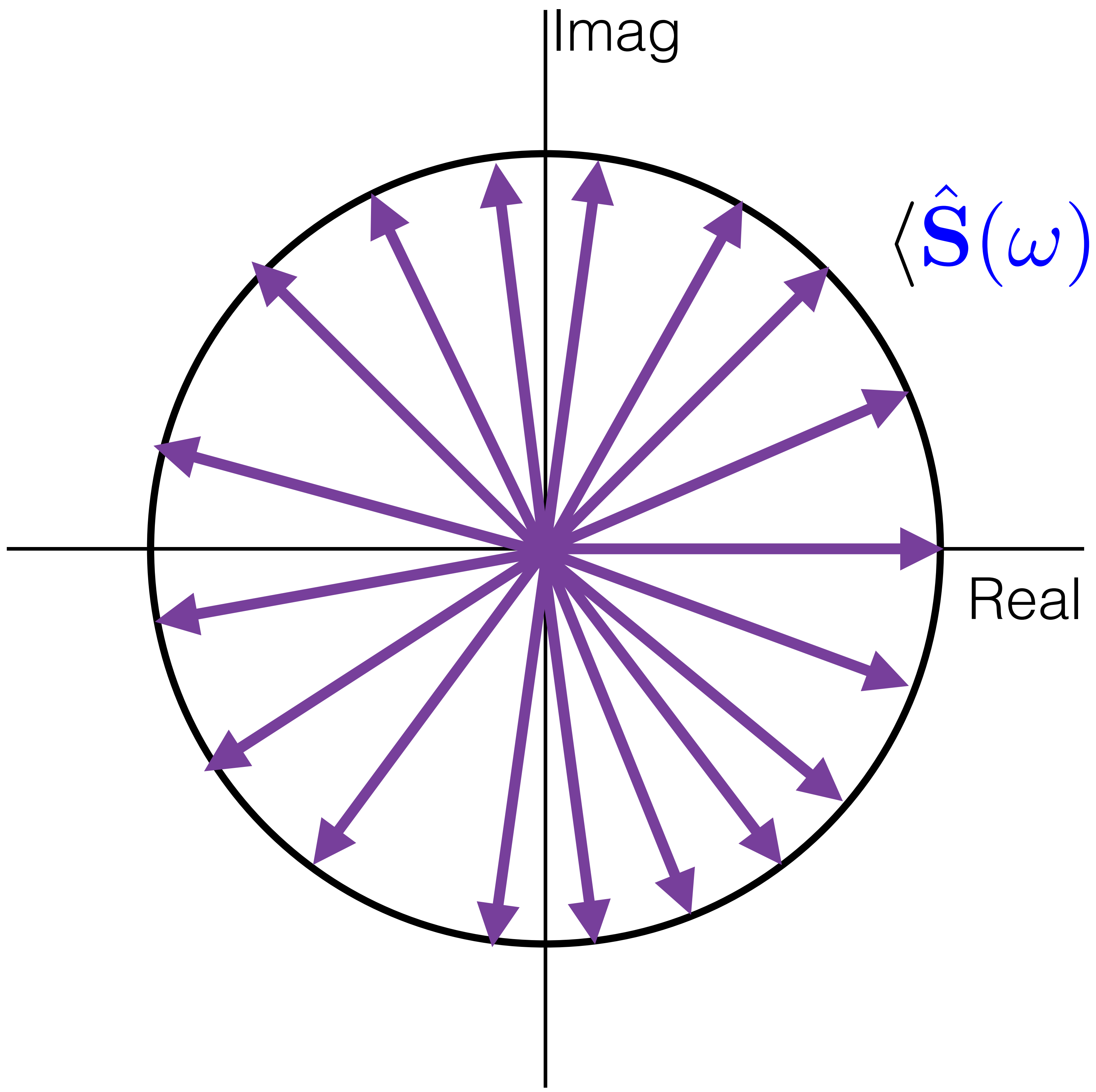

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

Phase change due to Random Shift

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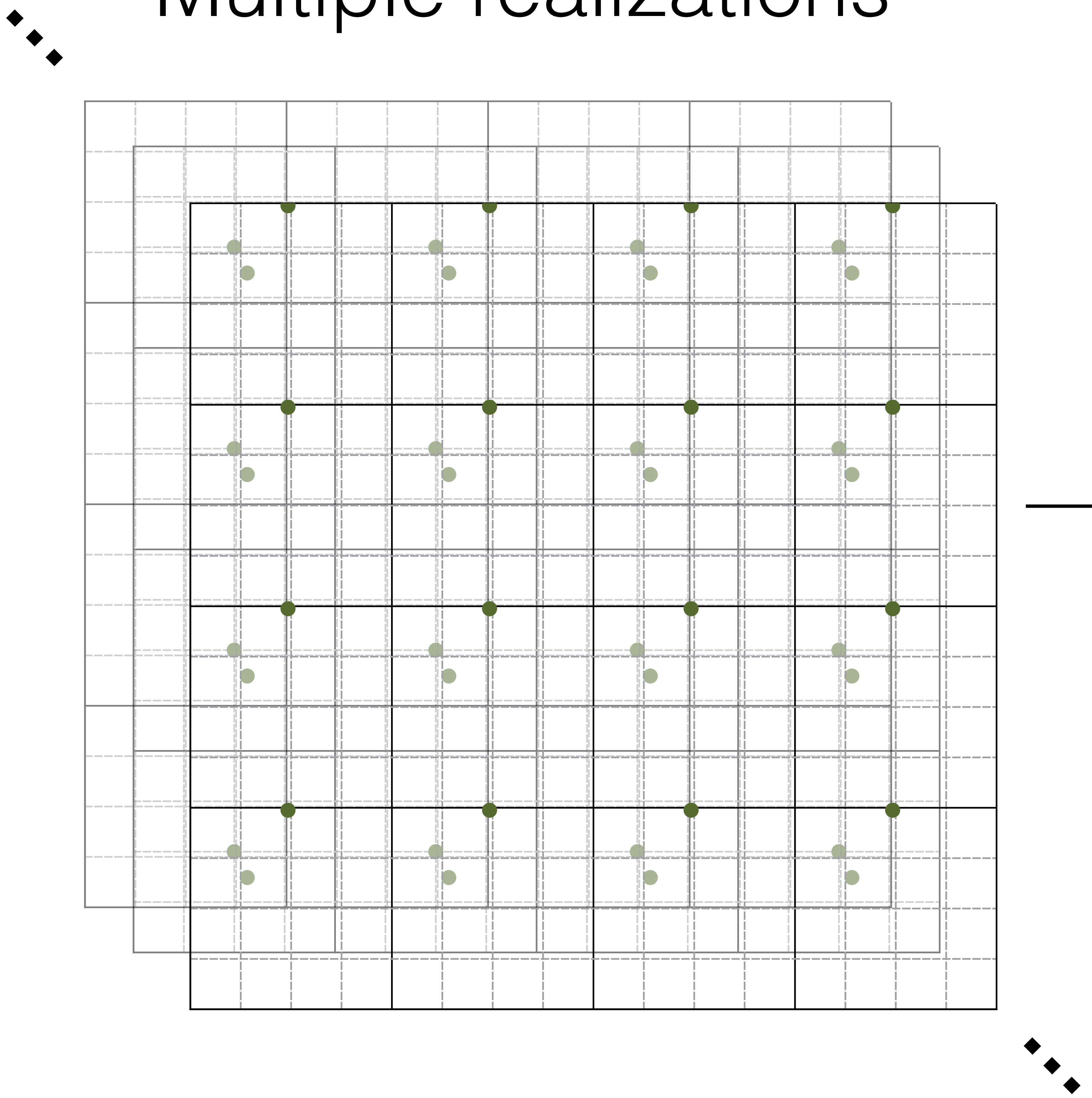

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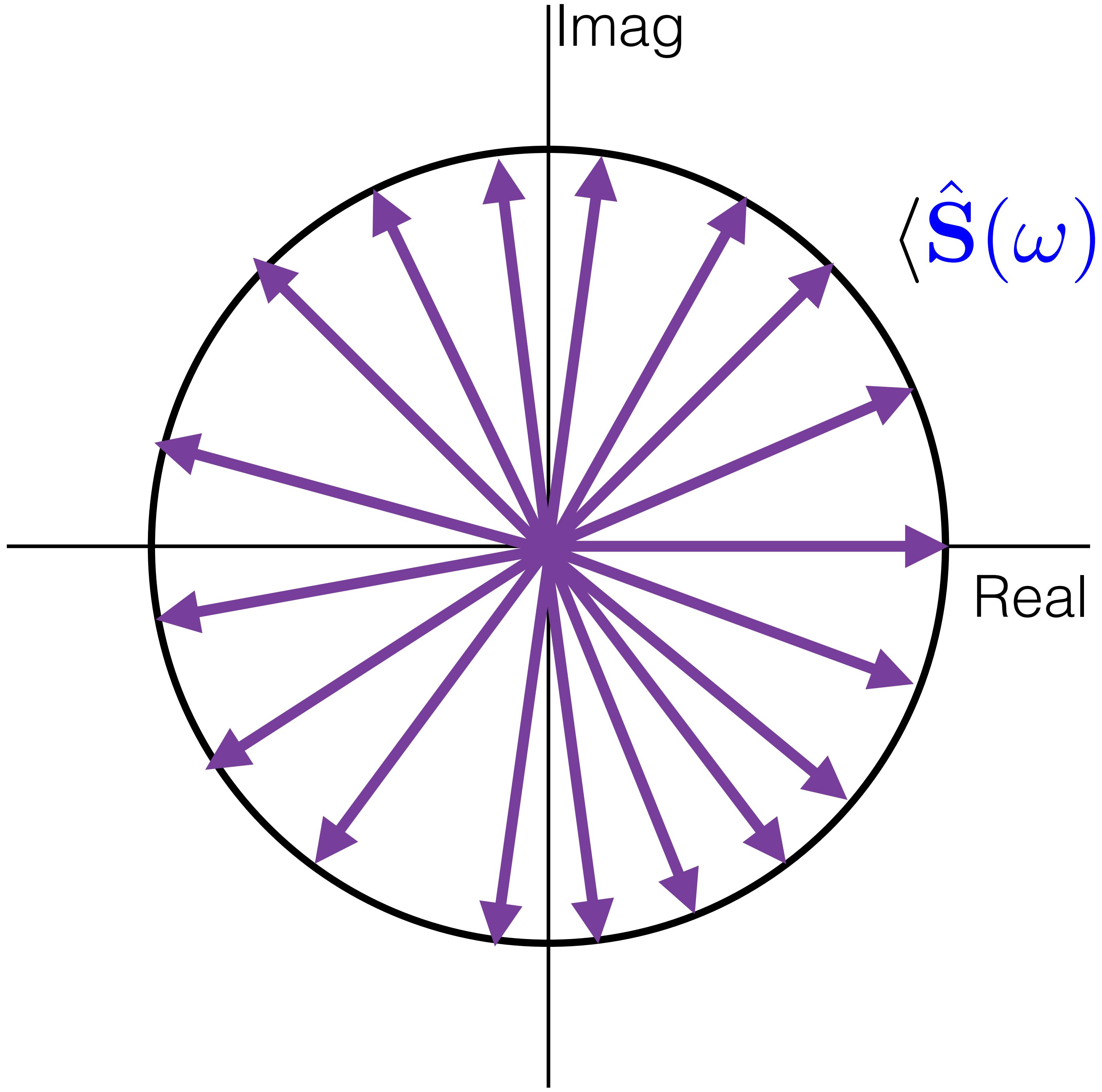

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0 \;\; \forall \omega \neq 0$$

$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

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- Homogenization allows representation of error only in terms of variance

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- Homogenization allows representation of error only in terms of variance
- We can take any sampling pattern and homogenize it to make the Monte Carlo estimator unbiased.

Variance in the Fourier domain

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Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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$$\mathrm{Var}(I - \tilde{\mu}_N)$$

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$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

where,

$$P_f(\omega) = |\hat{f}^*(\omega)|^2$$ Power Spectrum

Variance in the Fourier domain

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Subr and Kautz [2013]

Variance in the Fourier domain

$$\boxed{\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega}$$

Subr and Kautz [2013]

This is a general form, both for homogenised as well as non-homogenised sampling patterns

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Fredo Durand [2011]

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Fredo Durand [2011]

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Pilleboue et al. [2015]

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Variance in terms of n-dimensional Power Spectra

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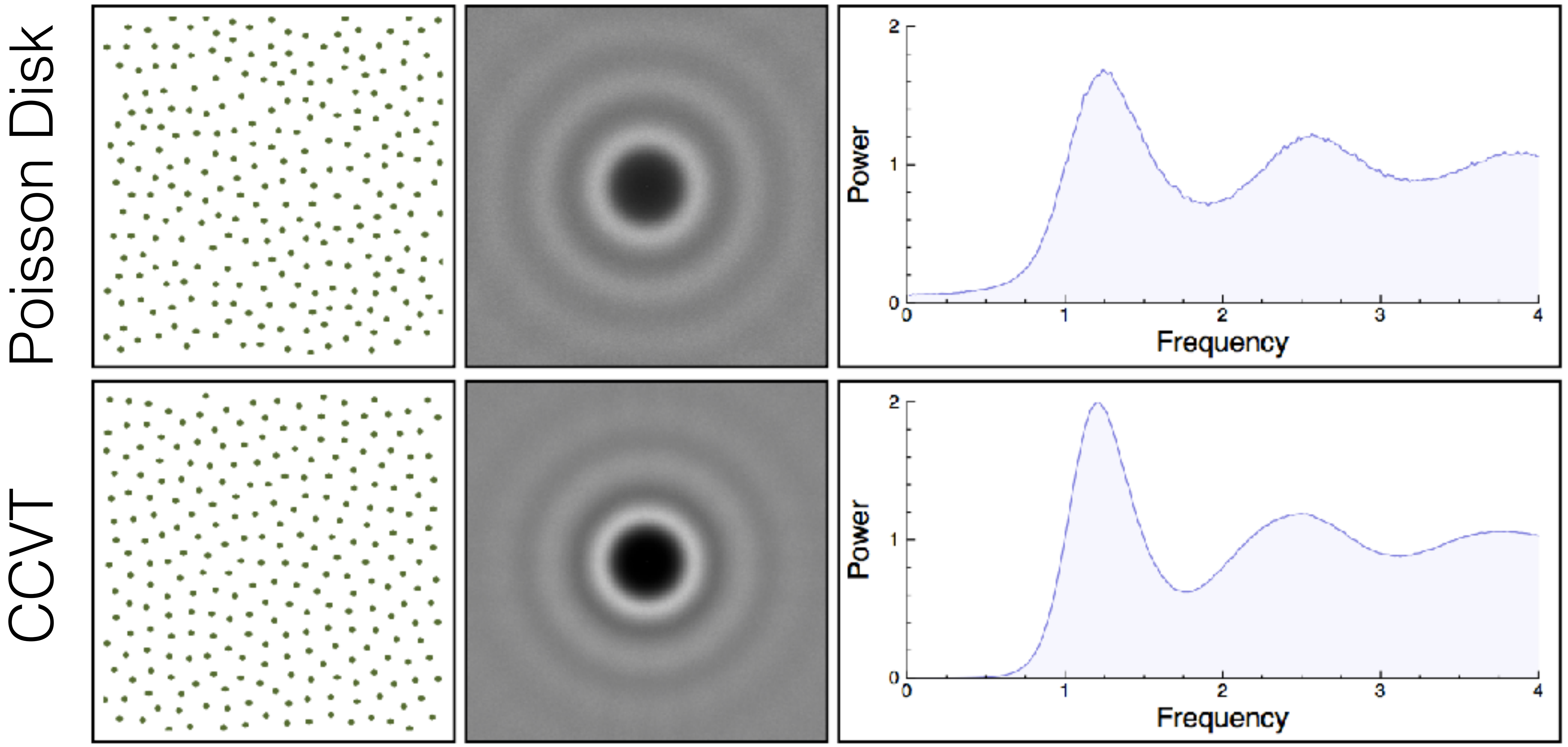

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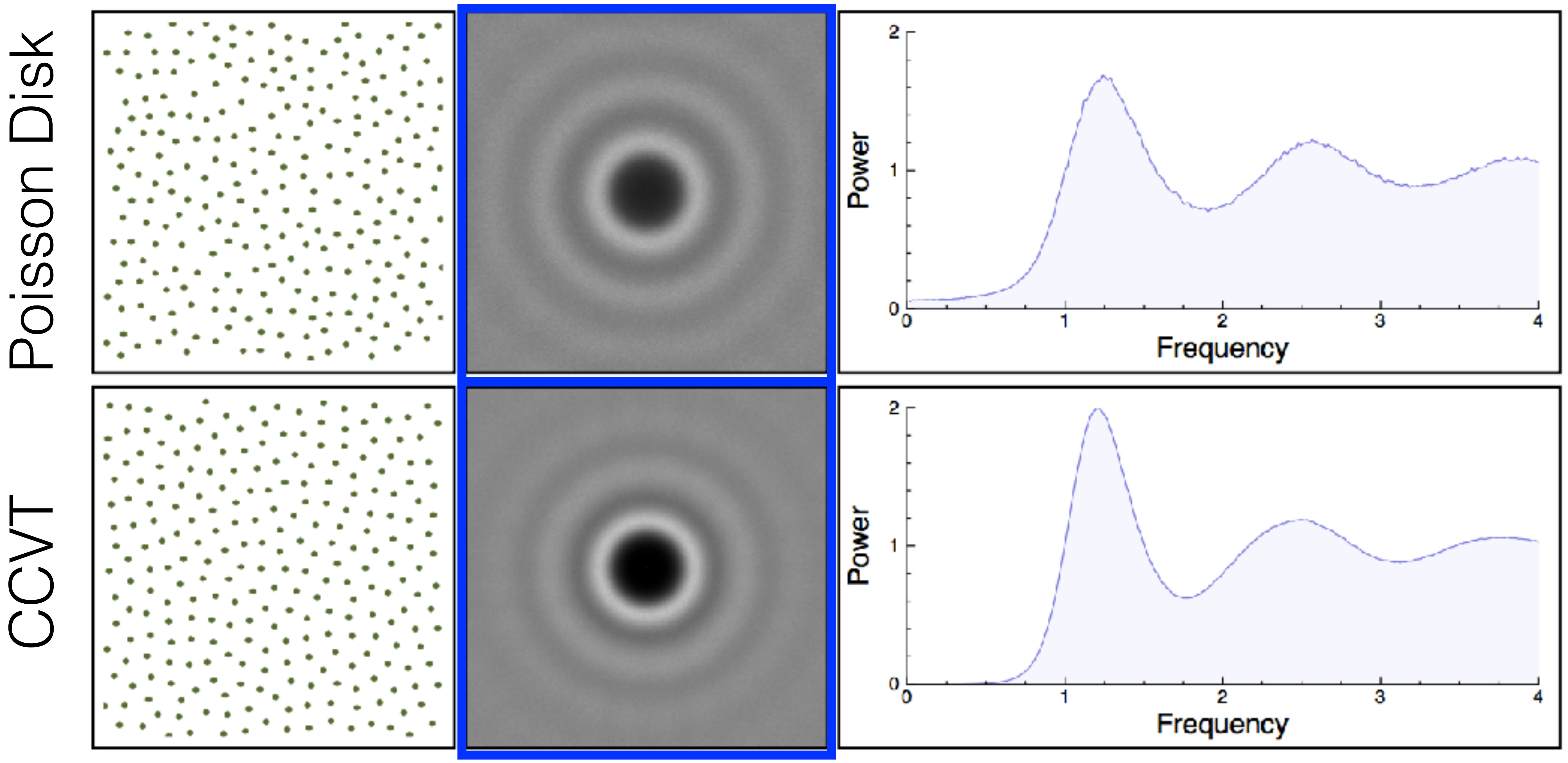

Variance in the Polar Coordinates

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$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho \mathbf{n}) \left\langle P_{\mathbf{S}}(\rho \mathbf{n}) \right\rangle d\mathbf{n}\, d\rho$$

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For isotropic power spectra:

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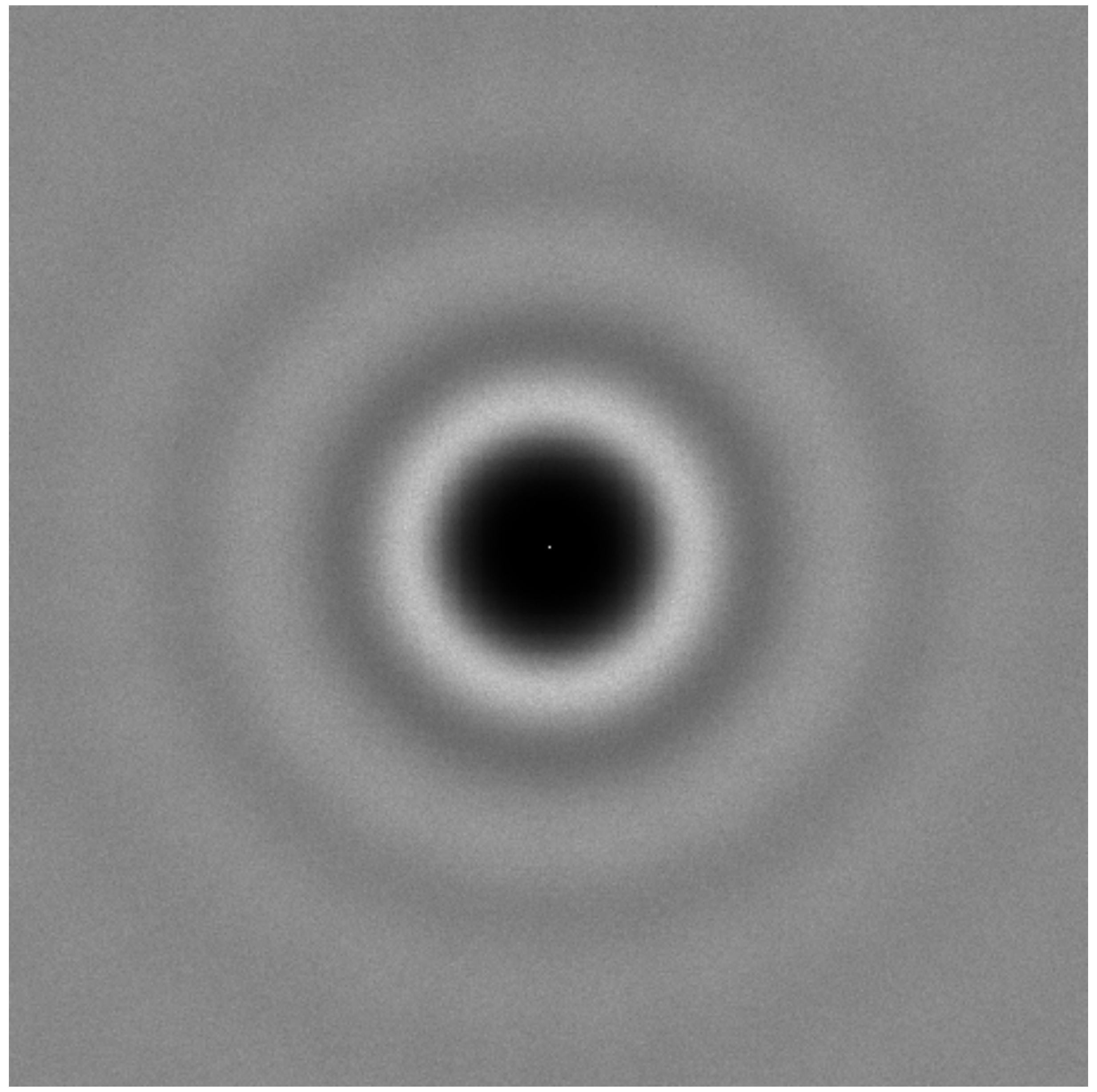

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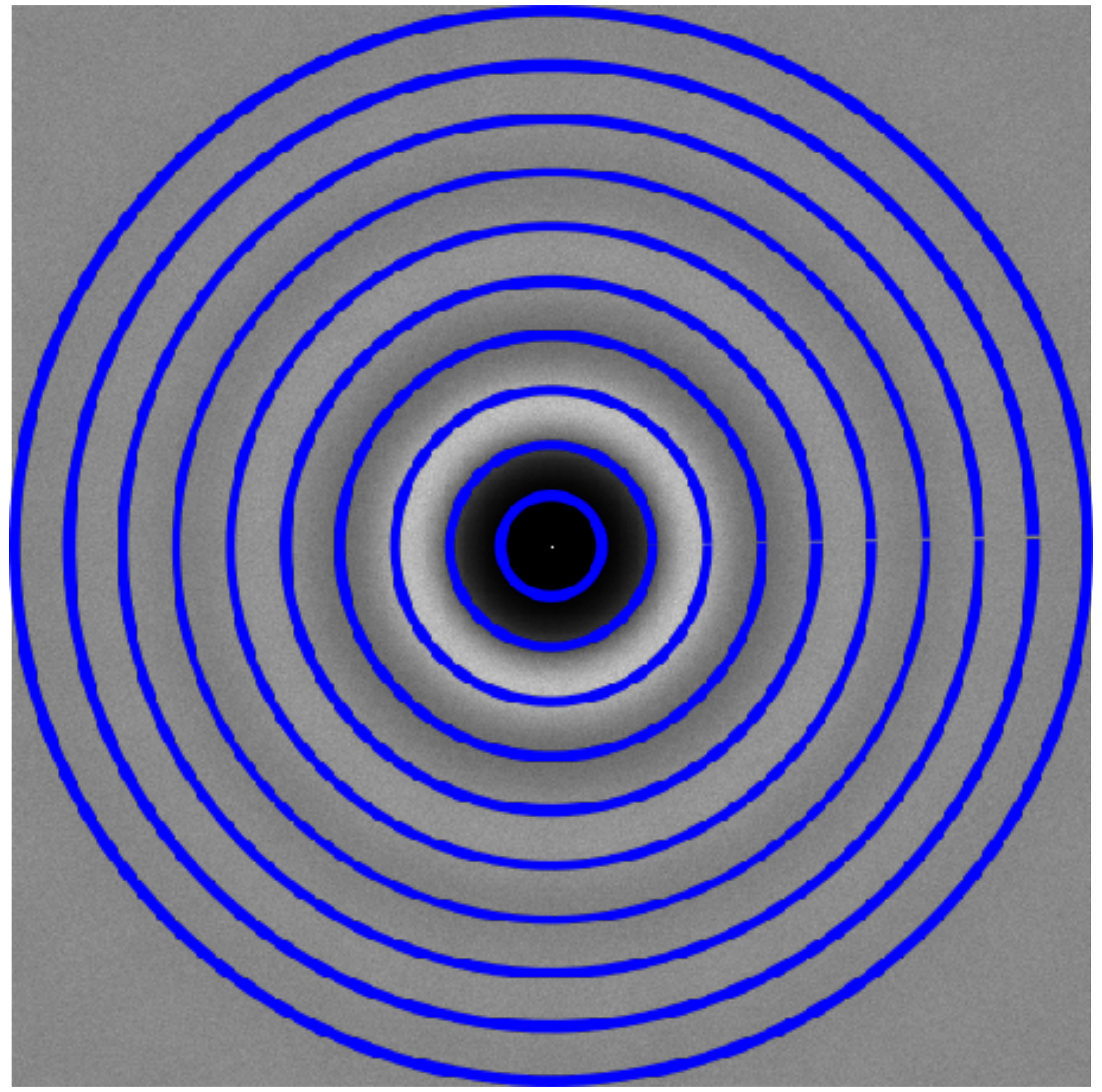

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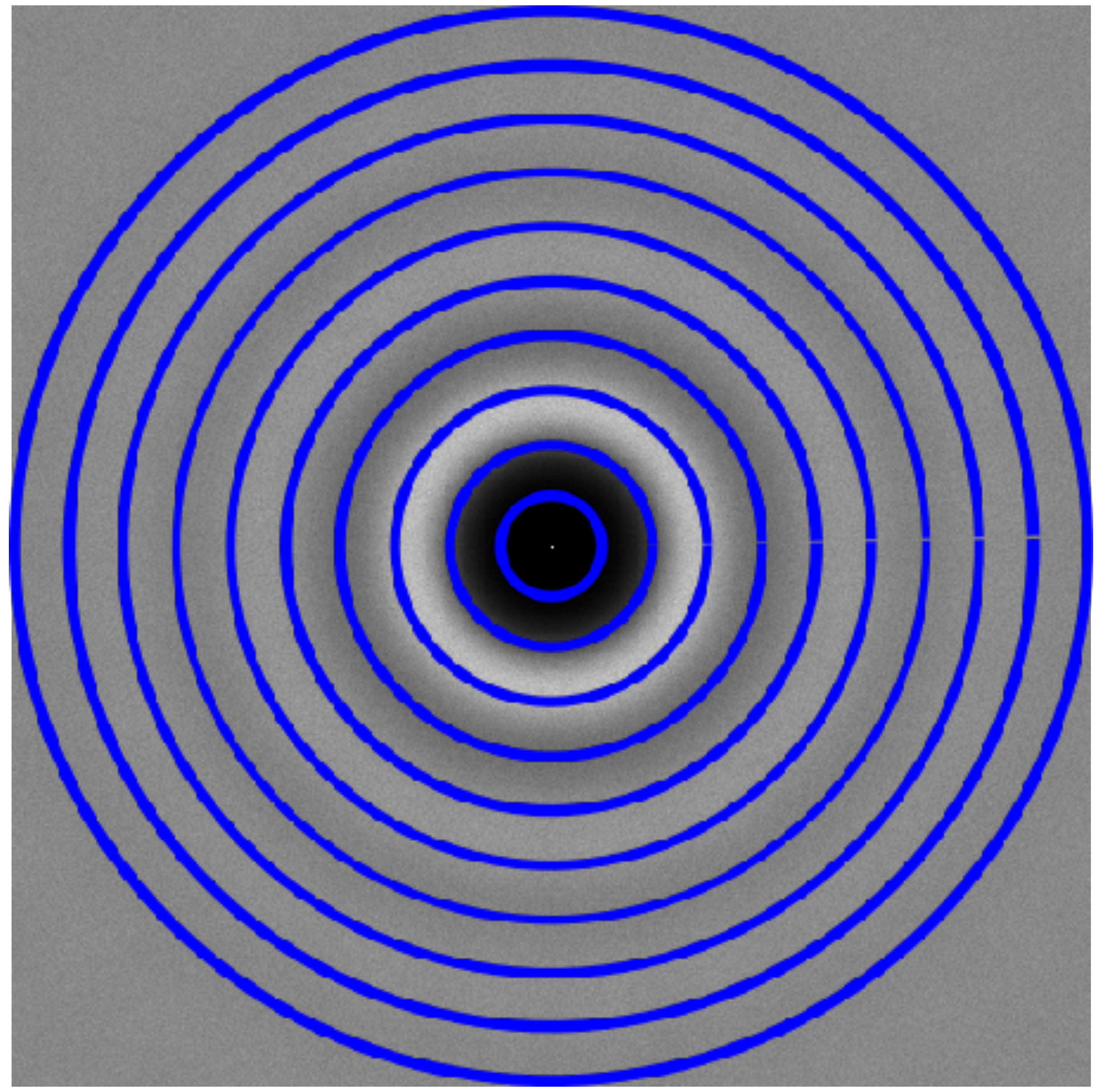

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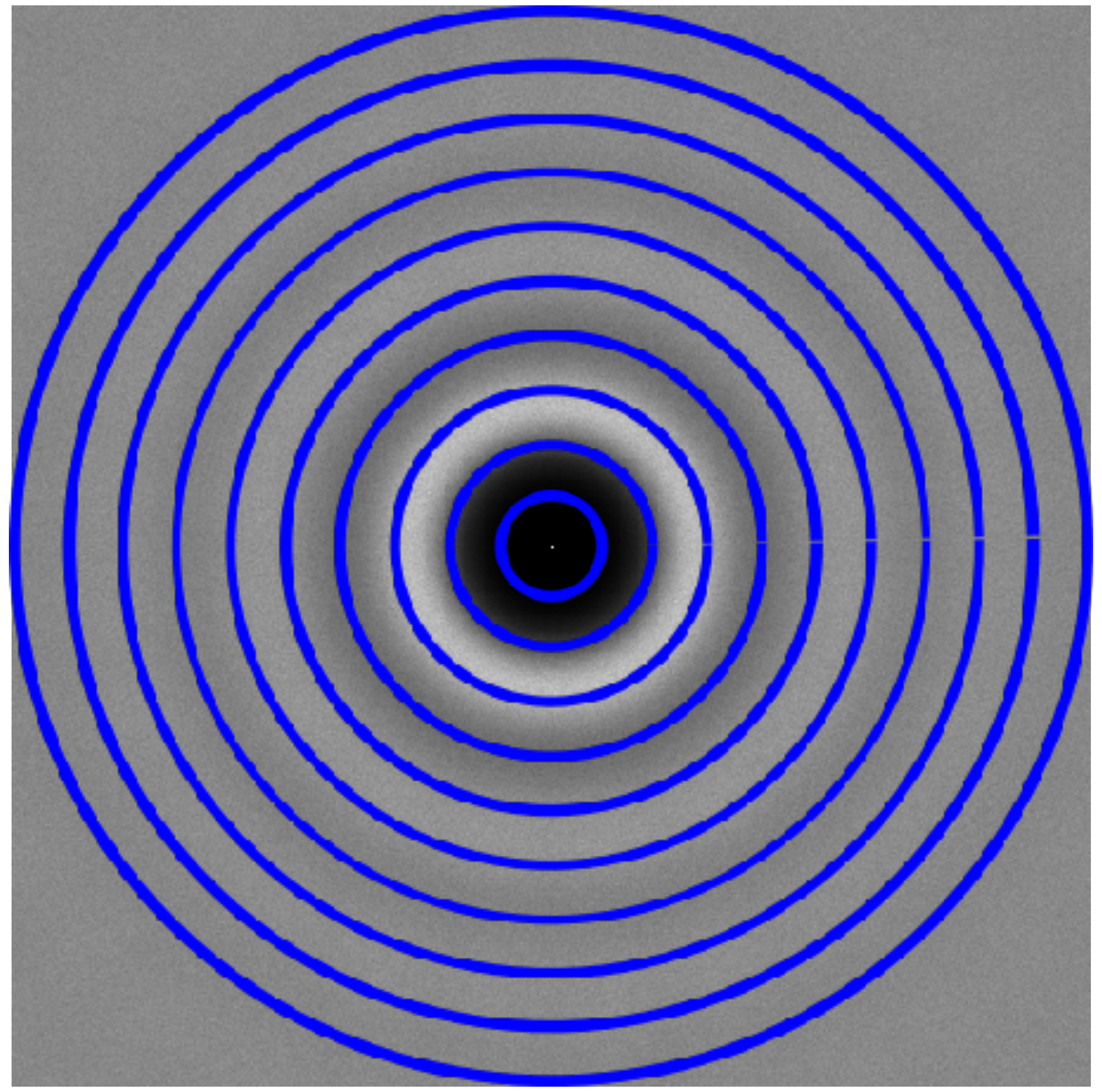

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Variance in terms of 1-dimensional Power Spectra

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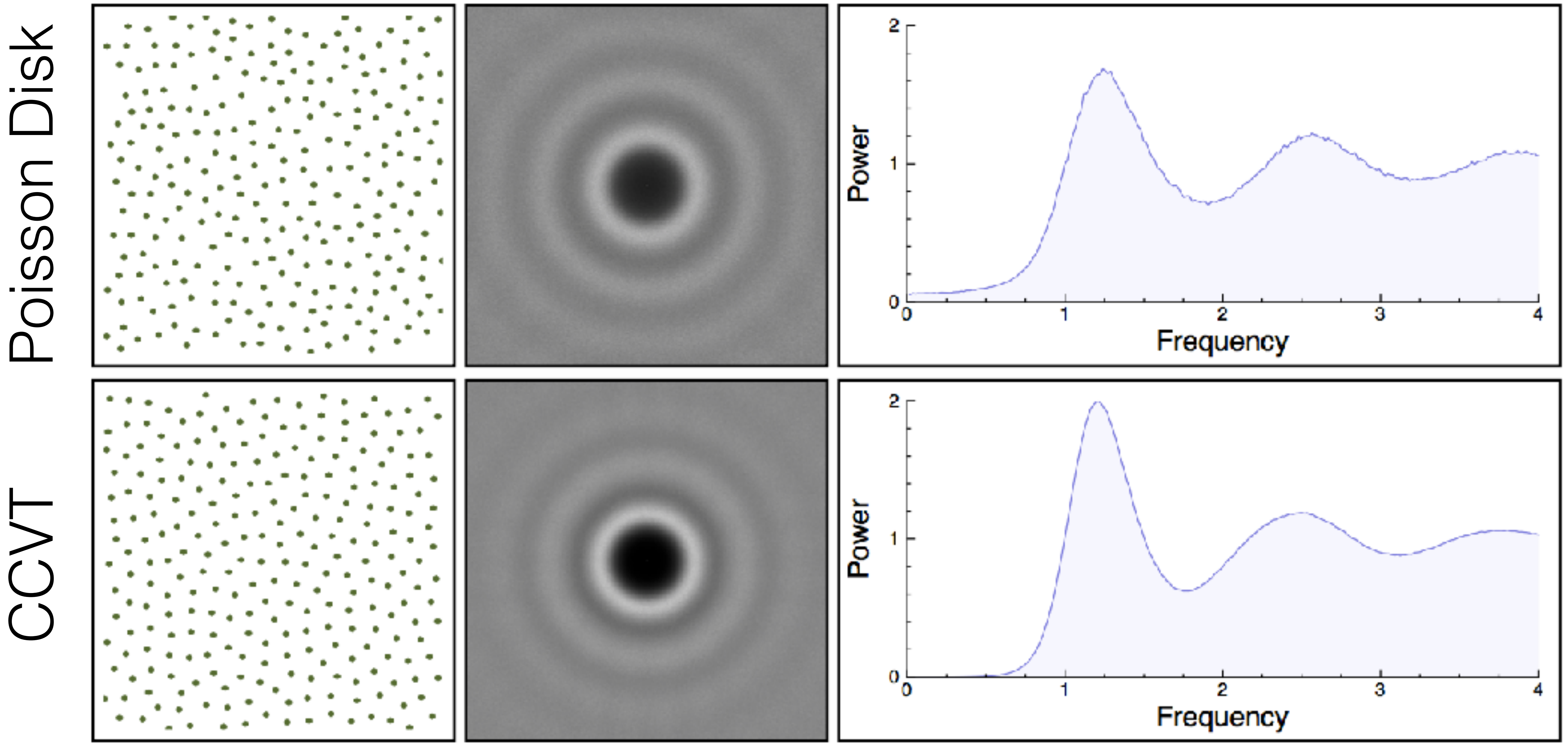

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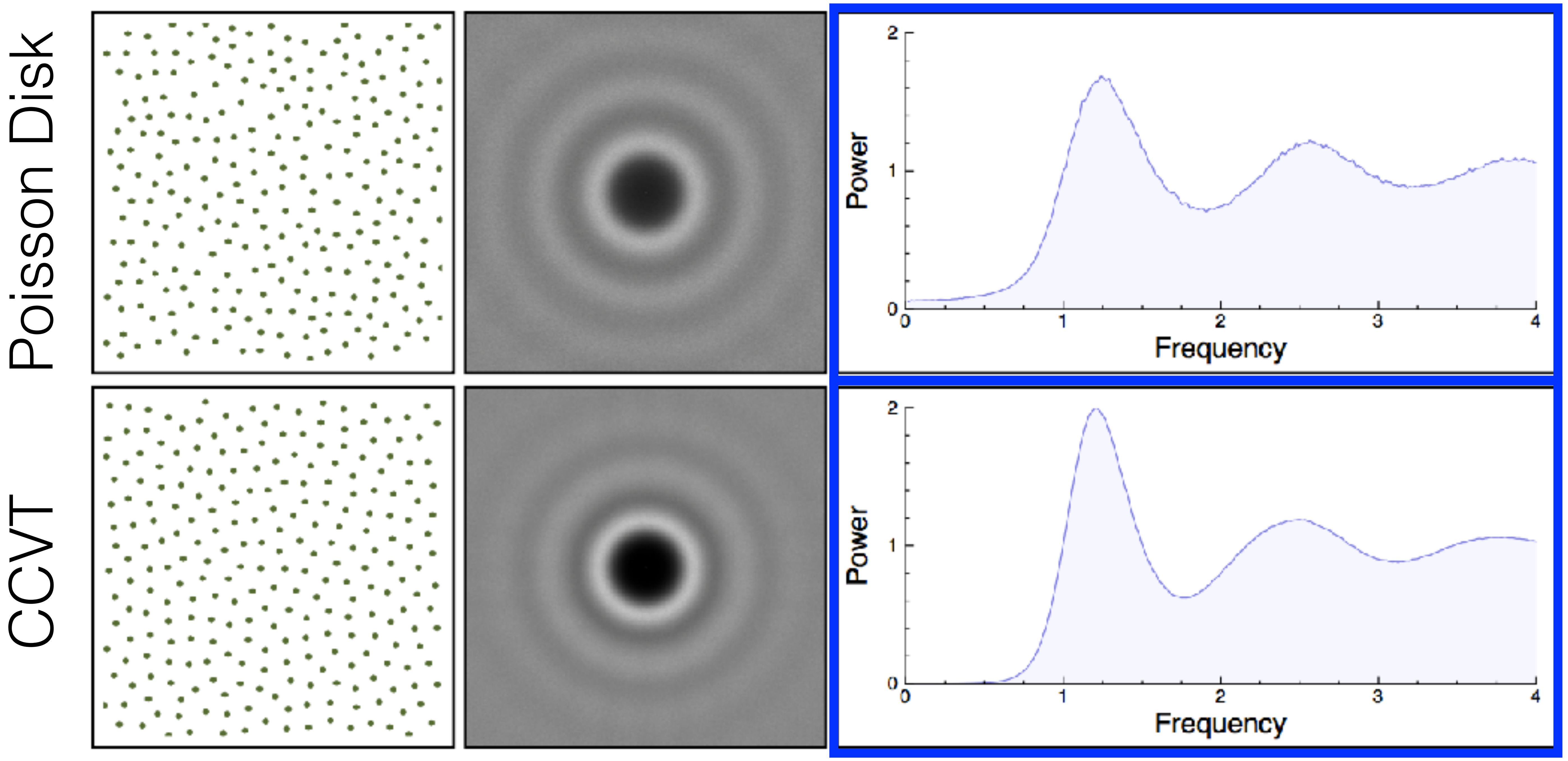

Variance: Integral over Product of Power Spectra

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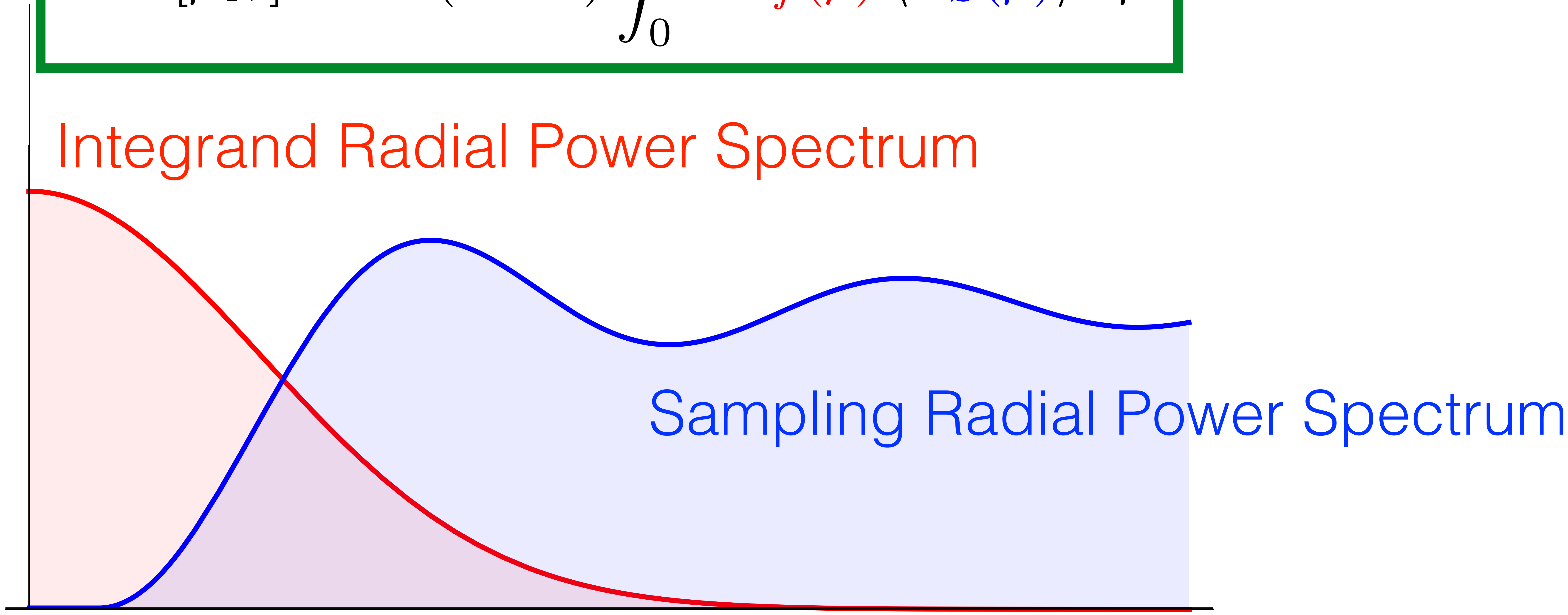

For given number of Samples

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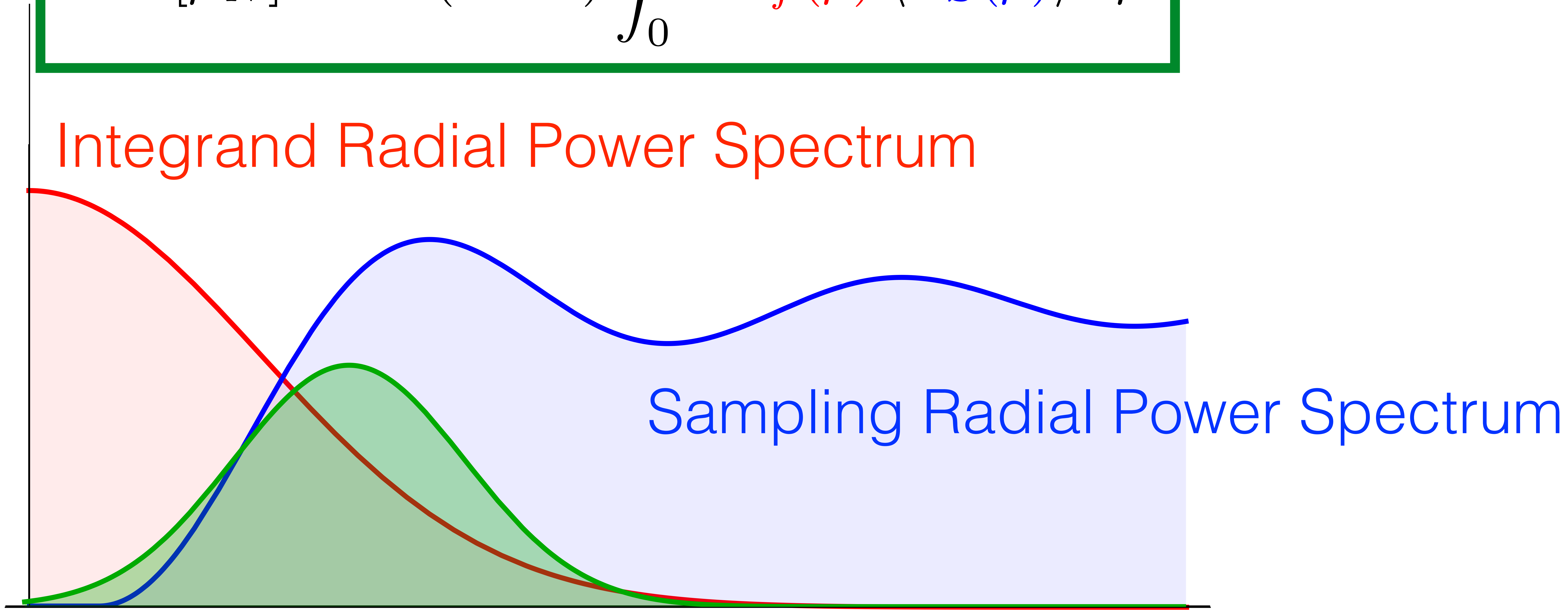

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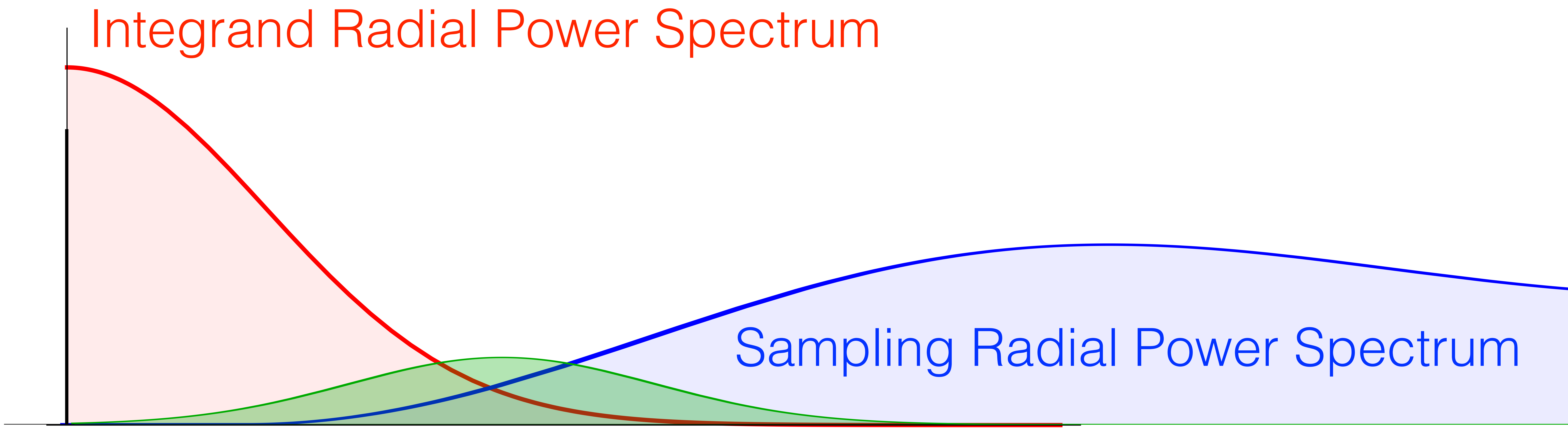

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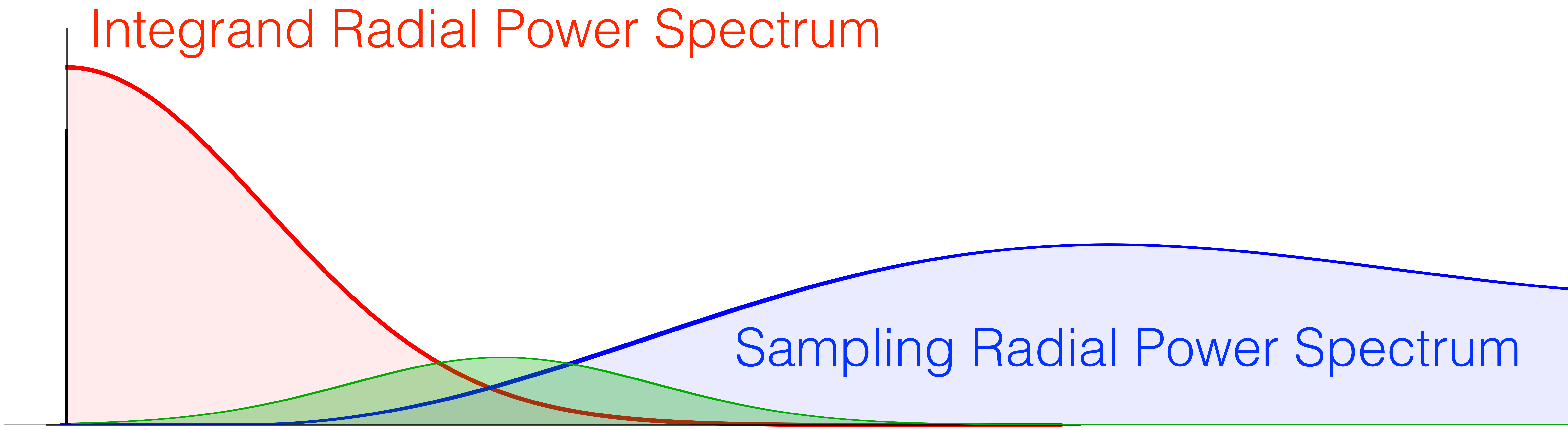

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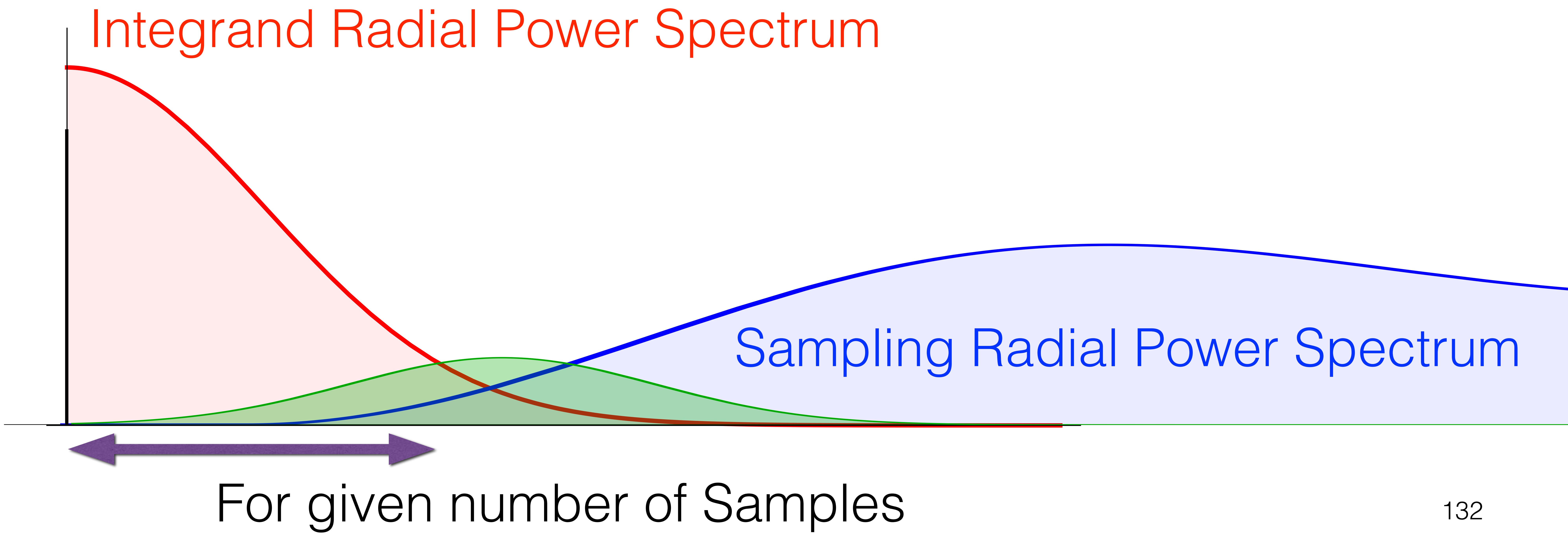
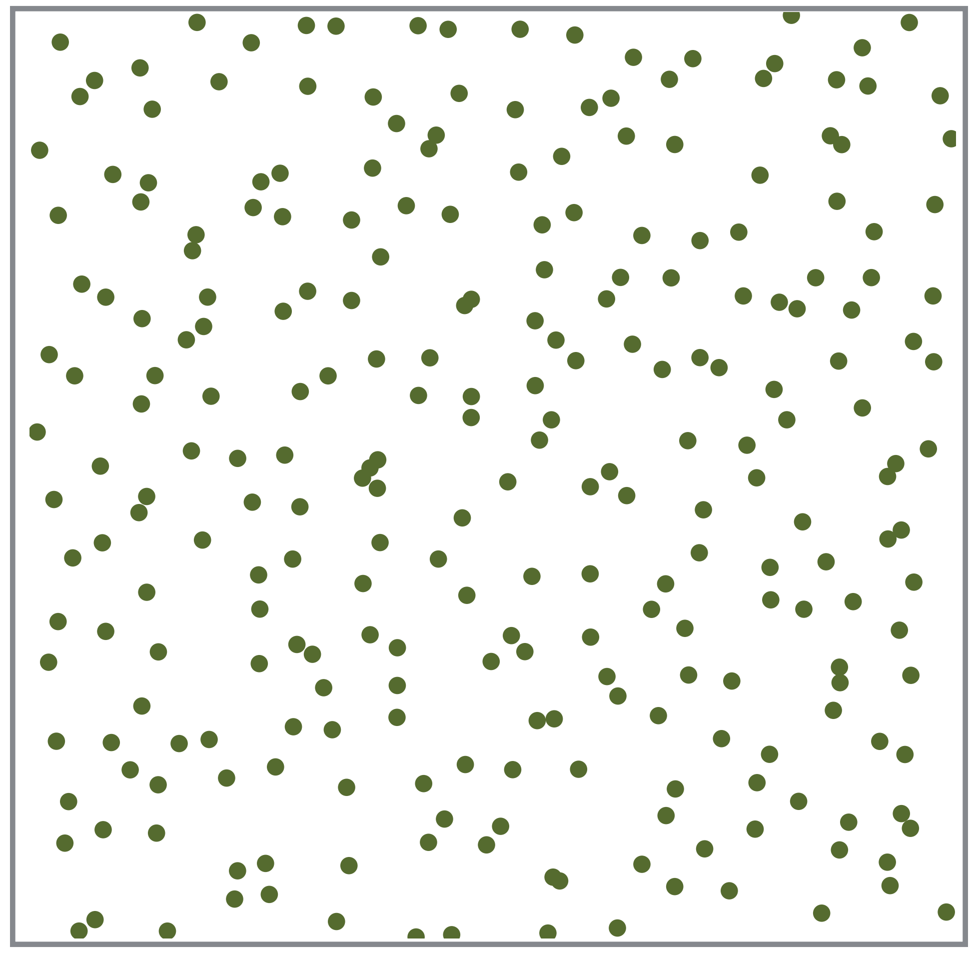
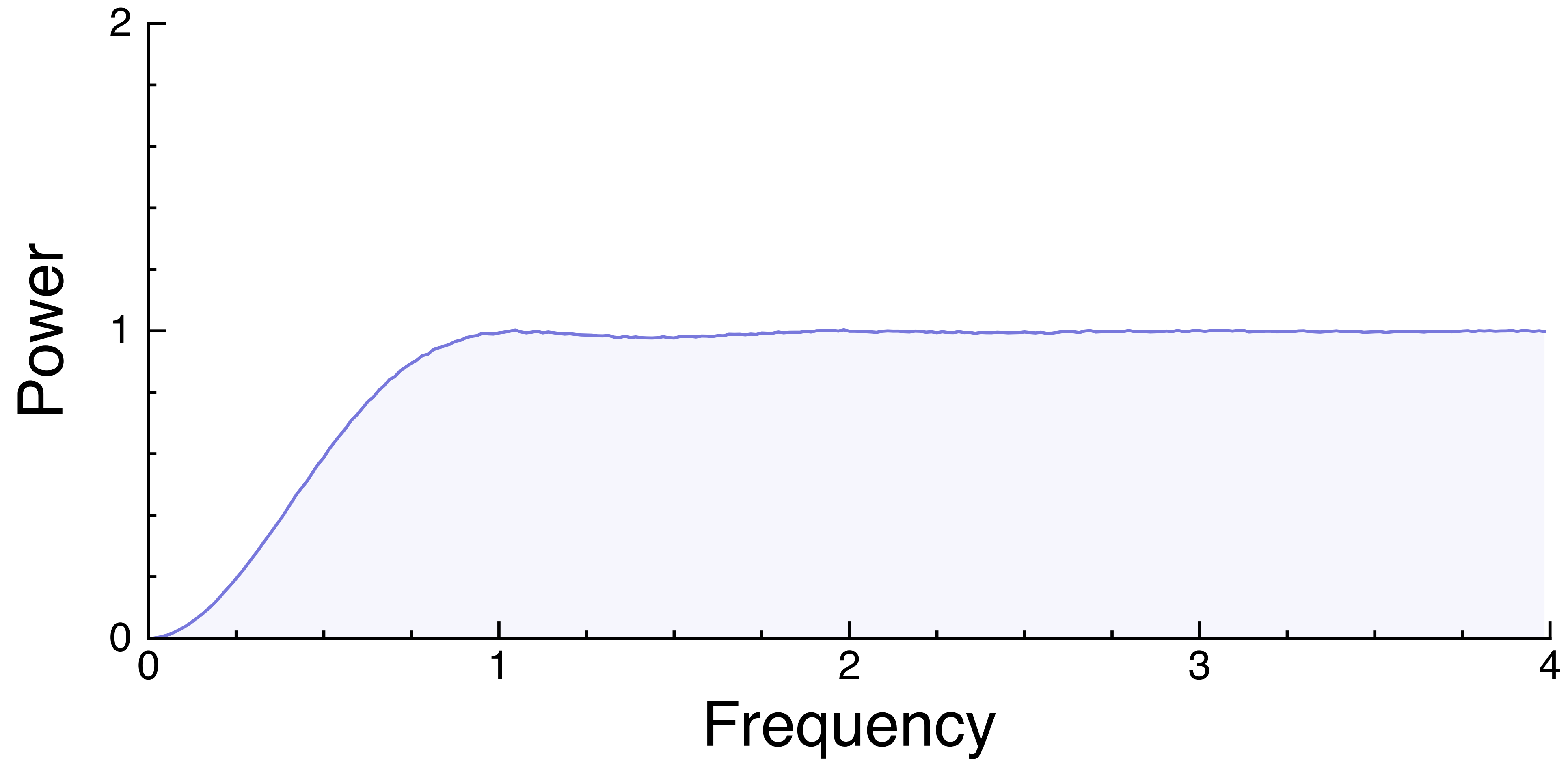


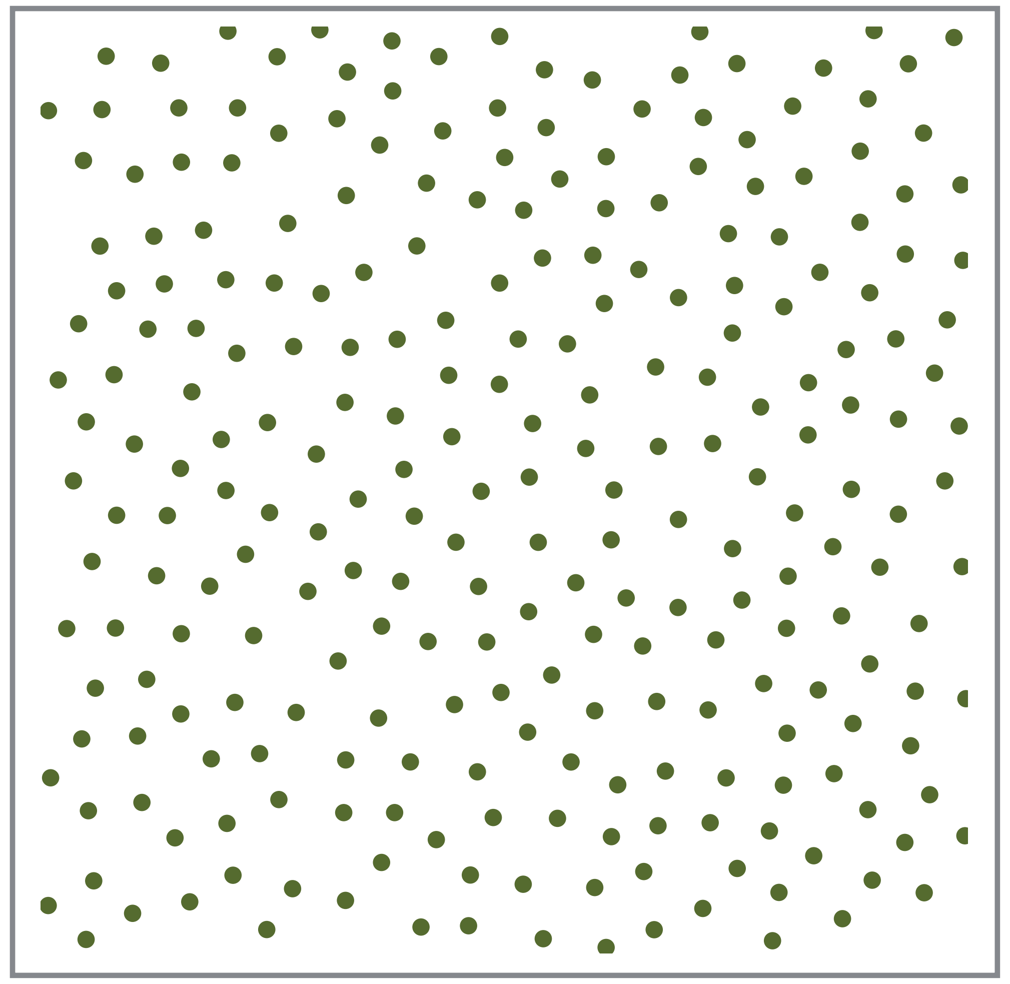
Spatial Distribution vs Radial Mean Power Spectra

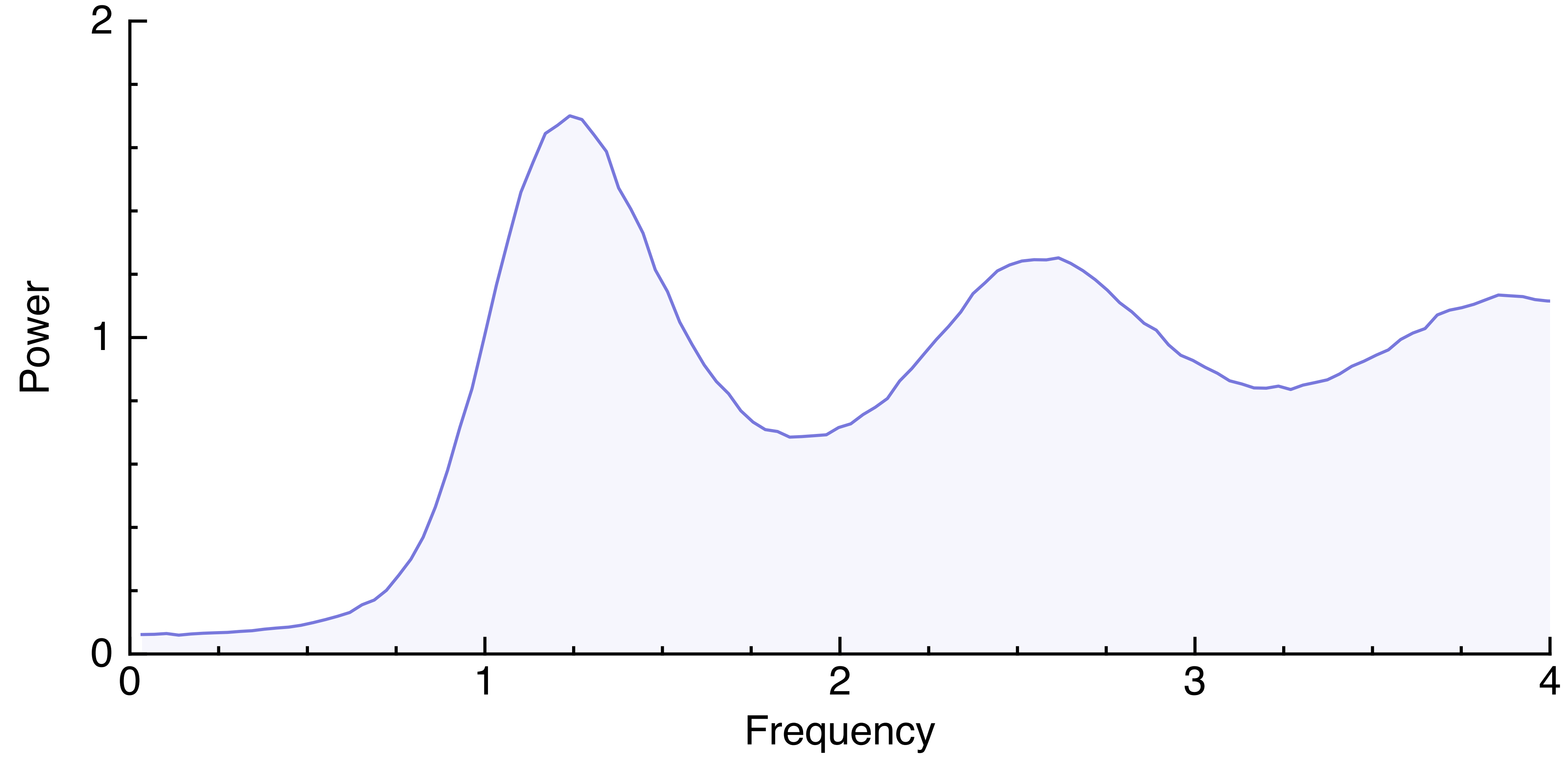
Jitter





Poisson Disk





For 2-dimensions

Samplers	Worst Case	Best Case
Random		
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	
CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
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CCVT		

Pilleboue et al. [2015]

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	

Pilleboue et al. [2015]

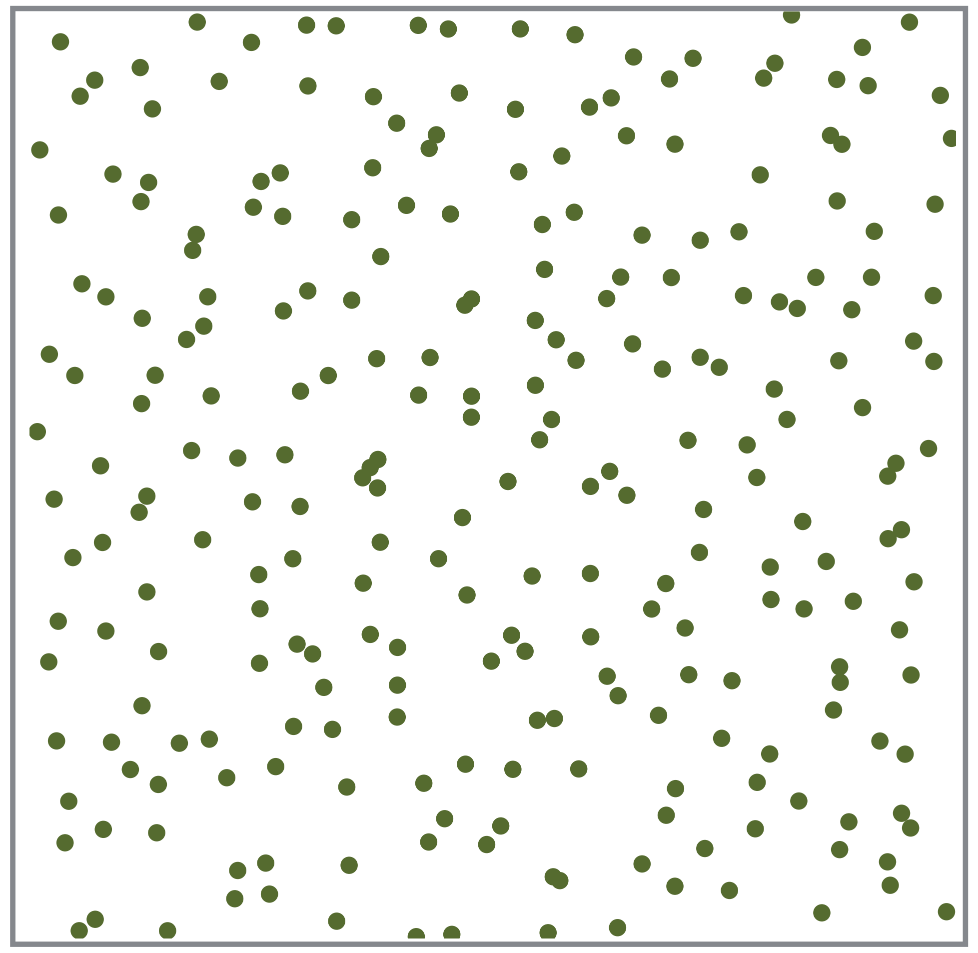
For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

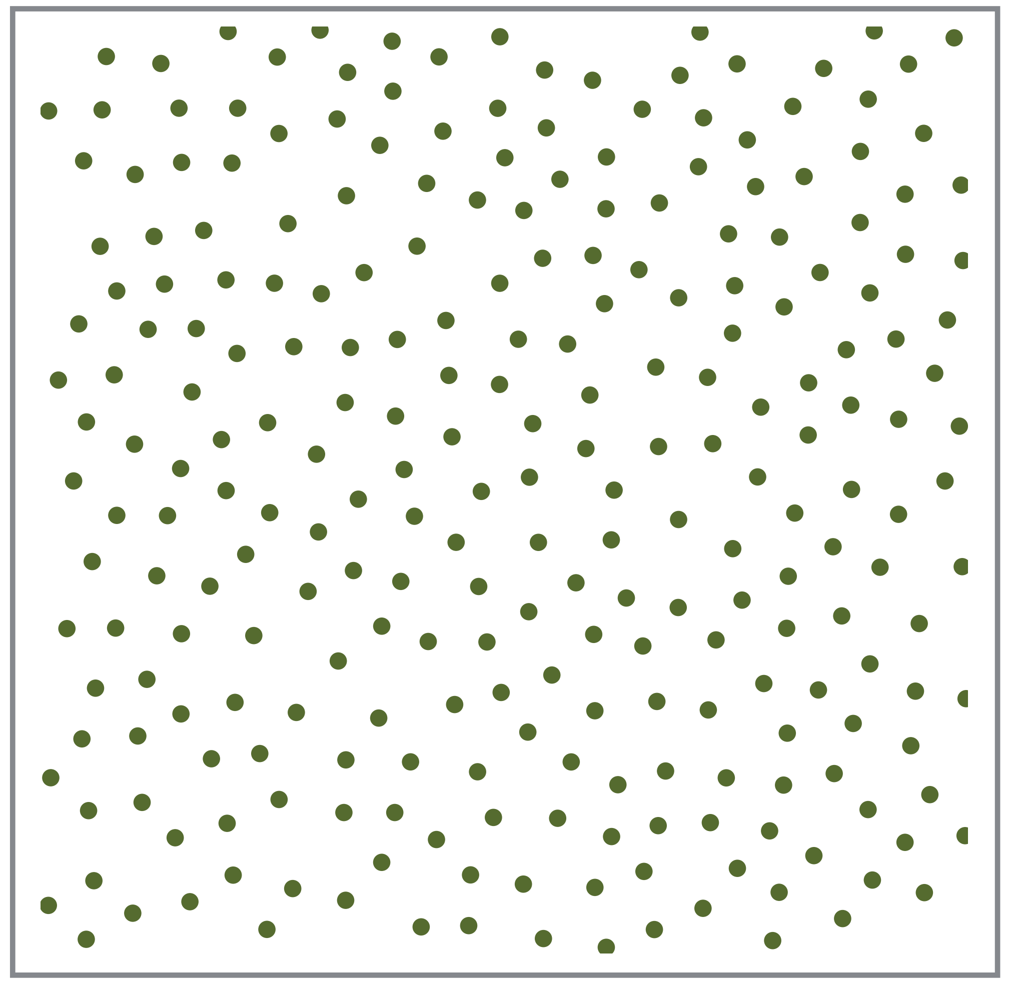
Pilleboue et al. [2015]

For 2-dimensions

Jitter



Poisson Disk

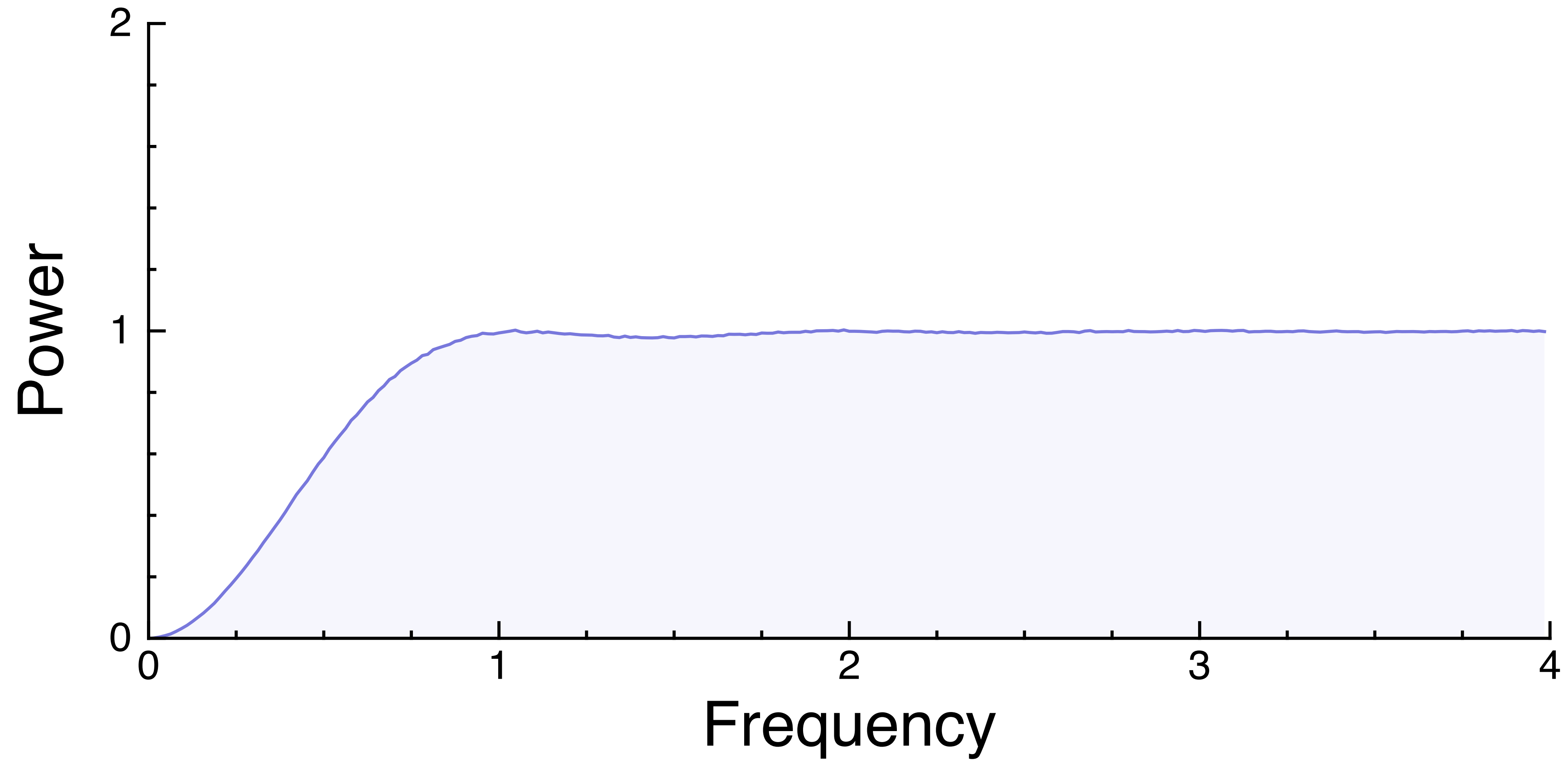


Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

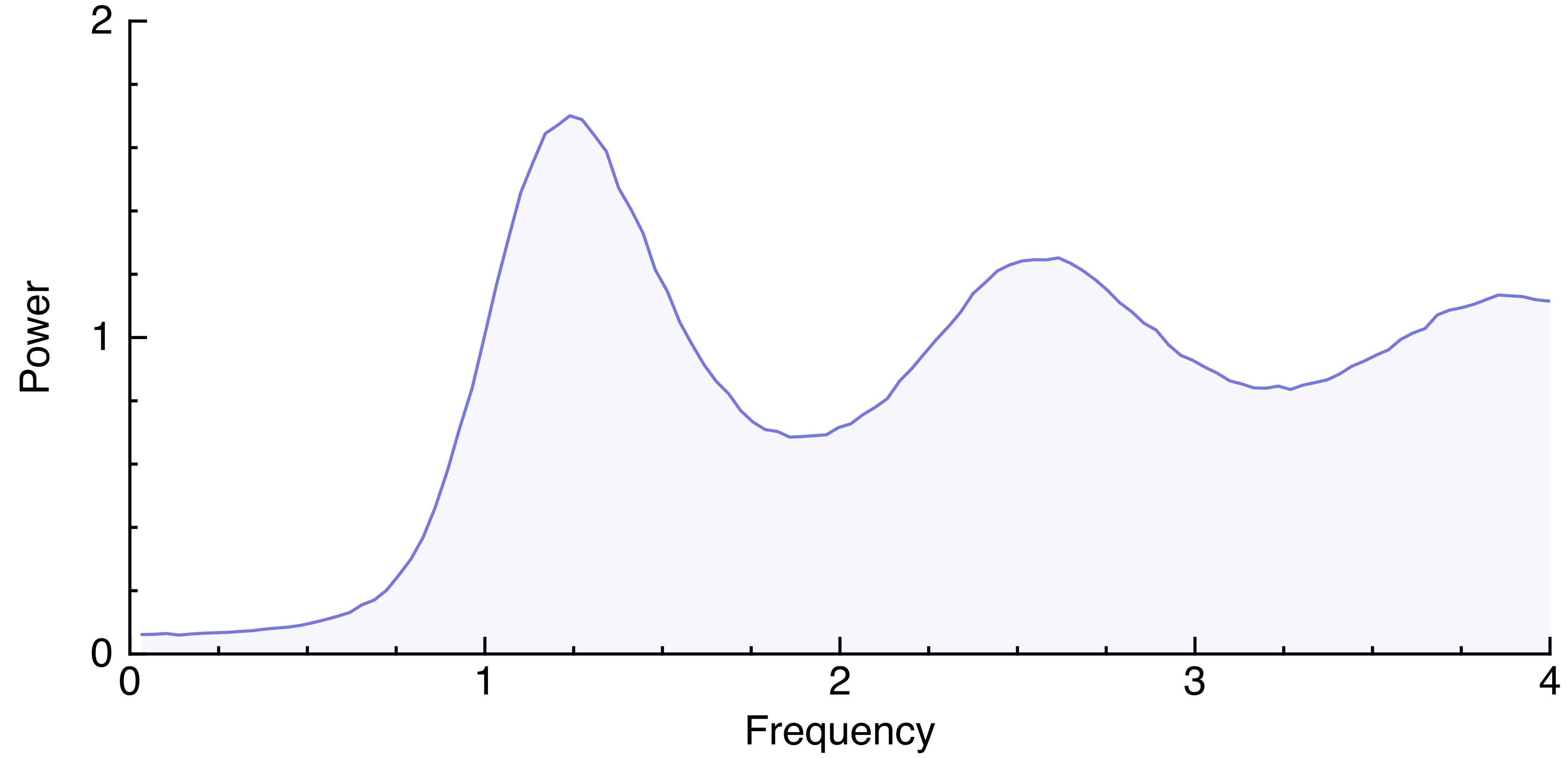
Pilleboue et al. [2015]

Low Frequency Region

Jitter



Poisson Disk



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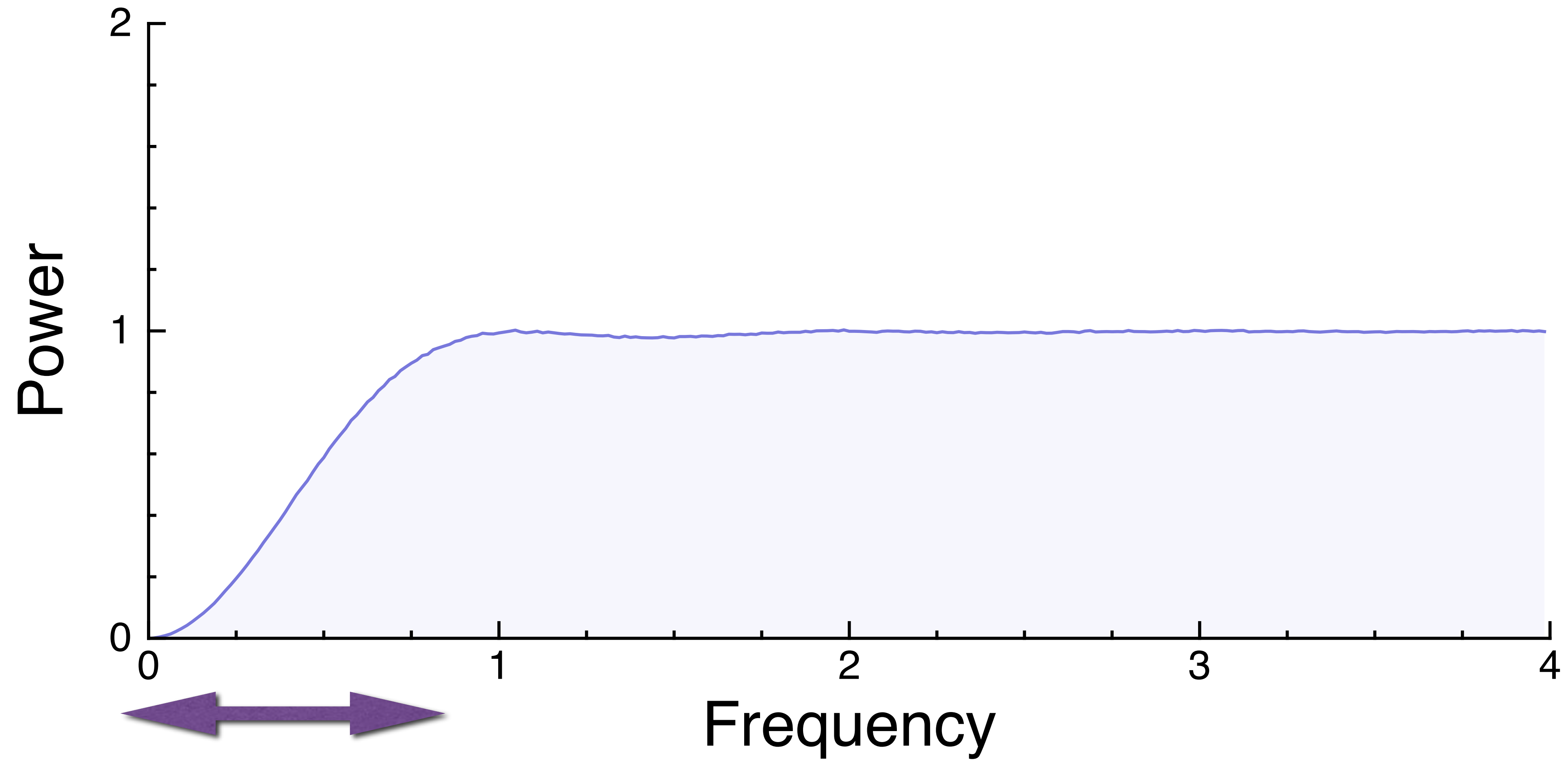
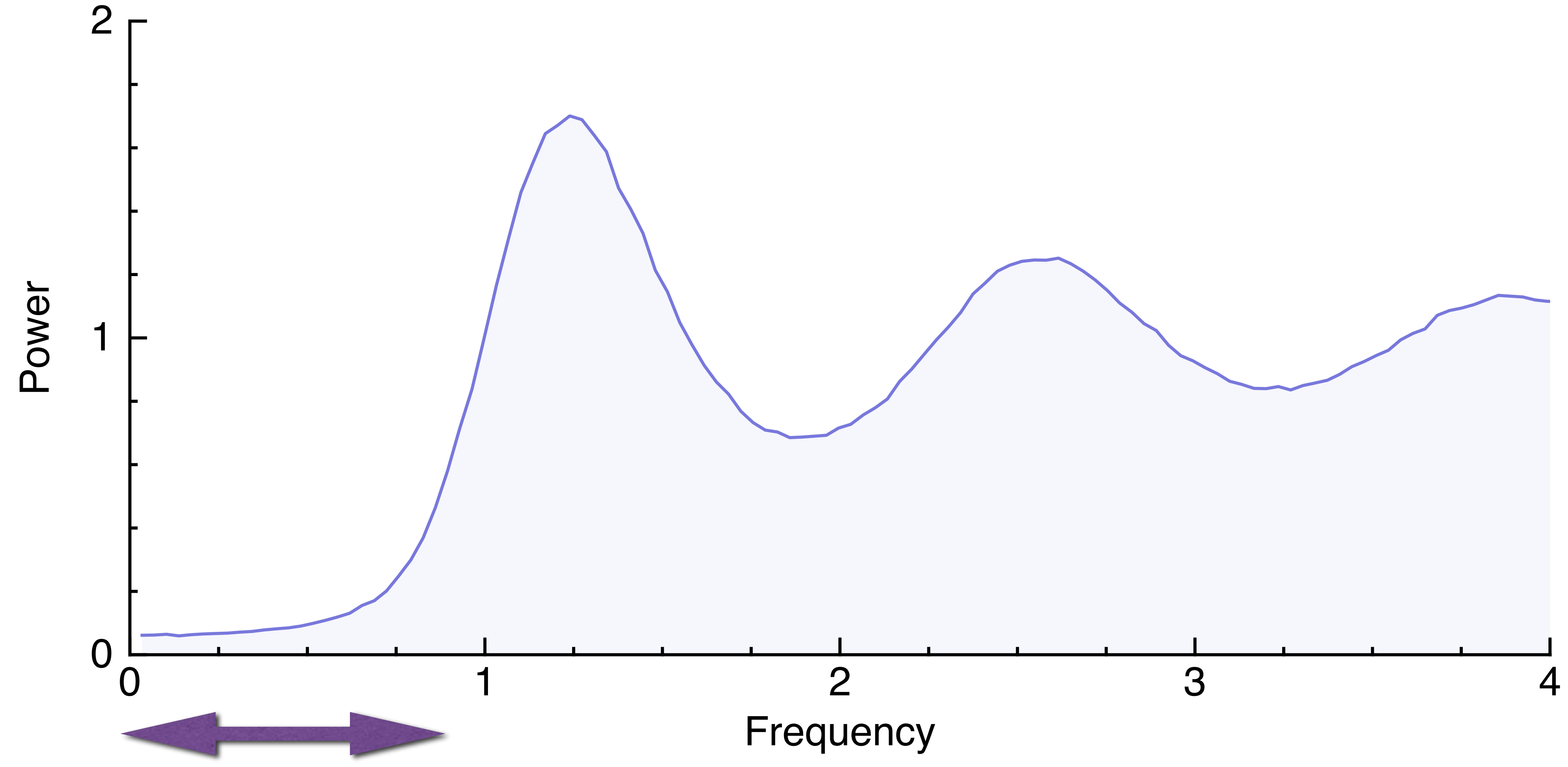
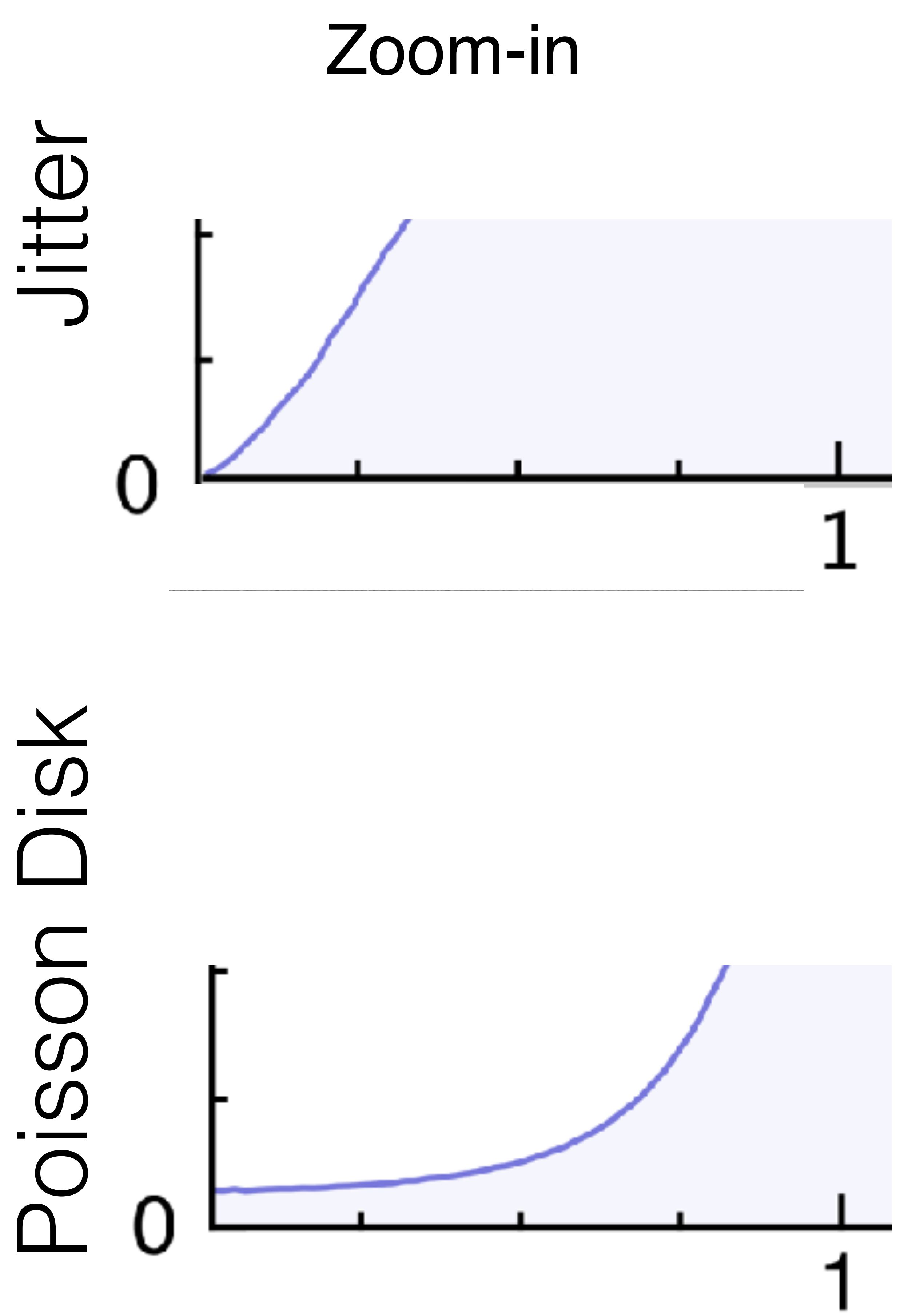
Jitter

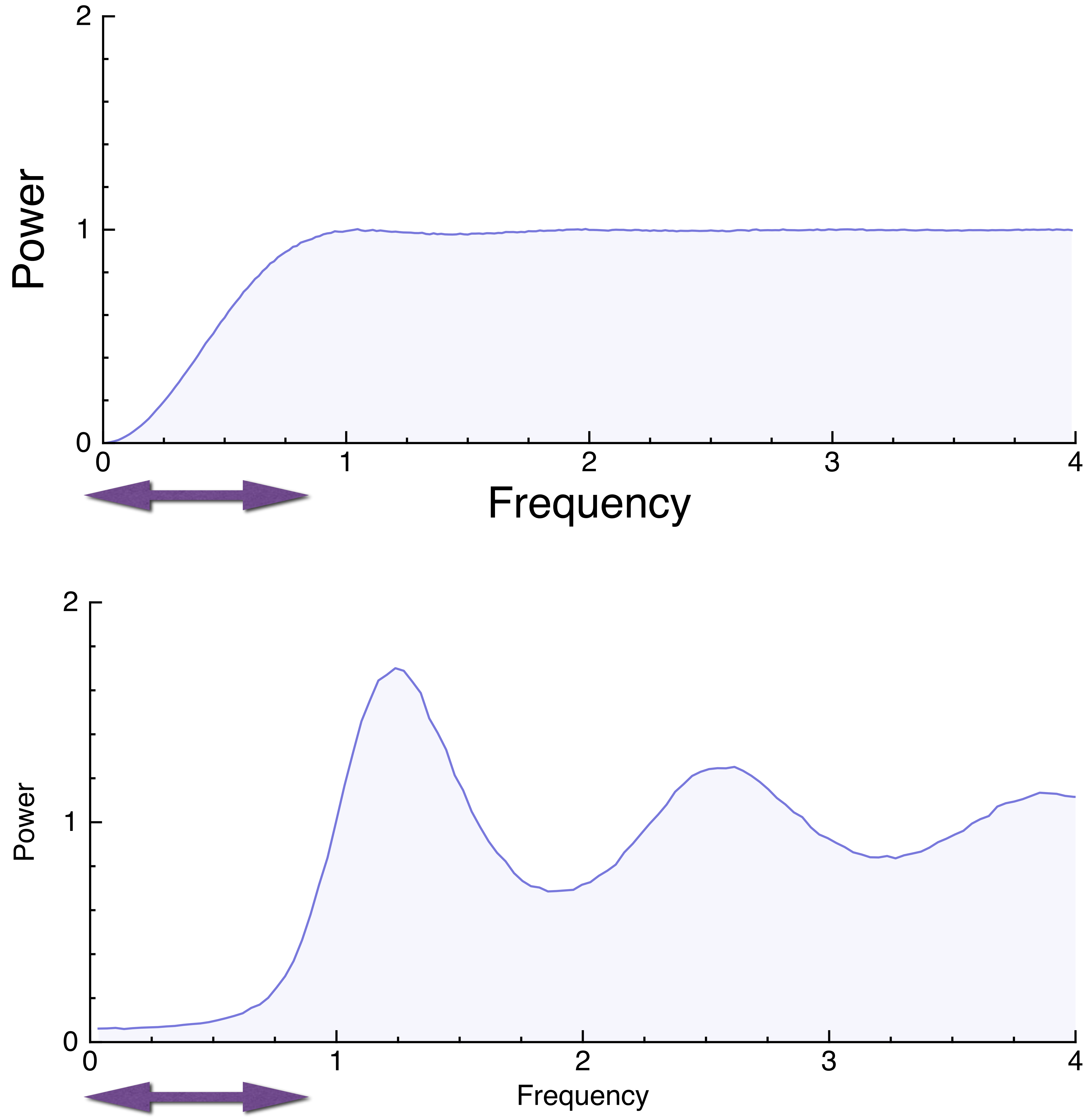


Poisson Disk

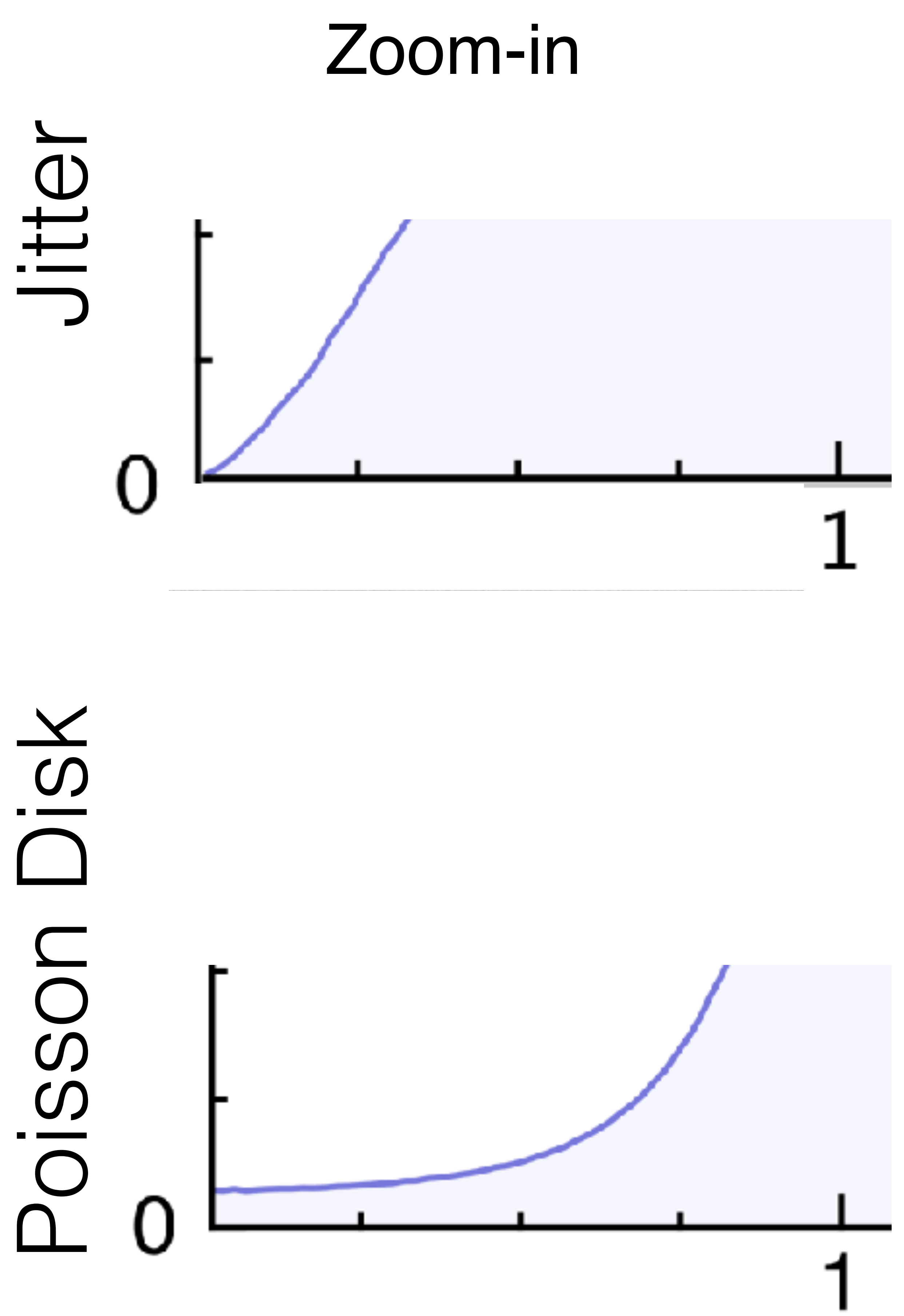


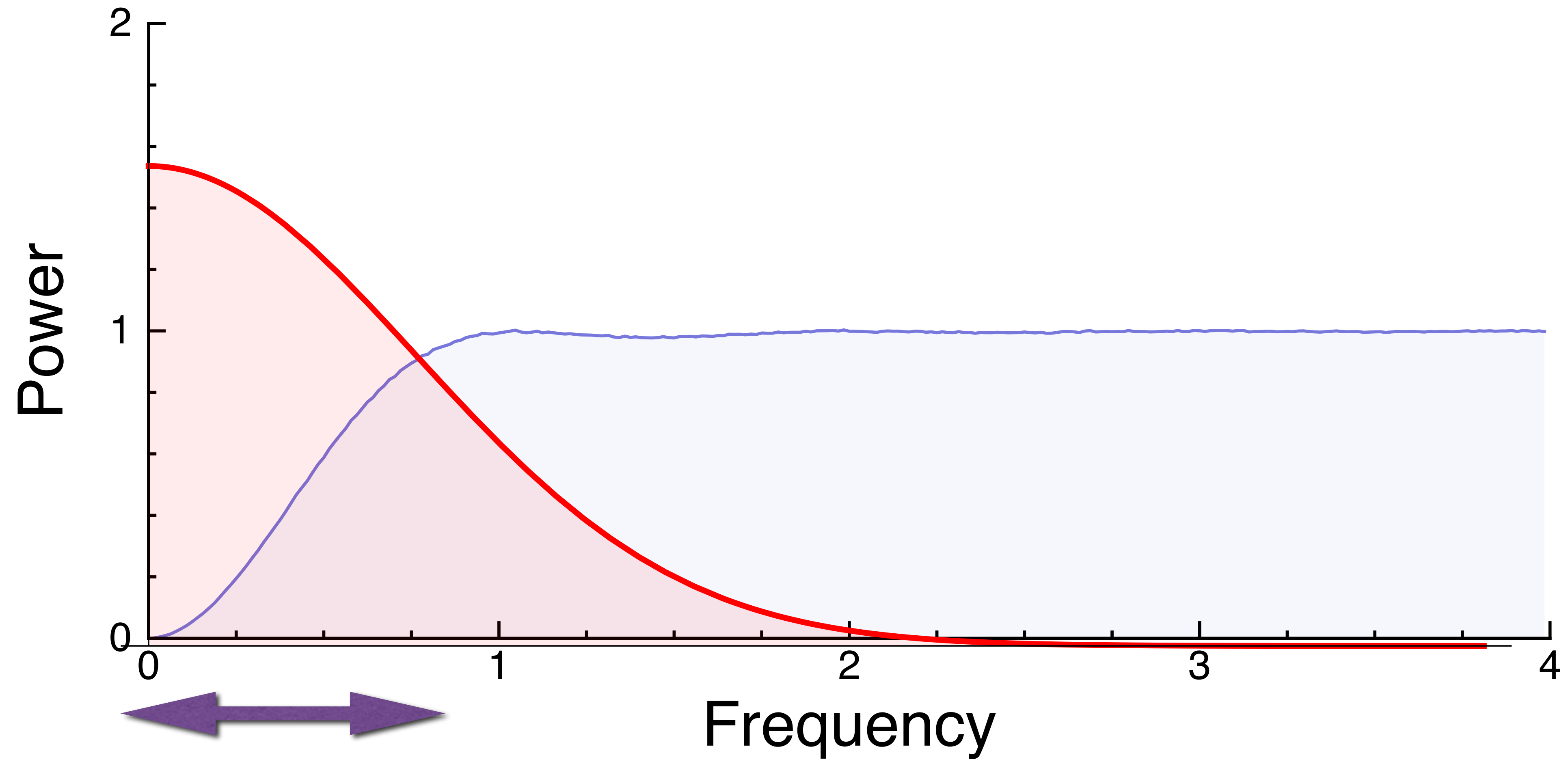
Low Frequency Region

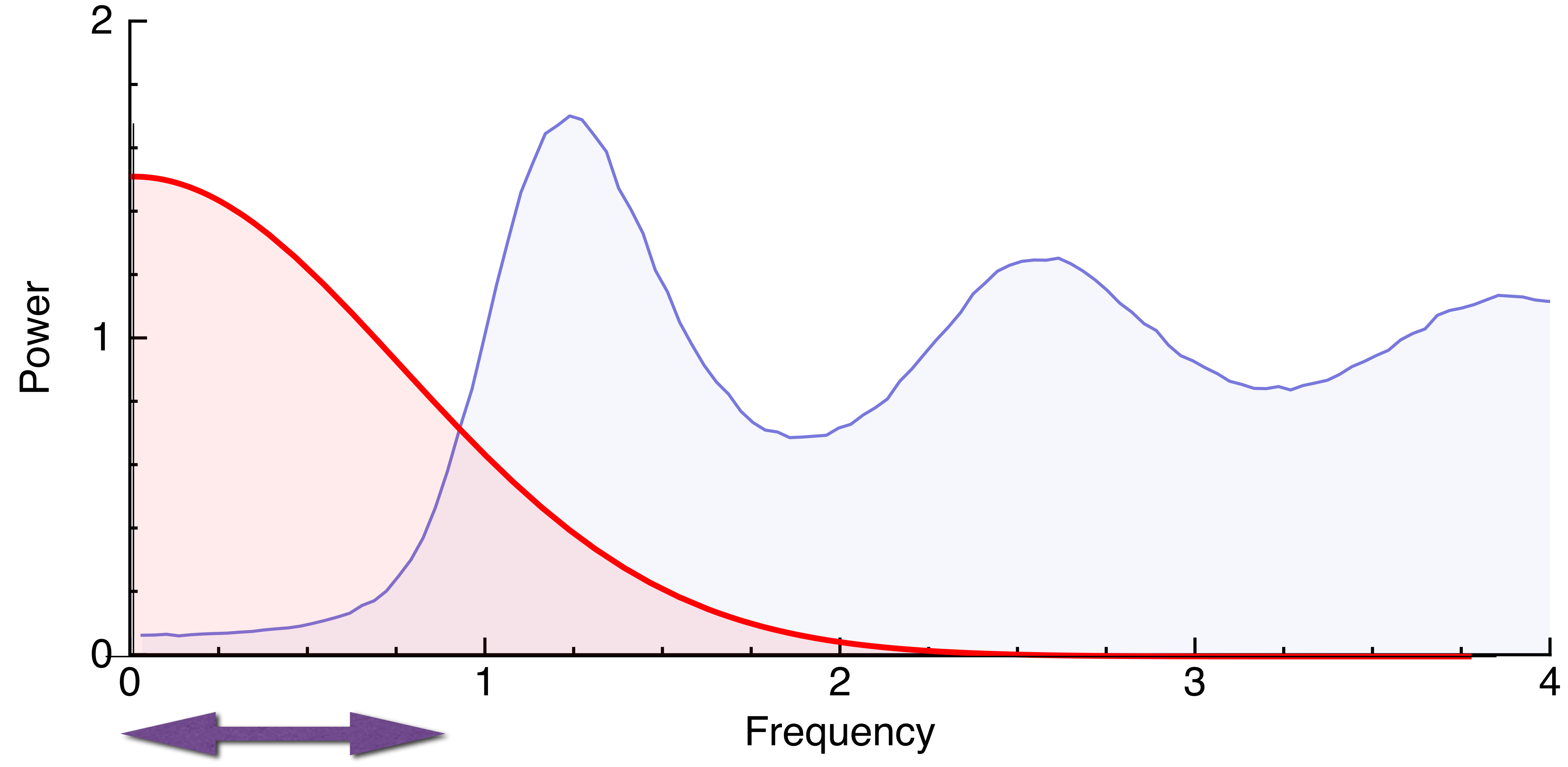




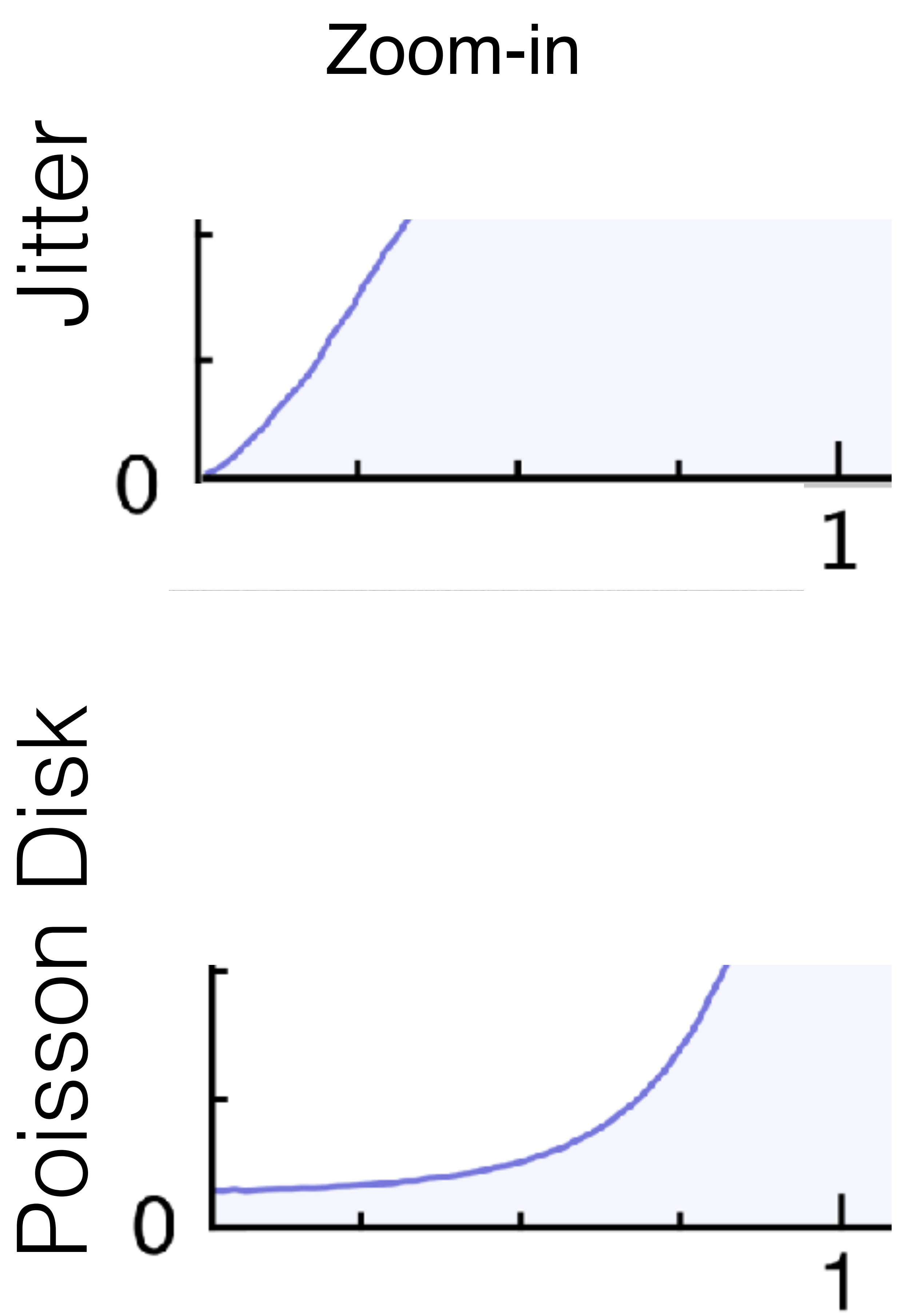
Variance for Low Sample Count

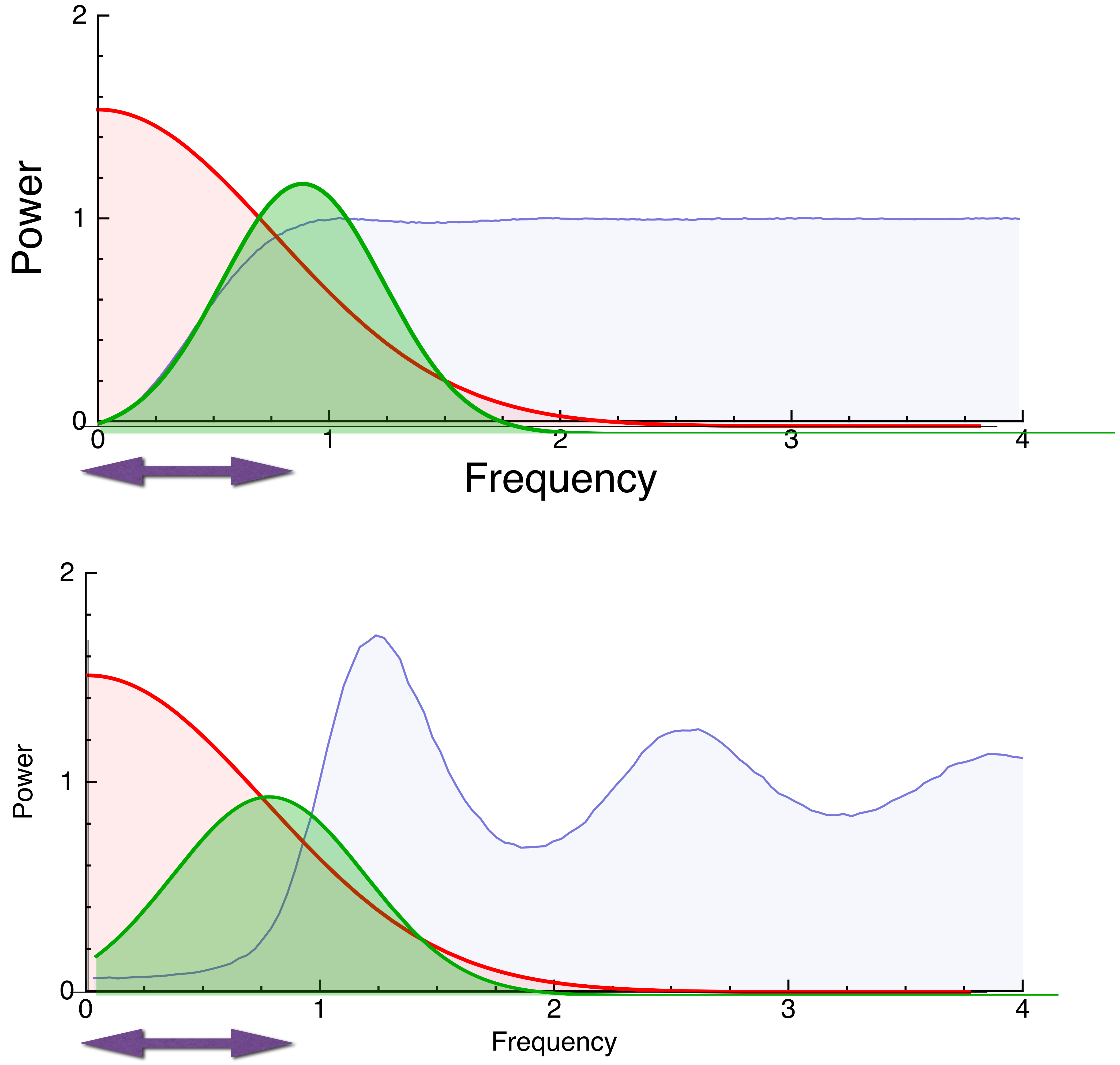






Variance for Low Sample Count

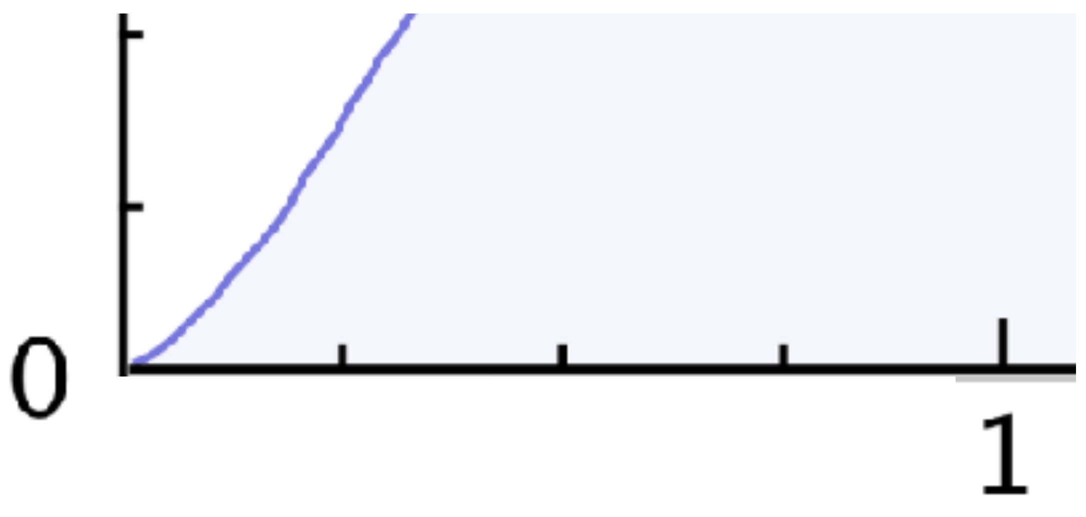


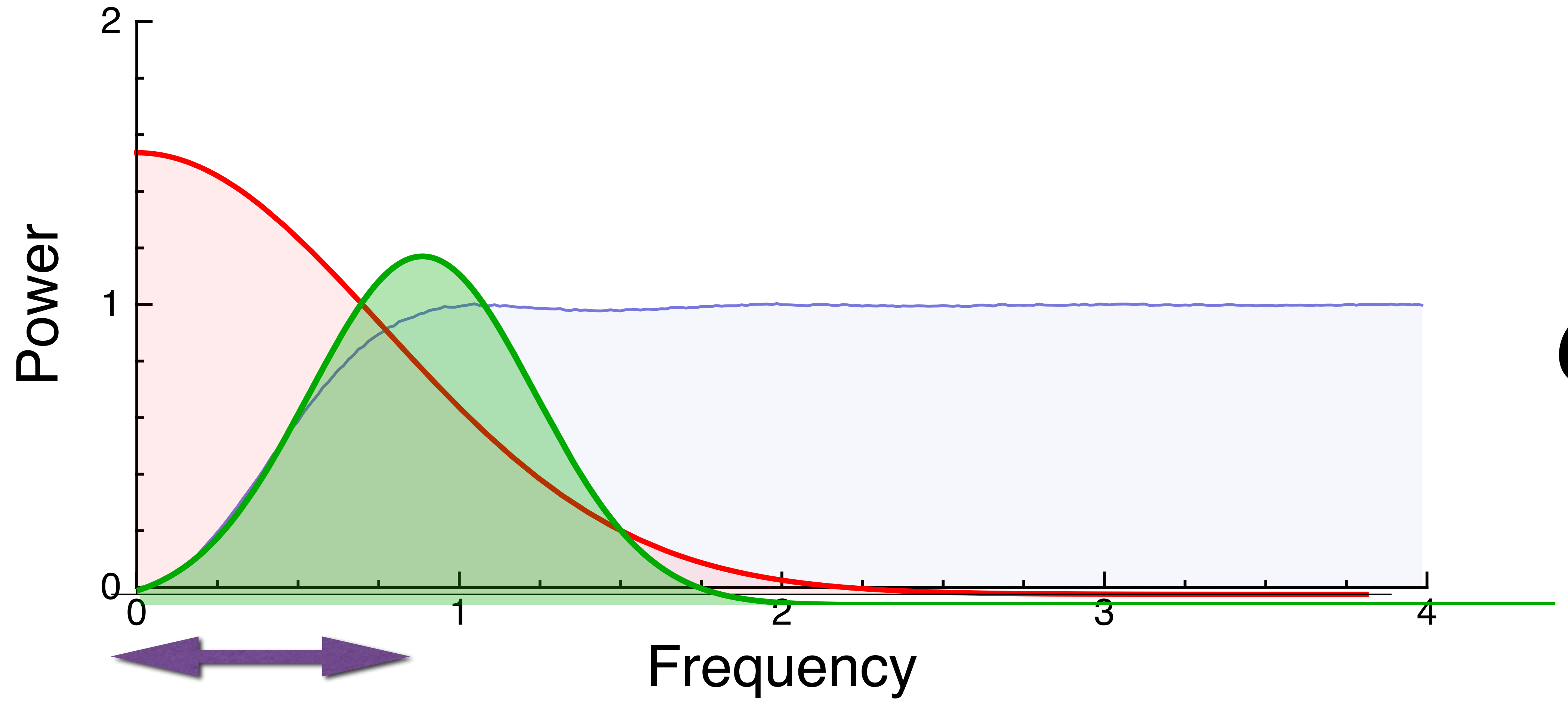


Variance for Increasing Sample Count

Zoom-in

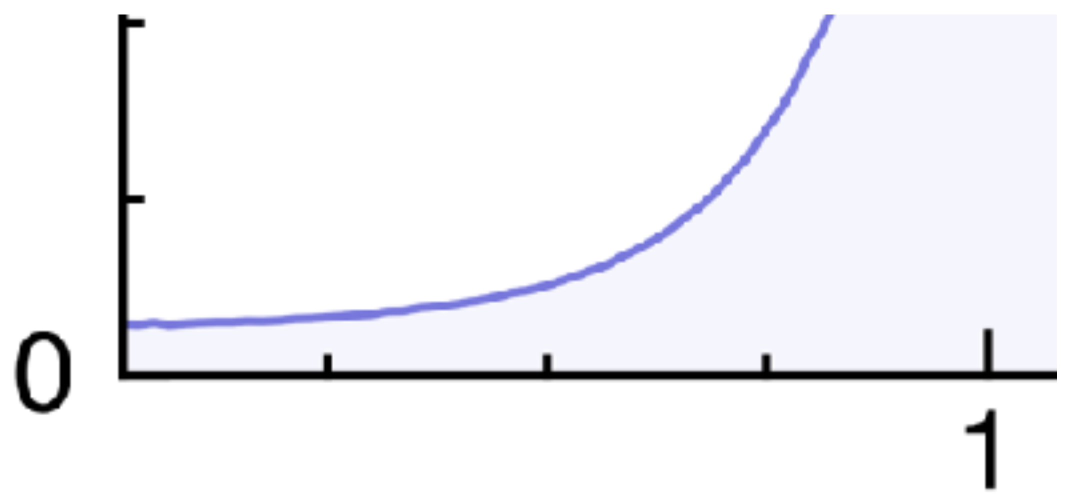
Jitter

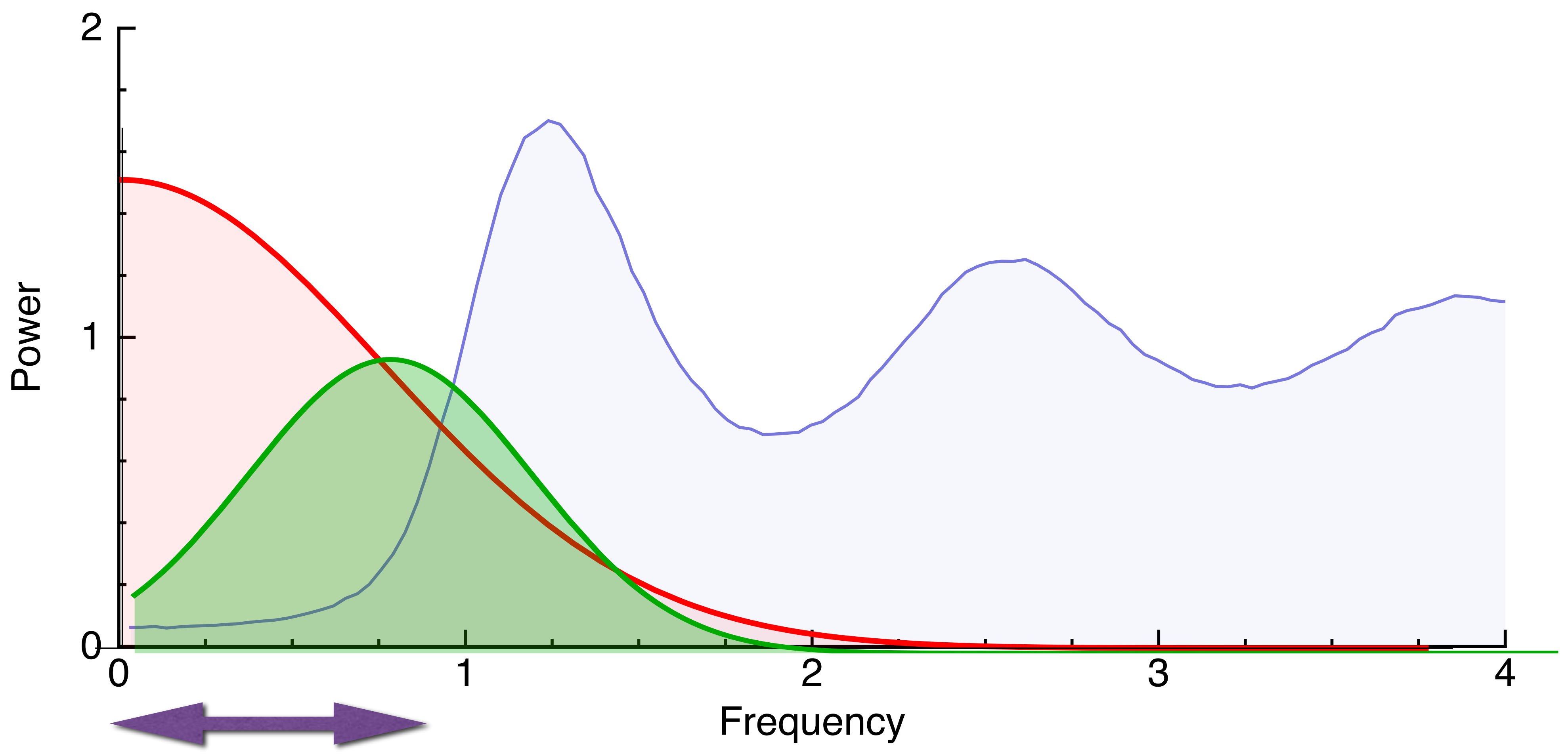




$\mathcal{O}(N^{-2})$

Poisson Disk

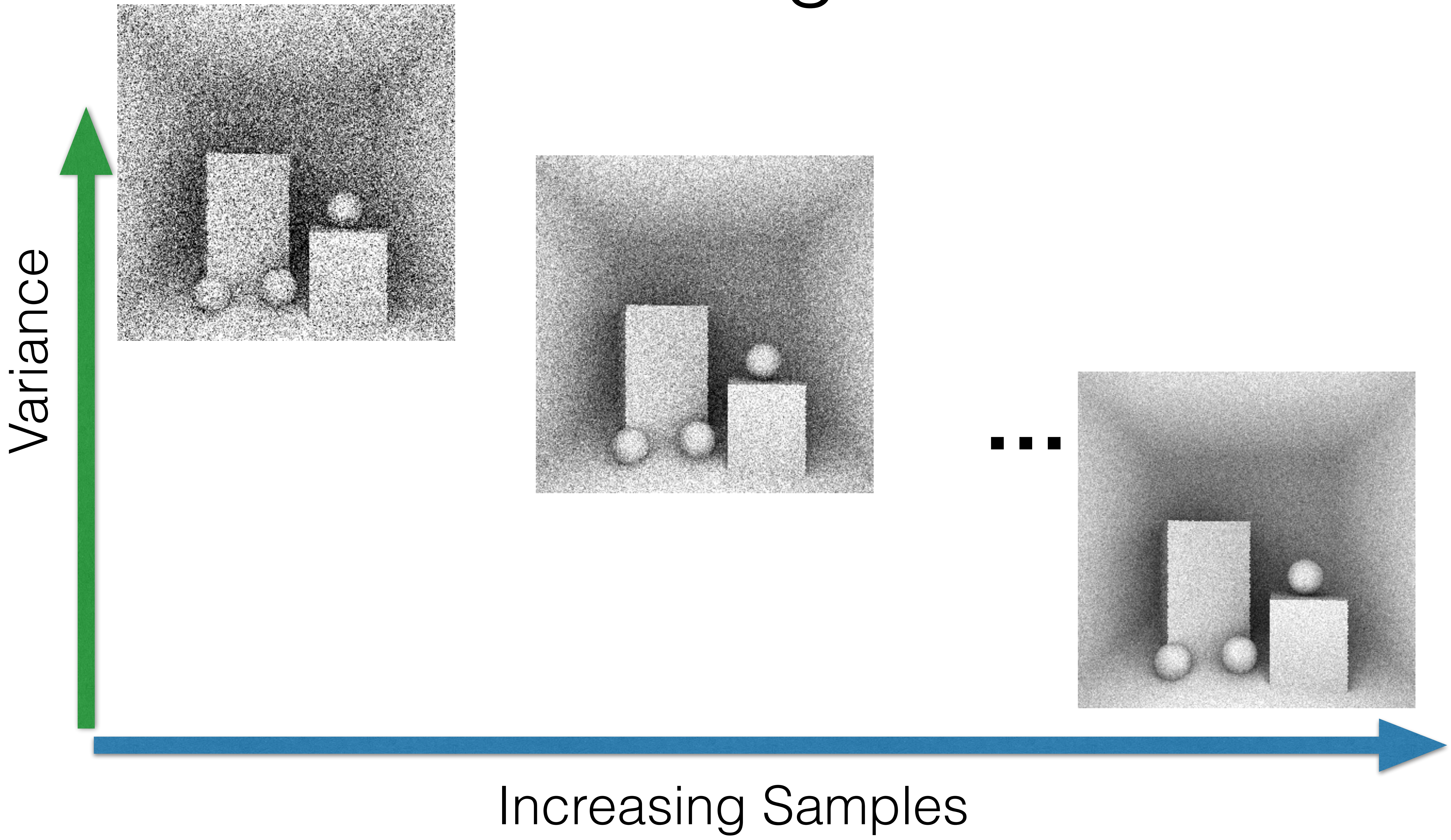




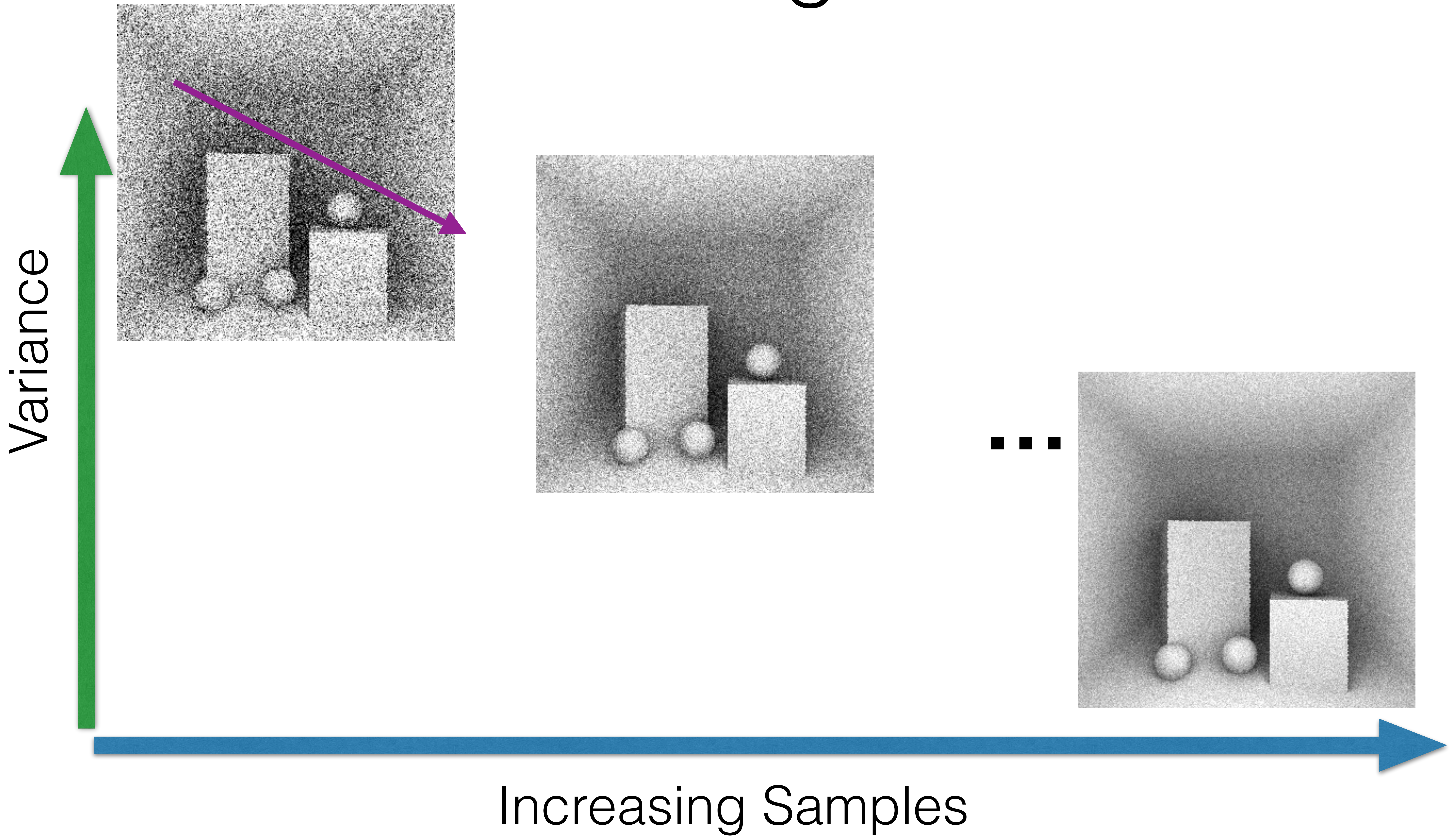
$\mathcal{O}(N^{-1})$

Experimental Verification

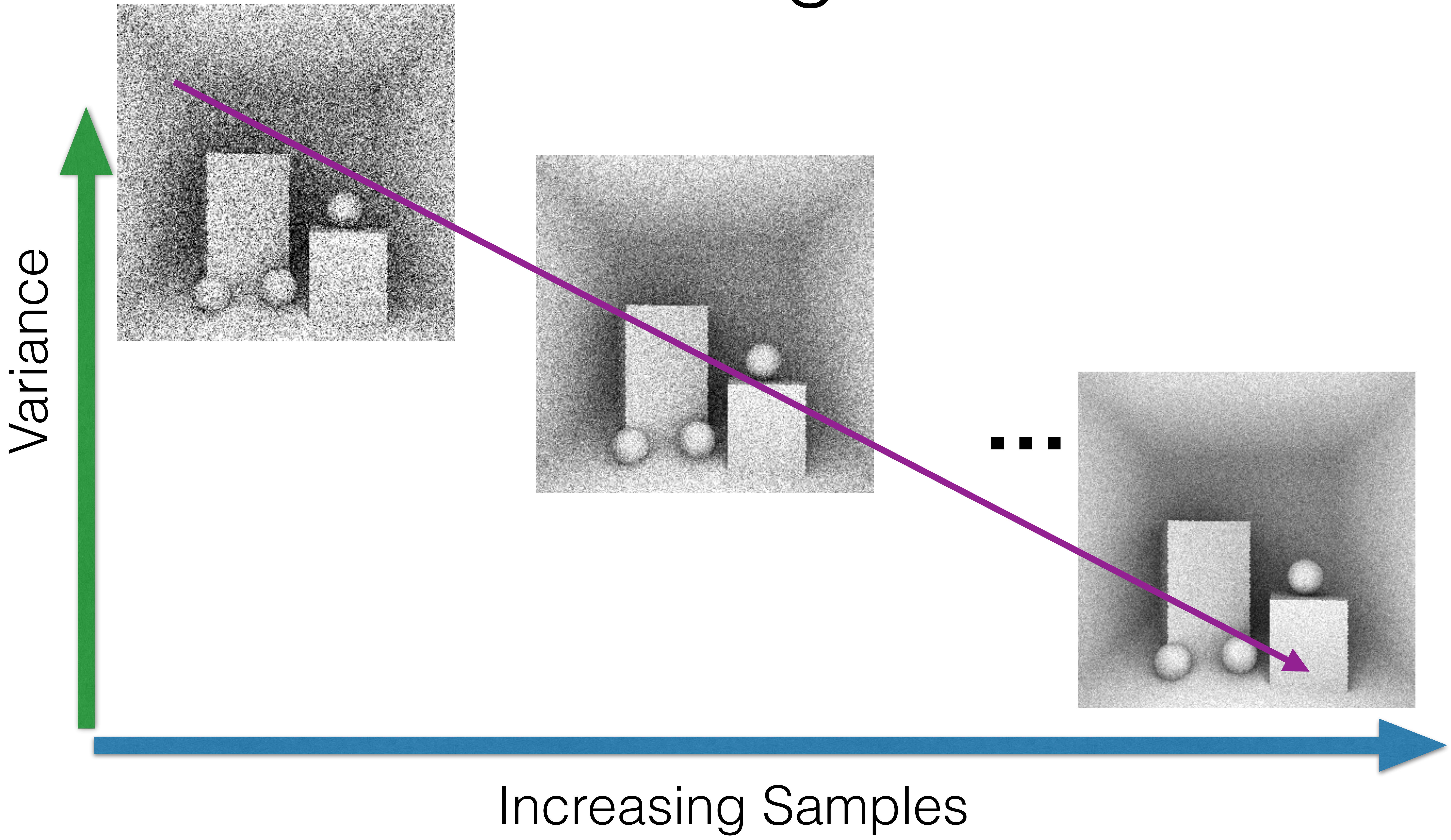
Convergence rate



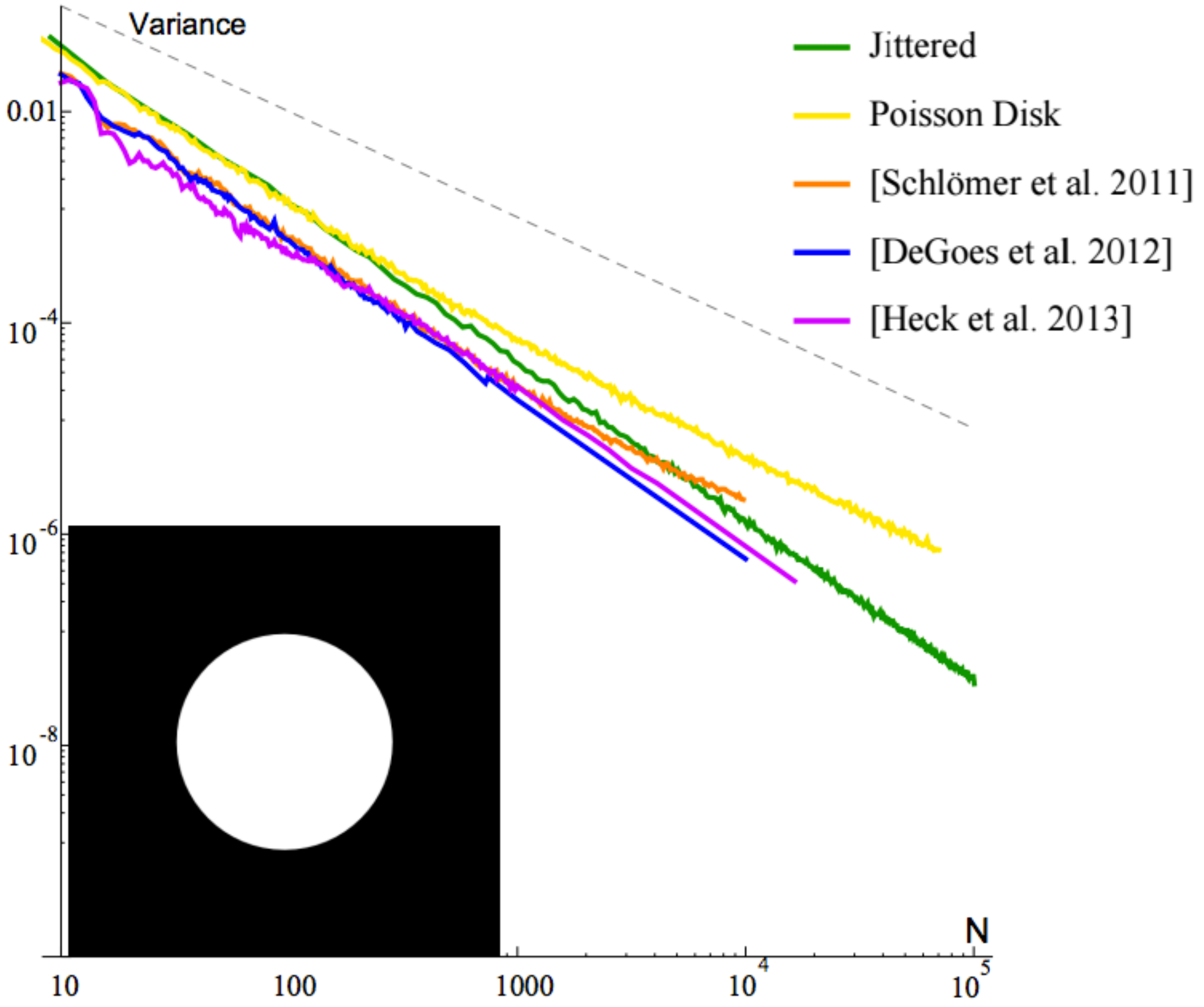
Convergence rate



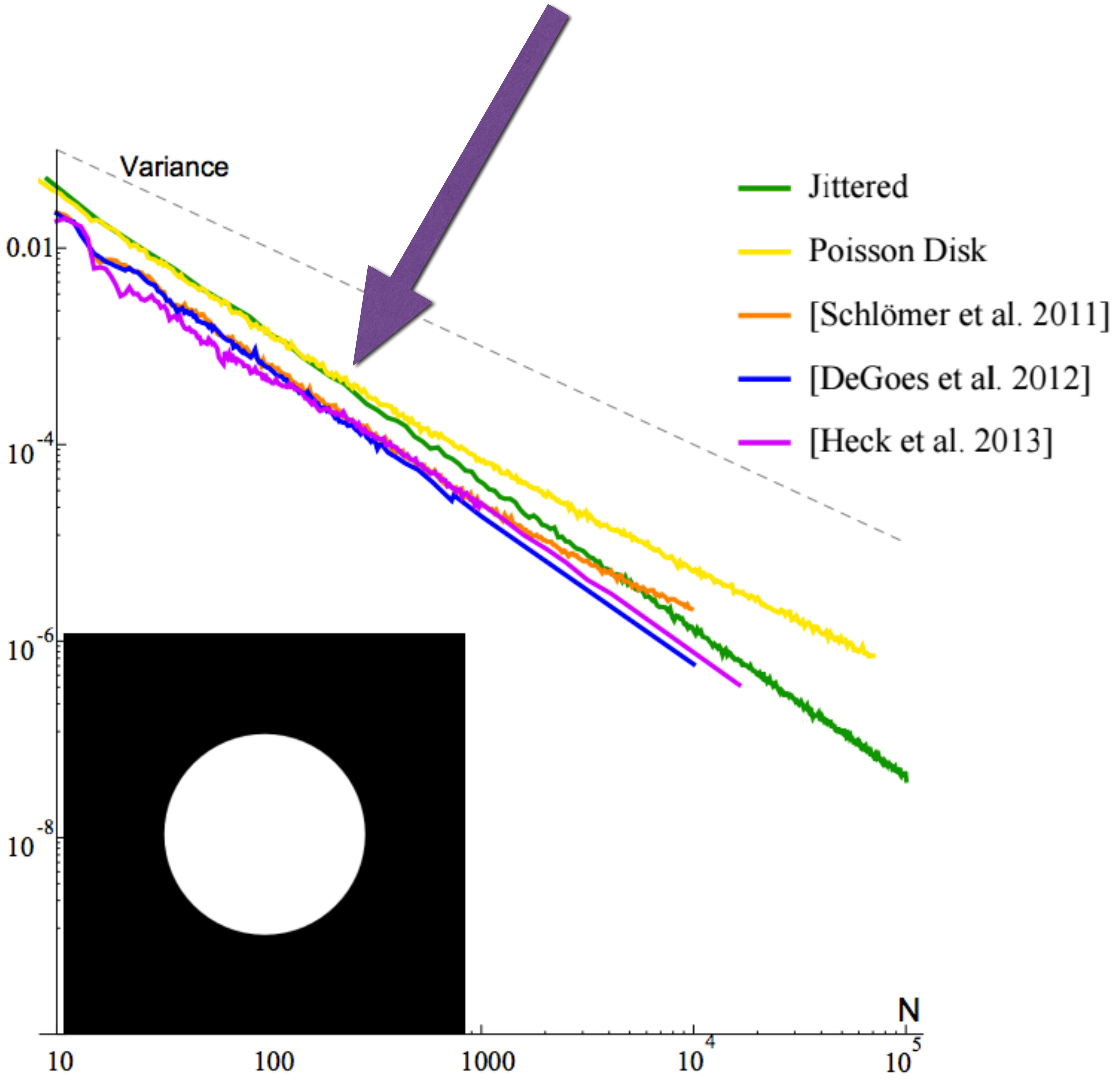
Convergence rate



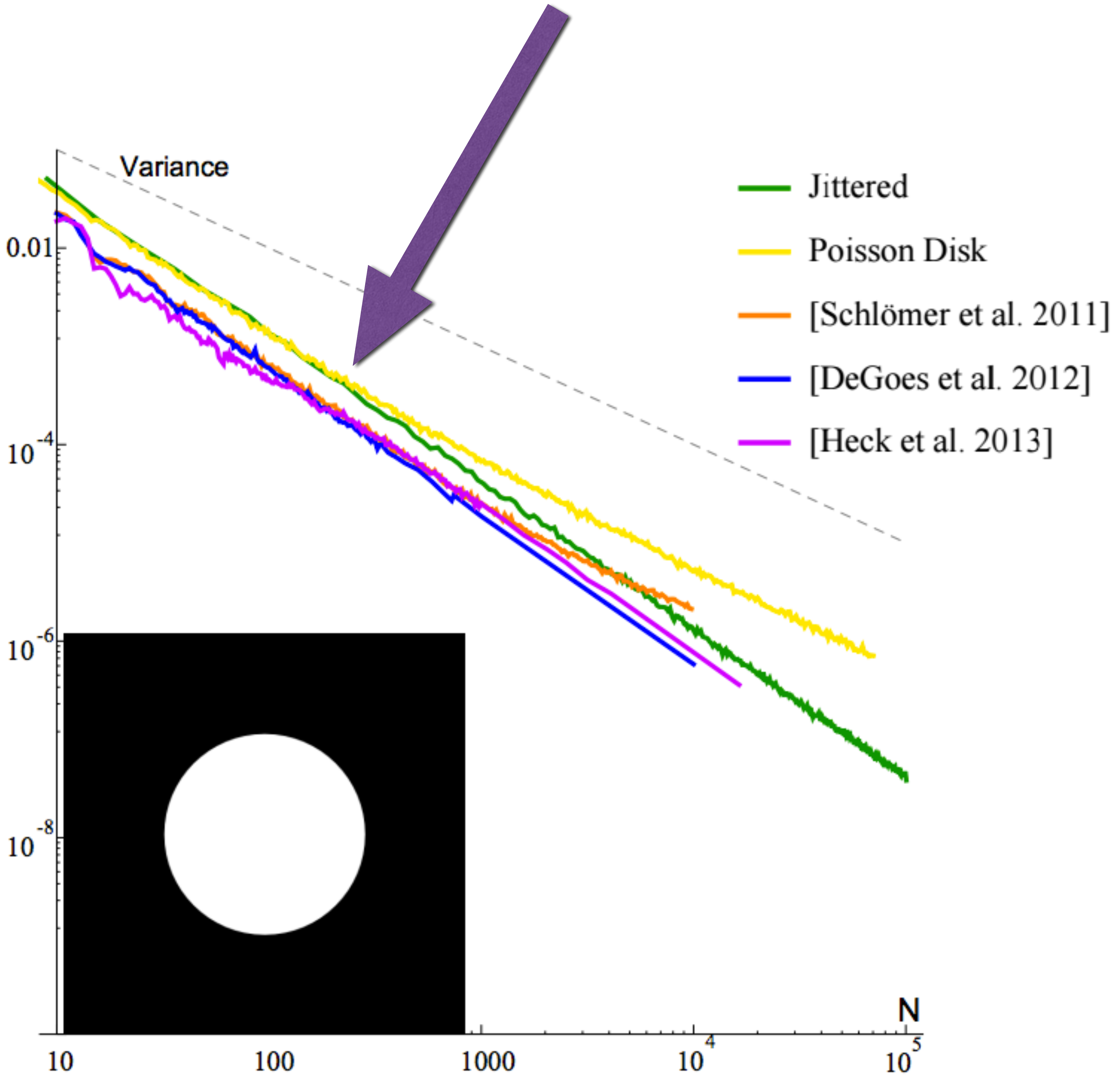
Disk Function as Worst Case



Disk Function as Worst Case



Disk Function as Worst Case



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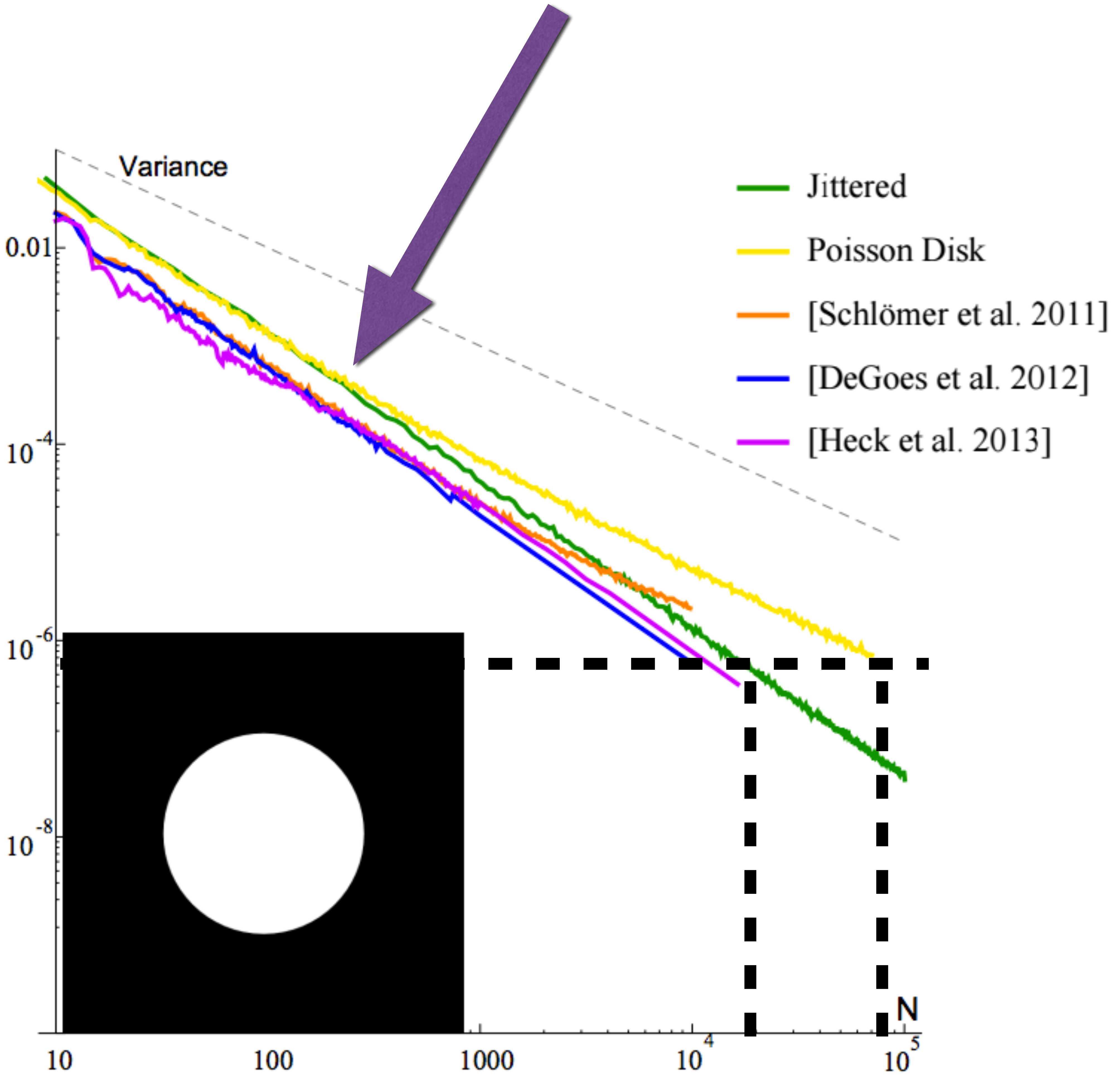
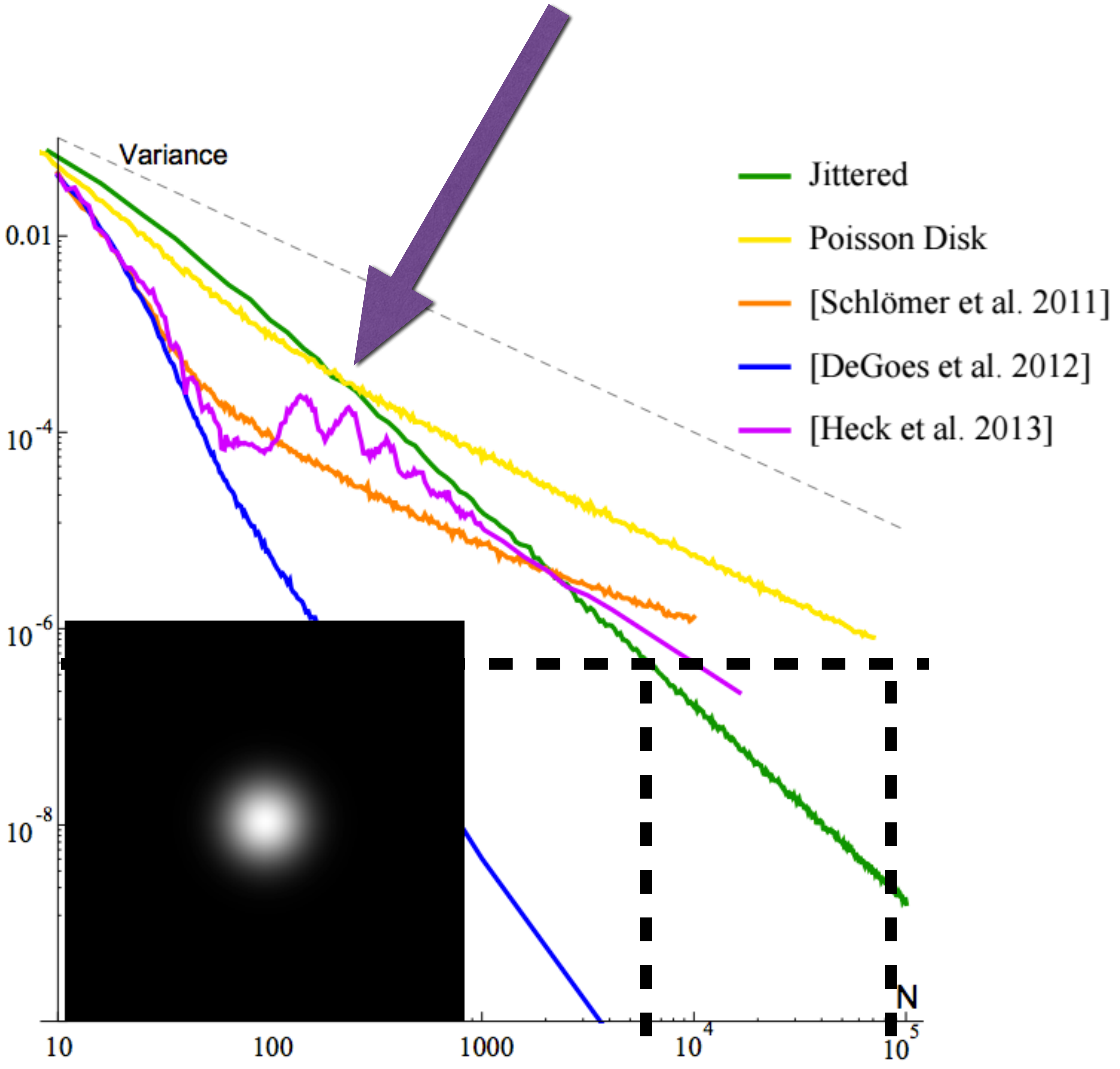
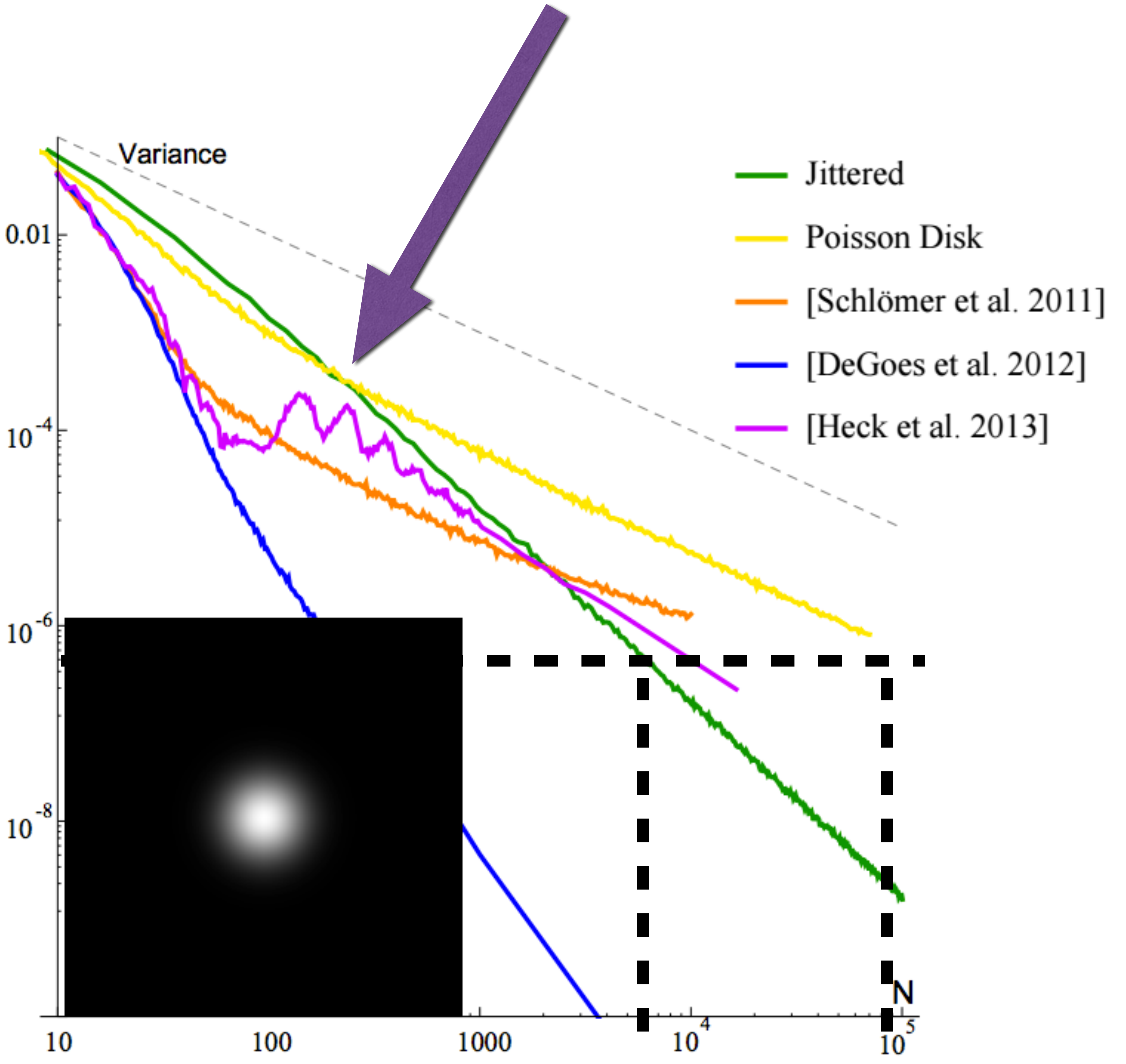


Gaussian as Best Case



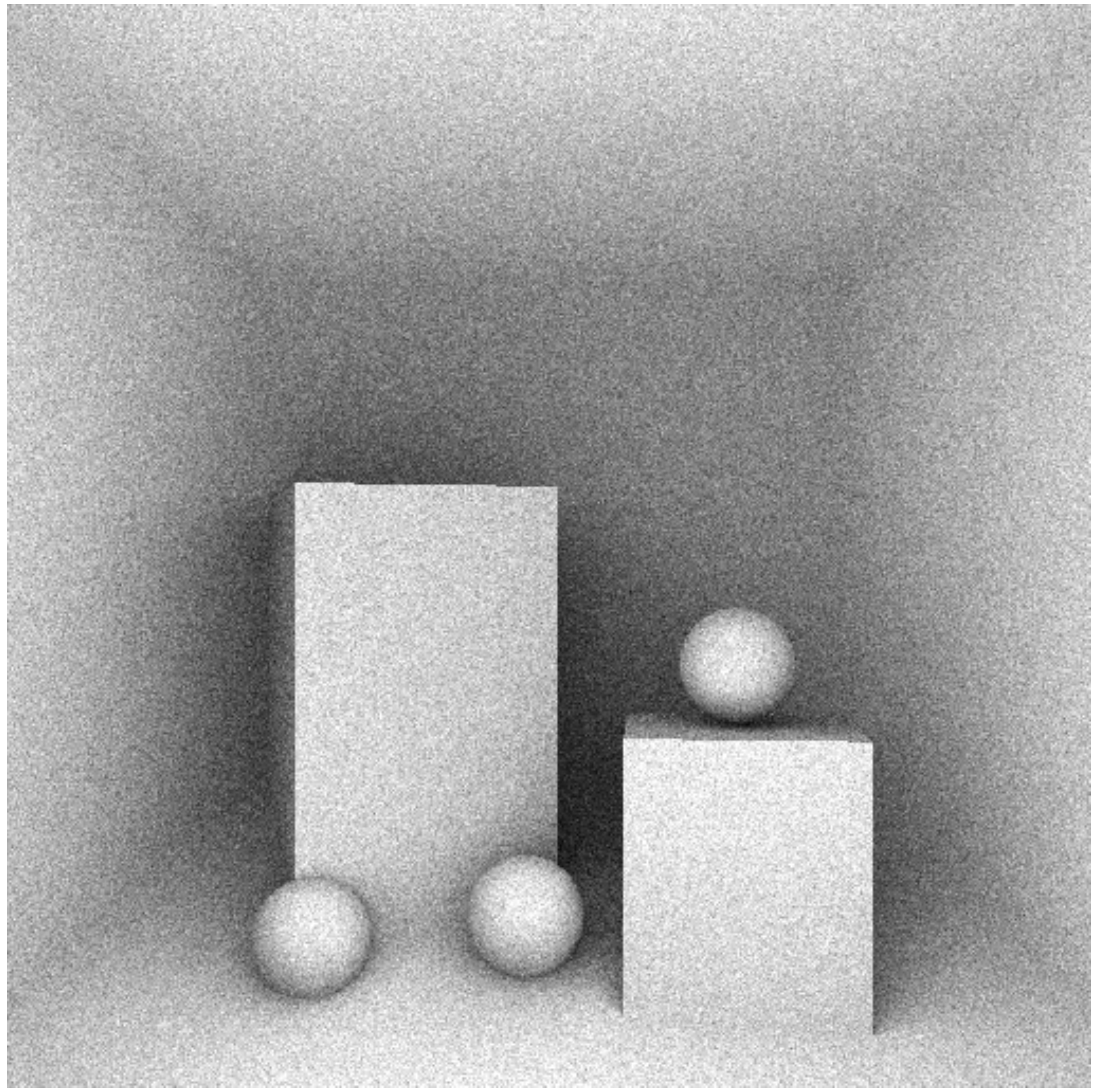
Gaussian as Best Case



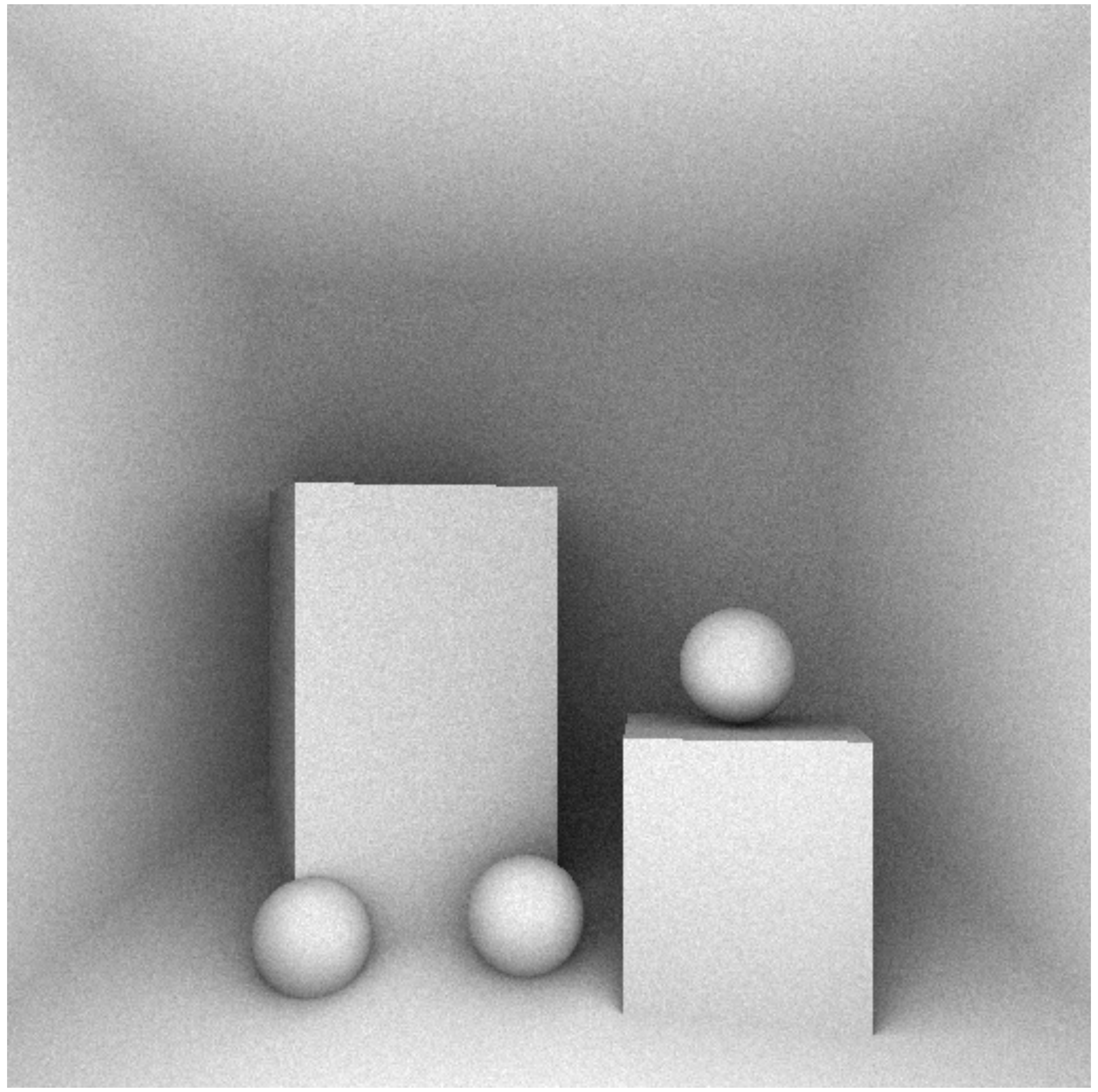
Ambient Occlusion Examples

Random vs Jittered

96 Secondary Rays



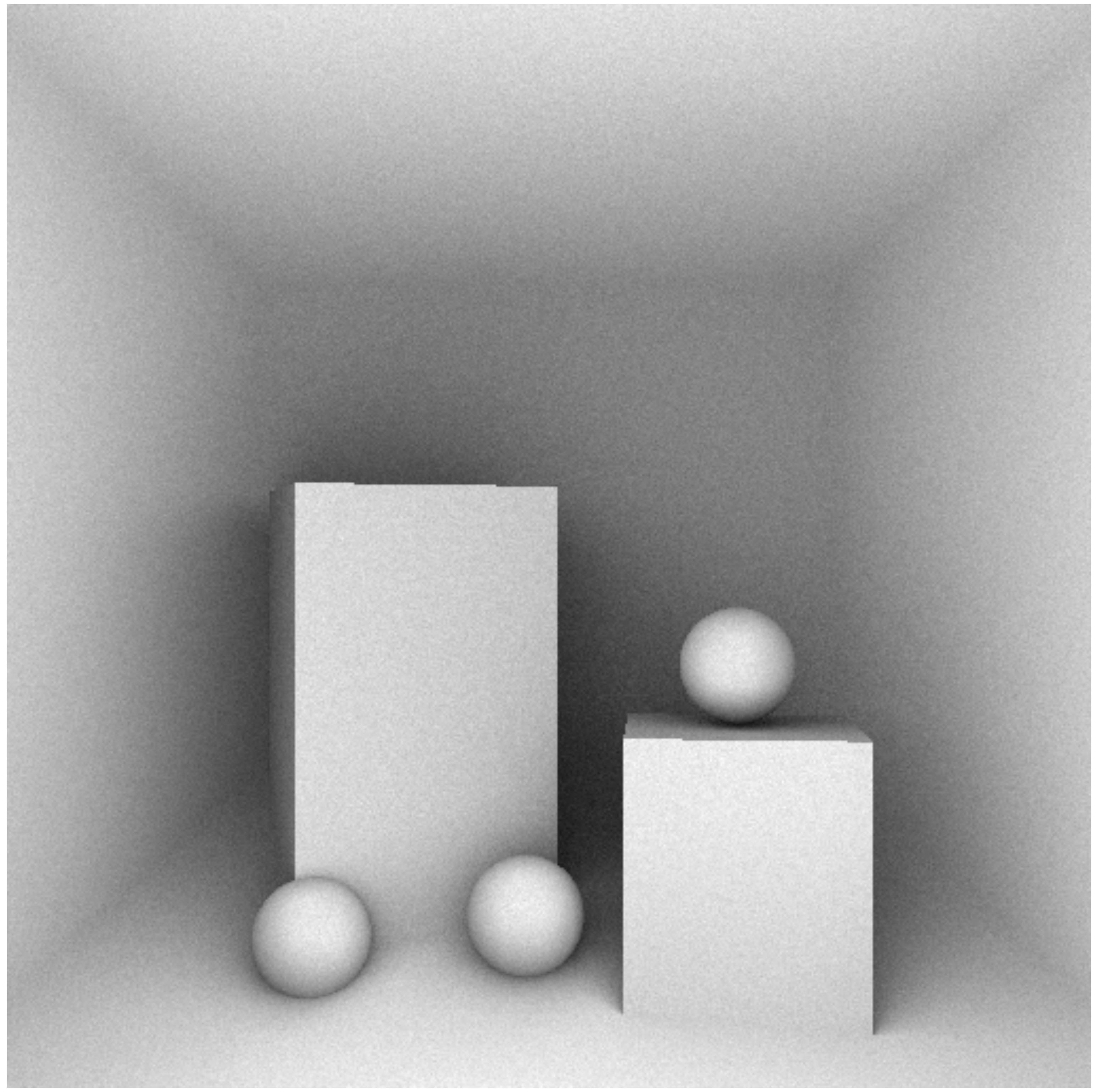
MSE: 4.74 x 10e-3



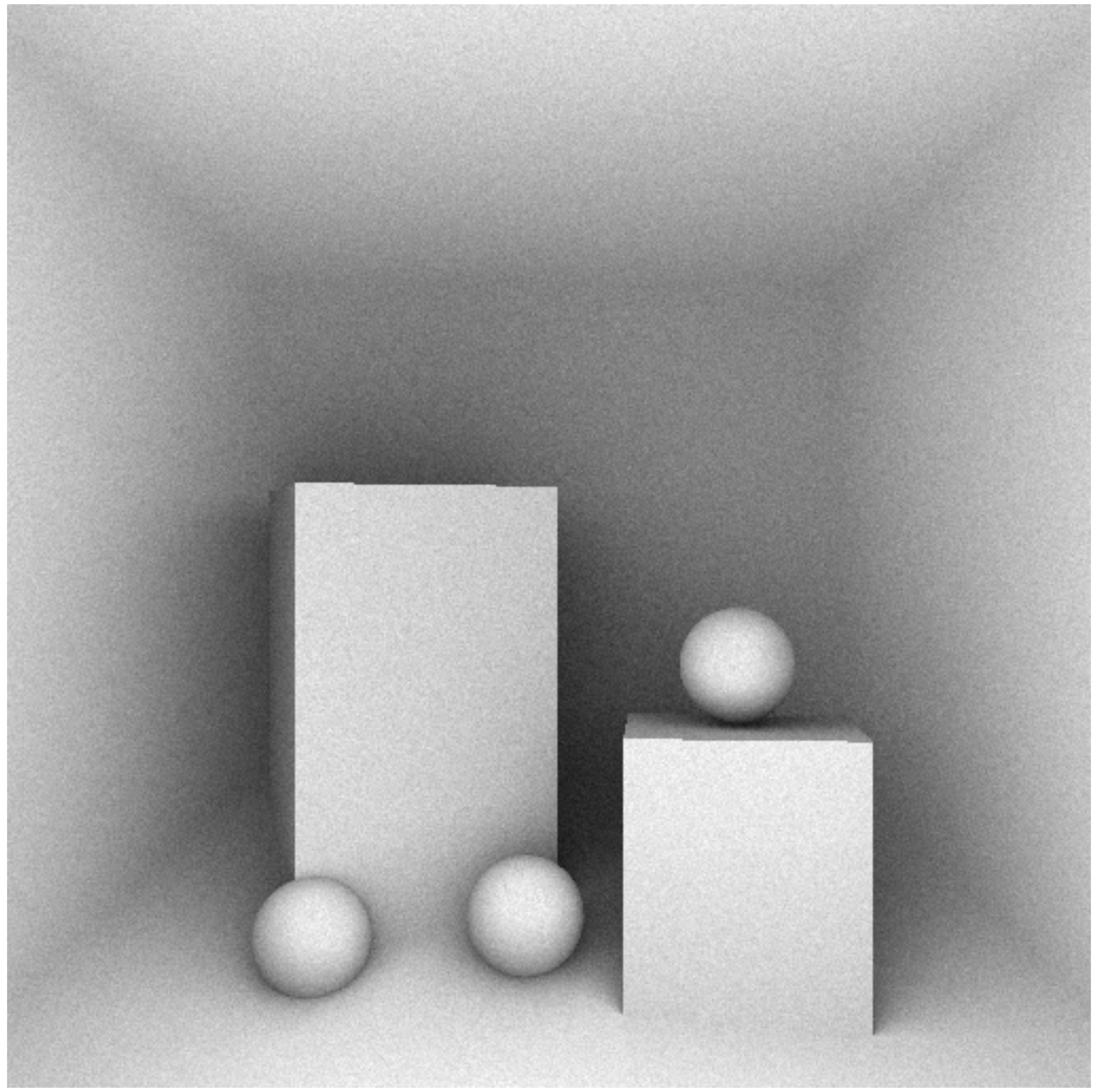
MSE: 8.56 x 10e-4

CCVT vs. Poisson Disk

96 Secondary Rays

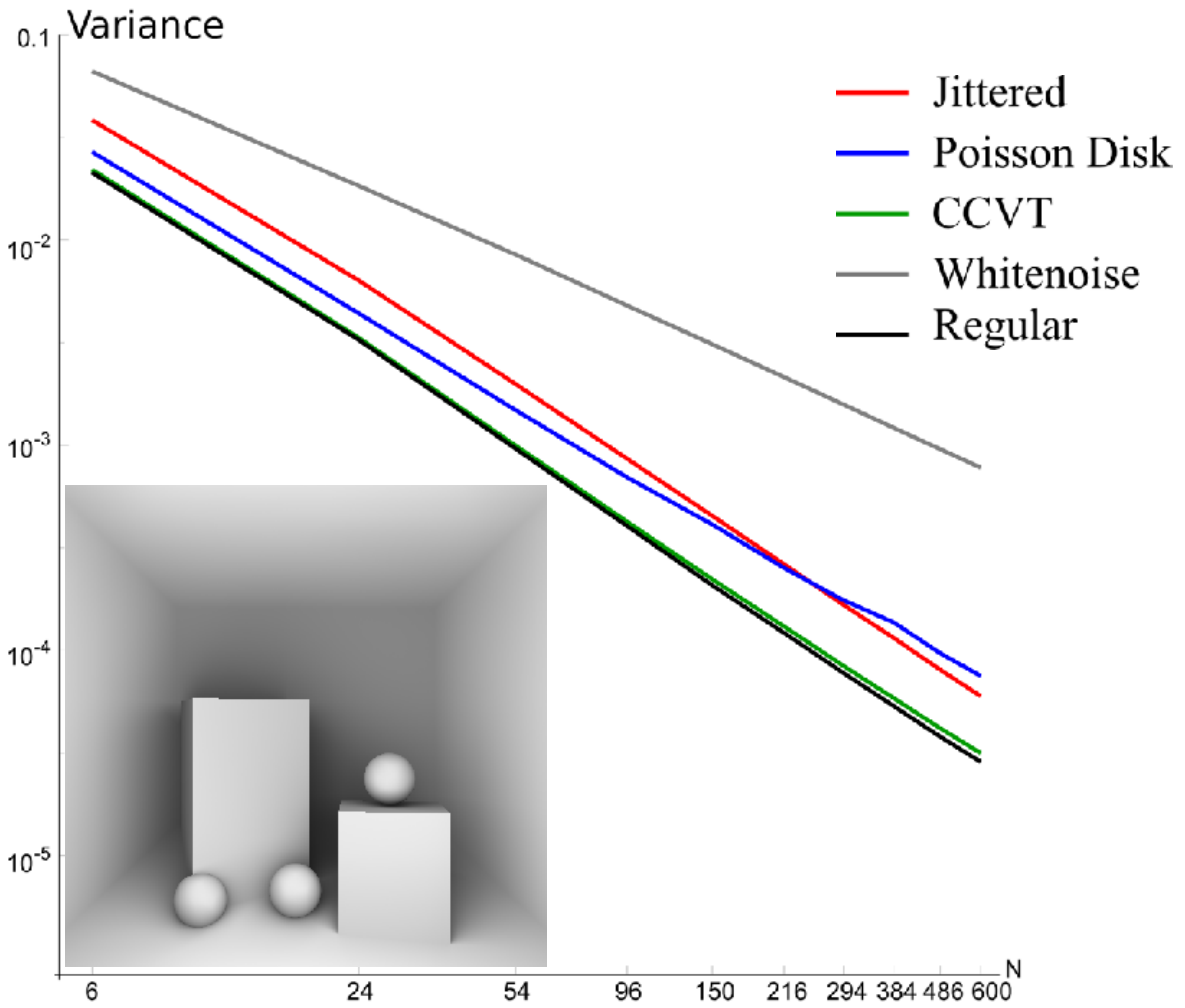


MSE: 4.24 x 10e-4

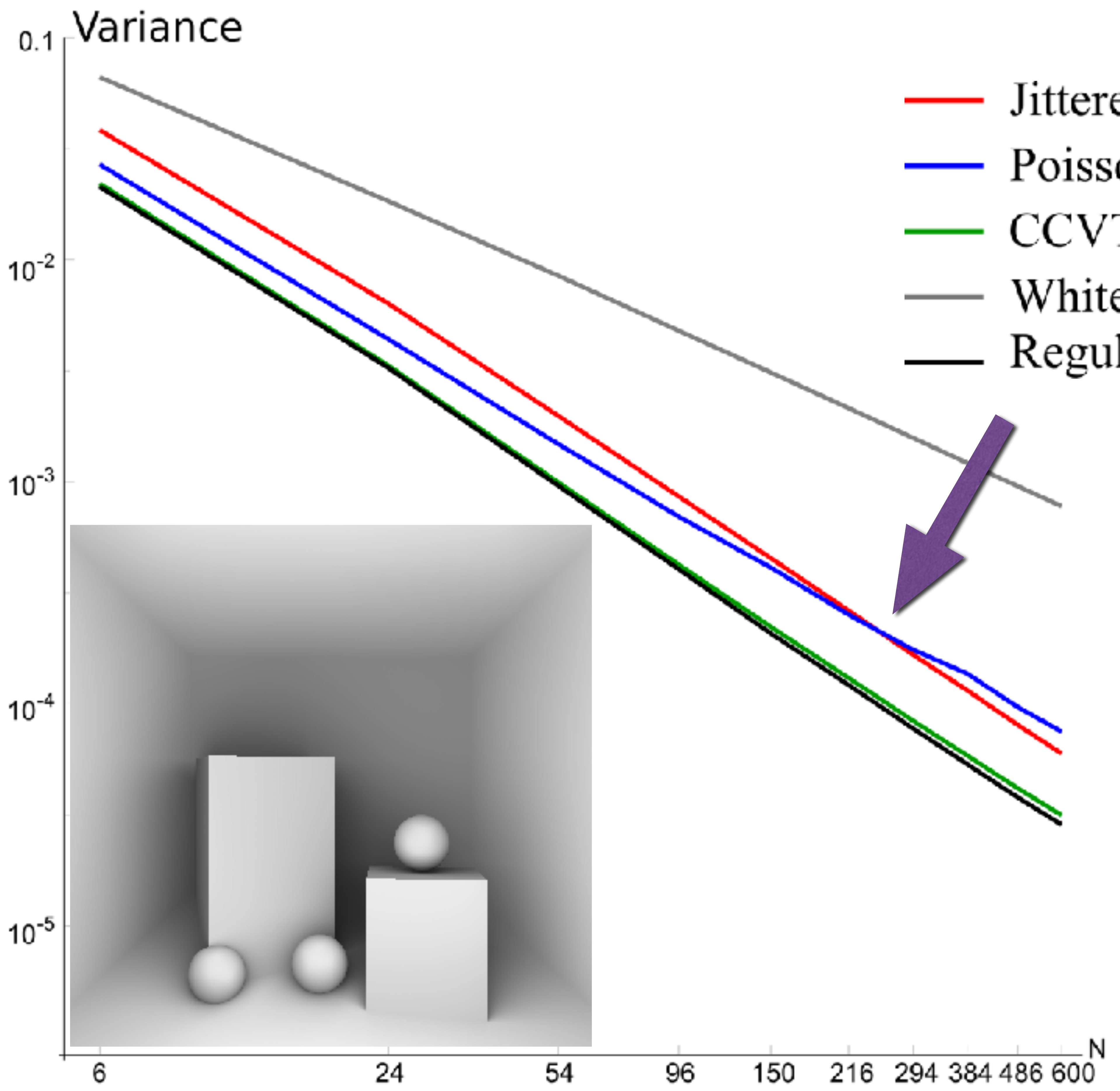


MSE: 6.95 x 10e-4

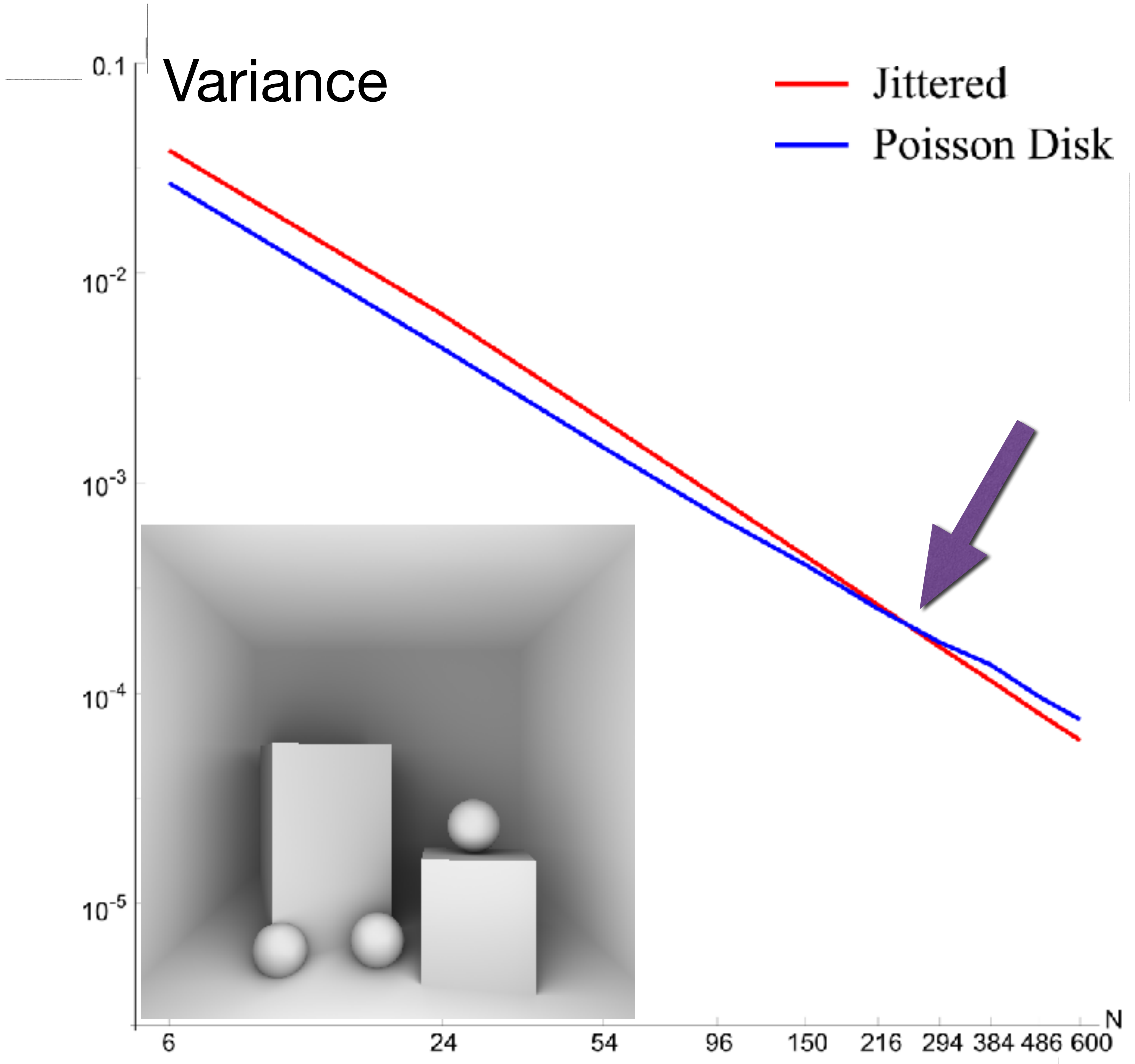
Convergence rates



Convergence rates



Jittered vs Poisson Disk



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- For real time rendering, blue noise samples are more effective in reducing variance for a given number of samples